ON THE STRAIN-ENERGY FUNCTION FOR ISOTROPIC BODIES

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Abstract

The strain-energy function ϕ is assumed to be made up of a sum of homogeneous polynomials of the first, second, third, etc. degrees in the components of the strain tensor $e_{\alpha\beta}$. The expansion is carried only so far as to include the third degree group of terms and then this expression for ϕ is studied from the point of view of the isotropy of space. Isotropy of space having been defined by means of rectangular trihedrons (or orthogonal ennuples), the effect on the "structure-tensor" $c^{\alpha\beta\gamma\delta\rho\sigma}$ (which forms with $e_{\alpha\beta}$ the third degree terms) of the assumption of isotropy is obtained. It is found that the third degree group introduces, for a completely isotropic state, only two coefficients of elasticity, which when taken together with the corresponding Lame's coefficients λ, μ of the second degree group gives only four elastic moduli in the complete expression for ϕ. The expression for ϕ thus developed is introduced into a set of equations obtained by F. D. Murnaghan connecting the stress and strain components by means of ϕ; and so we obtain relations between stress and strain which are applicable to deformations of higher order than those allowed by Hooke's law. These latter relations are then applied to the case of normal and uniform compression and these further applied to P. W. Bridgman's experiments on the change of volume of sodium under pressure at 30°C. In this application the four elastic moduli are reduced to two.

Introduction

If THE condition of a body from which the energy of deformation ϕ is reckoned be that of equilibrium under an arbitrary initial stress, then developing this function of the six strain components by Taylor's theorem, we have that[1]

$$\phi = \phi_0 + c^{\alpha\beta}e_{\alpha\beta} + c^{\alpha\beta\rho\sigma}e_{\alpha\beta}e_{\rho\sigma} + c^{\alpha\beta\rho\sigma\gamma\delta}e_{\alpha\beta}e_{\rho\sigma}e_{\gamma\delta} + \cdots \tag{1}$$

where ϕ_0 is an arbitrary constant of no interest as far as the deformations are concerned and the terms following ϕ_0 are respectively homogeneous functions of the first, second and third degrees in the components of the strain tensor $e_{\alpha\beta}$ ($e_{\alpha\beta} = e_{\beta\alpha}$ and $\alpha, \beta = 1, 2, 3$).

Hence, $\partial\phi/\partial e_{\alpha\beta}$ will be a linear function of the strain components together with functions of higher degrees obtained from the third degree, etc. terms. For isotropic bodies the second degree group $c^{\alpha\beta\rho\sigma}e_{\alpha\beta}e_{\rho\sigma}$ takes the simple form

$$2\phi_2 = (\lambda + 2\mu)I_1^2 - 4\mu I_2 \tag{2}$$

where λ and μ are two independent coefficients of elasticity characteristic of the body under consideration; also $I_1 = e_{xx} + e_{yy} + e_{zz}$ and $I_2 = (e_{yy}e_{zz} - e_{yz}^2)$

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[1] The repeated Greek labels denote summation, each "dummy" label taking all values 1, 2, 3.

$+e_{zz}e_{xx}-e_{zx}^2)+(e_{xx}e_{yy}-e_{xy}^2)$ are the first two fundamental invariants of elasticity theory, the third degree invariant $I_3=e_{xx}e_{yy}e_{zz}+2e_{yz}e_{zx}e_{xy}-(e_{xx}e_{yz}^2+e_{yy}e_{zx}^2+e_{zz}e_{xy}^2)$ not appearing in the theory unless the third degree group of Eq. (1) is taken into consideration. If we further suppose that the initial stress is a uniform hydrostatic pressure P_0, then Eq. (1) becomes (excluding terms beyond the second degree) for isotropic bodies

$$\phi = \phi_0 - P_0 I_1 + \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 \tag{3}$$

and we obtain Hooke's law of the linear dependence of stress on strain when the latter is small, from the relations

$$X_x = \frac{\partial \phi}{\partial e_{xx}}; \cdots ; Y_z = \frac{1}{2}\,\frac{\partial \phi}{\partial e_{yz}}; \cdots$$

so that

$$\begin{aligned} X_x &= -P_0 + YI_1 + 2\mu e_{xx}; \cdots \\ Y_z &= 2\mu e_{yz}; \cdots . \end{aligned} \tag{4}$$

We shall, in what follows, develop expressions corresponding to Eqs. (3) and (4) for the case where we include the group of terms of the third degree in the expression for ϕ, the strