THOMPSON'S GROUP F IS NOT STRONGLY AMENABLE

YAIR HARTMAN, KATE JUSCHENKO, OMER TAMUZ, AND POOYA VAHIDI FERDOWSI

Abstract. We show that Thompson's group F has a topological action on a compact metric space that is proximal and has no fixed points.

1. Introduction

In his book "Proximal Flows" [9] Glasner defines the notion of a *strongly amenable group*: A group is strongly amenable if each of its proximal actions on a compact space has a fixed point. A continuous action $G \curvearrowright X$ of a topological group on a compact Hausdorff space is proximal if for every $x, y \in X$ there exists a net $\{g_n\}$ of elements of G such that $\lim_n g_n x = \lim_n g_n y$.

Glasner shows that virtually nilpotent groups are strongly amenable and that non-amenable groups are not strongly amenable. He also gives examples of amenable — in fact, solvable — groups that are not strongly amenable. To the best of our knowledge there are no other such examples known. Furthermore, there are no other known examples of minimal proximal actions that are not also *strongly proximal*. An action $G \curvearrowright X$ is strongly proximal if the orbit closure of every Borel probability measure on G contains a point mass measure. This notion, as well as that of the related Furstenberg boundary [6–8], have been the object of a much larger research effort, in particular because a group is amenable if and only if all of its strongly proximal actions on compact spaces have fixed points.

Richard Thompson's group F has been alternatively "proved" to be amenable and non-amenable (see, e.g., [4]), and the question of its amenability is currently unresolved. In this paper we pursue the less ambitious goal of showing that is it not strongly amenable, and do so by directly constructing a proximal action that has no fixed points. This action does admit an invariant measure, and thus does not provide

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any information about the amenability of F. It is a new example of a proximal action which is not strongly proximal.

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2. Proofs

Let F denote Thompson's group F. In the representation of F as a group of piecewise linear transformations of $\mathbb{R}$ (see, e.g., [3]), it is generated by a and b which are given by

$$a(x) = x - 1$$
$$b(x) = \begin{cases} x & x \leq 0 \\ x/2 & 0 \leq x \leq 2 \\ x - 1 & 2 \leq 2 \end{cases} \cdot$$

The set of dyadic rationals $\Gamma = \mathbb{Z}[\frac{1}{2}]$ is the orbit of 0. The Schreier graph of the action $G \curvearrowright \Gamma$ with respect to the generating set $\{a, b\}$ is shown in Figure 1 (see [10,11]). The solid lines denote the a action and the dotted lines denote the b action; self-loops (i.e., points stabilized by a generator) are omitted. This graph consists of a tree-like structure (the blue and white nodes) with infinite chains attached to each node (the red nodes).

Equipped with the product topology, $\{-1, 1\}^\Gamma$ is a compact space on which F acts continuously by shifts:

$$[fx](\gamma) = x(f^{-1}\gamma). \tag{2.1}$$

A natural metric which induces the product topology is the one obtained by picking a base point, and considering two elements as close when they agree on a large ball around the base point.

Proposition 2.1. *Let $x_{-1}, x_{+1} \in \{-1, 1\}^\Gamma$ be the constant functions. Then for any $x \in \{-1, 1\}^\Gamma$ it holds that at least one of x_{-1}, x_{+1} is in the orbit closure $\overline{Fx}$.*

Proof. It is known that the action $F \curvearrowright \Gamma$ is strongly transitive (Lemma 4.2 in [5]), i.e. if we have V, W finite subsets of dyadic rationals and $|V| = |W|$, then there exists a $f \in F$ such that $f(V) = W$. Let $x \in \{-1, 1\}^\Gamma$. There is at least one of -1 and 1, say α, for which we have infinitely many $\gamma \in \Gamma$ with $x(\gamma) = \alpha$. Now pick a growing sequence of balls in Γ covering Γ, i.e. $B_1 \subset B_2 \subset \dots$ with $\bigcup_n B_n = \Gamma$.

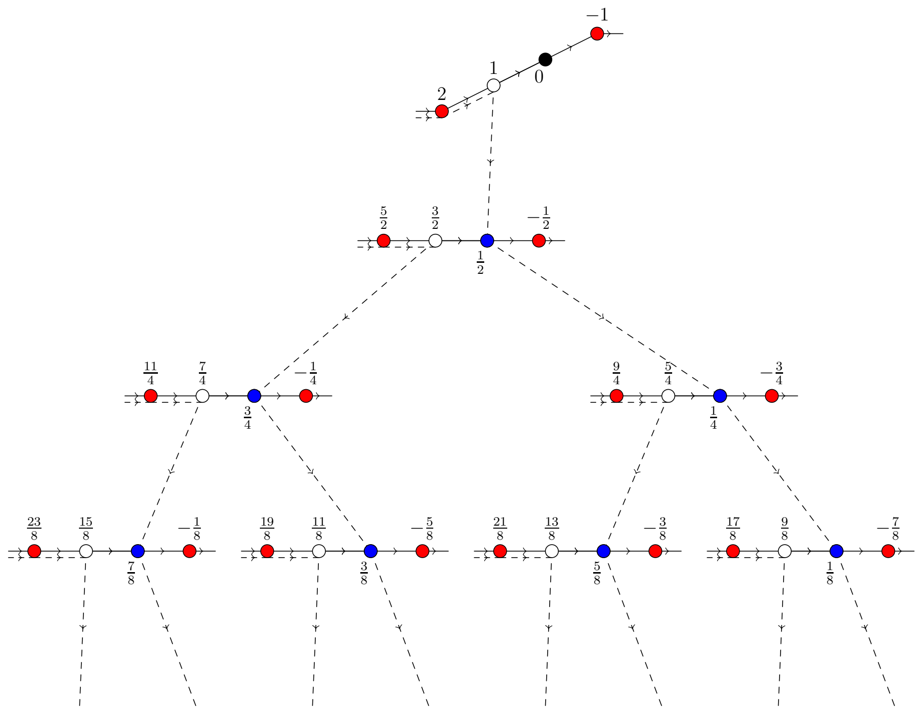

FIGURE 1. The action of F on Γ.

For each n we can choose $C_n \subset \Gamma$ with $x(C_n) = \{\alpha\}$ and $|C_n| = |B_n|$. By strong transitivity of the action of F on Γ, for any n, there exists an element $f_n \in F$ with $f_n(C_n) = B_n$. We get that:

$$[f_n x](B_n) = x(f_n^{-1} B_n) = x(C_n) = \{\alpha\}$$

and hence $f_n x \to x_\alpha$. $\square$

Given $x, y \in \{-1, 1\}^\Gamma$, let z be their pointwise product, given by $z(\gamma) = x(\gamma) \cdot y(\gamma)$. By Proposition 2.1 there exists a sequence $\{f_n\}$ of elements in F such that either $\lim_n f_n z = x_{+1}$ or $\lim_n f_n z = x_{-1}$. In the first case $\lim_n f_n x = \lim_n f_n y$, while in the second case $\lim_n f_n x = -\lim_n f_n y$, and so this action resembles a proximal action. In fact, by identifying each $x \in \{-1, 1\}^\Gamma$ with $-x$ one attains a proximal action, and indeed we do this below. However, this action has a fixed point — the constant functions — and therefore does not suffice to prove our result. We spend the remainder of this paper in deriving a new action from this one. The new action retains proximality but does not have fixed points.

Consider the path $({}^1/_2, {}^1/_4, {}^1/_8, \ldots, {}^1/_{2^n}, \ldots)$ in the Schreier graph of Γ (Figure 1); it starts in the top blue node and follows the dotted edges through the blue nodes on the rightmost branch of the tree. The

Benjamini-Schramm limit [1, 2] of this sequence of rooted graphs is given in Figure 2, and hence is also a Schreier graph of some transitive F-action $F \curvearrowright F/K$. In terms of the topology on the space $\mathrm{Sub}_F \subset \{0,1\}^F$ of the subgroups of F, the subgroup K is the limit of the subgroups K_n, where K_n is the stabilizer ${}^1/_{2^n}$. It is easy to verify that K is the subgroup of F consisting of the transformations that stabilize 0 and have right derivative 1 at 0 (although this fact will not be important).

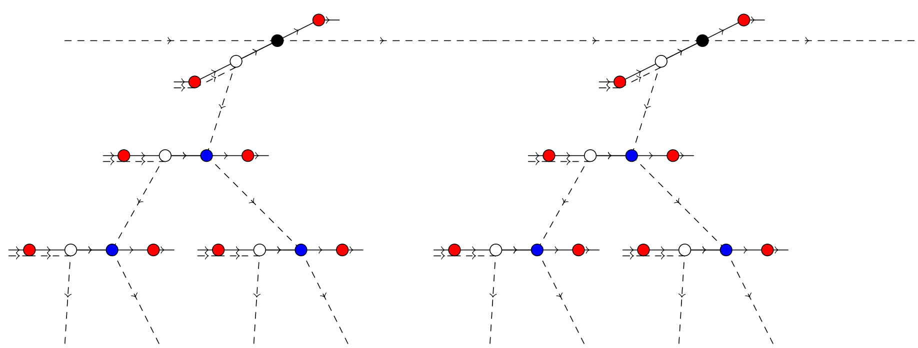

FIGURE 2. The action of F on Λ.

We can naturally identify with $\mathbb{Z}$ the black nodes at the top of Λ. Let Λ' be the subgraph of Λ in which the dotted edges connecting the black nodes have been removed. Given a black node $z \in \mathbb{Z}$, denote by T_z the connected component of z in Λ'; this includes the black node z, the hair that can be reached from it using solid edges, and the entire tree that hangs from it. Each graph T_z is isomorphic to the Schreier graph of Γ, and so the graph Λ is a covering graph of Γ (in the category of Schreier graphs). Let

$$\Psi \colon \Lambda \to \Gamma$$

be the covering map. That is, Ψ is a graph isomorphism when restricted to each T_z, with the black nodes in Λ mapped to the black node $0 \in \Gamma$.

Using the map Ψ we give names to the nodes in Λ. Denote the nodes in T_0 as $\{v^0_\gamma \,:\, \gamma \in \Gamma\}$ so that $\Psi(v^0_\gamma) = \gamma$. Likewise, in each T_z denote by v^z_γ the unique node in T_z that Ψ maps to γ. Hence

$$\Lambda = \{v^z_\gamma \,:\, z \in \mathbb{Z}, \gamma \in \Gamma\}$$

and the F-action is given by

$$av^z_\gamma = v^z_{a\gamma} \tag{2.2}$$

$$bv^z_\gamma = \begin{cases} v^z_{b\gamma} & \text{if } \gamma \neq 0 \\ v^{z+1}_0 & \text{if } \gamma = 0 \end{cases} \tag{2.3}$$

Using this notation we can define the “shift” σ, an automorphism of Λ, as follows: $\sigma(v_\gamma^z) = v_\gamma^{z+1}$. Note that $\Psi \circ \sigma = \Psi$.

As usual, the F-action on Λ (given explicitly in 2.2) defines an action on $\{-1,1\}^\Lambda$. It is straightforward to check that the actions of σ and F on $\{-1,1\}^\Lambda$ commute.

Equip $\{-1,1\}^\Lambda$ with the product topology. Let $Y \subset \{-1,1\}^\Lambda$ be the set of configurations y such that $y(v_\gamma^z) = y(v_\gamma^{z'})$ iff $z = z' \mod 2$. Equivalently

$$Y = \left\{ y \in \{-1,1\}^\Lambda \,:\, y(v_\gamma^z) \cdot y(v_\gamma^{z'}) = (-1)^{z+z'} \text{ for all } z, z' \in \mathbb{Z} \text{ and } \gamma \in \Gamma \right\}.$$

In words, Y is the set of $\{-1,1\}$-configurations on Λ in which the configuration of T_z is identical to that of T_0 if z is even, and is its mirror image if z is odd. Note that all σ-orbits in Y have two elements, and that $\sigma(x) = -x$ for all $x \in \{-1,1\}^\Gamma$.

Claim 2.2. *Y is a closed subset of $\{-1,1\}^\Lambda$ that is invariant to the F action.*

Proof. Let $\{y_m\}_m$ be a sequence of elements in Y having a pointwise limit in $\{-1,1\}^\Lambda$, and let $y = \lim_m y_m$. We want to show that $y \in Y$, and indeed for any $v_\gamma^z, v_\gamma^{z'} \in \Lambda$

$$\begin{aligned} y(v_\gamma^z) \cdot y(v_\gamma^{z'}) &= \lim_m y_m(v_\gamma^z) \cdot \lim_m y_m(v_\gamma^{z'}) \\ &= \lim_m y_m(v_\gamma^z) \cdot y_m(v_\gamma^{z'}) \\ &= \lim_m (-1)^{z+z'} \\ &= (-1)^{z+z'} \end{aligned}$$

So $y \in Y$.

Now we want to show that Y is invariant to the F action. For that, it is enough to show that Y is invariant to the actions of a, b, a^{-1}, b^{-1}. Let $y \in Y$ and $z, z' \in \mathbb{Z}$.

For $f \in \{a, b, a^{-1}, b^{-1}\}$ and $\gamma \neq 0$, or $f \in \{a, a^{-1}\}$ and $\gamma = 0$, we have

$$\begin{aligned} [fy](v_\gamma^z) \cdot [fy](v_\gamma^{z'}) &= y(f^{-1} v_\gamma^z) \cdot y(f^{-1} v_\gamma^{z'}) \\ &= y(v_{f^{-1}\gamma}^z) \cdot y(v_{f^{-1}\gamma}^{z'}) \\ &= (-1)^{z+z'} \end{aligned}$$

For $f \in \{b, b^{-1}\}$ and $\gamma = 0$ we have:

$$\begin{aligned}
[fy](v_0^z) \cdot [fy](v_0^{z'}) &= y(f^{-1}v_0^z) \cdot y(f^{-1}v_0^{z'}) \\
&= y(v_0^{z\pm1}) \cdot y(v_0^{z'\pm1}) \\
&= (-1)^{(z\pm1)+(z'\pm1)} \\
&= (-1)^{z+z'}
\end{aligned}$$

So in all cases we verified that for all $f \in \{a, b, a^{-1}, b^{-1}\}$ we have $[fy](v_\gamma^z) \cdot [fy](v_\gamma^{z'}) = (-1)^{z+z'}$. So $fy \in Y$. $\square$

Any $y \in Y$ is determined by its values on T_0, which can be identified with a configuration on Γ using Ψ. Hence there is a natural isomorphism π between the sets Y and $\{-1,1\}^\Gamma$, given by $[\pi y](\gamma) = y(\lambda)$ where λ is the unique element of $T_0 \cap \Psi^{-1}(\gamma)$. In other words, π is the restriction map to T_0. Using π we can define a new action on $\{-1,1\}^\Gamma$: For $f \in F$ and $x \in \{-1,1\}^\Gamma$, let $f(x) = \pi(f(\pi^{-1}(x)))$.

Explicitly, this action on is given as follows: for $x \in \{-1,1\}^\Gamma$,

$$\begin{aligned}
[ax](\gamma) &= x(a^{-1}\gamma) \\
[bx](\gamma) &= \begin{cases} x(b^{-1}\gamma) & \text{if } \gamma \neq 0 \\ -x(0) & \text{if } \gamma = 0 \end{cases}
\end{aligned} \tag{2.4}$$

To differentiate this action from the shift action on $\{-1,1\}^\Gamma$ (Eq. 2.1) we refer to the space $\{-1,1\}^\Gamma$ as $\{-1,1\}_*^\Gamma$ when equipped with this action. That is the action $F \curvearrowright \{-1,1\}^\Gamma$ is given by (2.1), whereas the action $F \curvearrowright \{-1,1\}_*^\Gamma$ is given by (2.4). The proof that the latter is indeed an F-action is the fact that it is isomorphic, by construction, to $F \curvearrowright Y$.

The last F-space we define is Z, the set of pairs of mirror image configurations in $\{-1,1\}_*^\Gamma$:

$$Z = \left\{\{x_1, x_2\} \subset \{-1,1\}_*^\Gamma \,:\, x_1(\gamma) = -x_2(\gamma) \text{ for all } \gamma \in \Gamma\right\}. \tag{2.5}$$

So Z is the space of σ-orbits in Y. Equivalently, we can describe Z in the the following way

Let $\varphi : \{-1,1\}_*^\Gamma \to \{-1,1\}_*^\Gamma$ be the flipping map, that is

$$[\varphi(x)](\gamma) = -x(\gamma).$$

Then $\pi(\sigma(x)) = \varphi(\pi(x))$. Since σ commutes with the F-action on Y, φ commutes with the F-action on $\{-1,1\}_*^\Gamma$. Hence we can realize Z as the space of φ-orbits in $\{-1,1\}_*^\Gamma$. Note that each φ-orbit consists of only two points.

Now it is clear that equipped with the quotient topology, Z is a compact F-space. Furthermore, we now observe that Z admits an invariant measure. Consider the i.i.d. Bernoulli $1/2$ measure on $\{-1,1\}^\Gamma_*$. Clearly, it is an invariant measure and hence it is pushed forward to an invariant measure on Z. In particular, this shows that Z is not strongly proximal.

Claim 2.3. *The action $F \curvearrowright Z$ does not have any fixed points.*

Proof. The element $-1 \in \Gamma$ is fixed under the action of b. So if we have $z = \{x_1, x_2\} \in Z$ that is fixed under the action of F, then we should have $bx_1 = x_1$. On the other hand, by the definition of the action $F \curvearrowright \{-1,1\}^\Gamma_*$,

$$[bx_1](0) = -x_1(0) \neq x_1(0),$$

and so $bx_1 = x_2$, which is a contradiction. Hence $F \curvearrowright Z$ does not have any fixed points.

□

Before showing that $F \curvearrowright Z$ is proximal, it would be useful to introduce ϕ, a factor of $\{-1,1\}^\Gamma_* \times \{-1,1\}^\Gamma_*$ into $\{-1,1\}^\Gamma$:

$$\begin{aligned}\phi\colon \{-1,1\}^\Gamma_* \times \{-1,1\}^\Gamma_* &\to \{-1,1\}^\Gamma\\ (y_1, y_2) &\mapsto y_1 \cdot y_2\end{aligned}$$

where the $y_1 \cdot y_2$ is the pointwise product.

It is easy to verify that this is indeed an F-factor, when acting by shifts on $\{-1,1\}^\Gamma$ and as defined in (2.4) on $\{-1,1\}^\Gamma_*$.

Proposition 2.4. *The action $F \curvearrowright Z$ is proximal.*

Proof. Let $\hat{y} = \{y_1, y_2\}$ and $\hat{z} = \{z_1, z_2\}$ be two points in Z, and let $d = \phi(y_1, z_1)$.

Now by Proposition 2.1 we know that either $x_{-1} \in \overline{Fd} \subset \{-1,1\}^\Gamma$ or $x_{+1} \in \overline{Fd} \subset \{-1,1\}^\Gamma$. So there is a sequence of elements $\{f_n\}_n$ in F such that $\{f_n d\}_n$ tends to either x_{-1} or x_{+1} in $\{-1,1\}^\Gamma$. That is,

$$f_n d = f_n \phi(y_1, z_1) = \phi(f_n y_1, f_n z_1) = [f_n y_1] \cdot [f_n z_1]$$

tends to a constant function in $\{-1,1\}^\Lambda$. If it tends to the constant function 1, then $\{f_n y_1\}_n$ and $\{f_n z_1\}_n$ have the same limit (if necessary we can go to a subsequence so that both $\{f_n y_1\}_n$ and $\{f_n z_1\}_n$ have limits). If, on the other hand, it tends to the constant function -1, then $\{f_n y_1\}_n$ and $\{f_n z_2\}_n$ have the same (subsequential) limit. So in any case $\{f_n \hat{y}\}_n$ and $\{f_n \hat{z}\}_n$ have the same limit. Hence the action on Z is proximal. □

Theorem 2.5. *Thompson's group F is not strongly amenable.*

Proof. As mentioned in the introduction, a strongly amenable group must have a fixed point in any proximal action. Since the space Z we constructed above is proximal (Proposition 2.4), and has no fixed points (Claim 2.3), we conclude that F is not strongly amenable. ☐

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(Y. Hartman, K. Juschenko) Northwestern University

(O. Tamuz, P. Vahidi Ferdowsi) California Institute of Technology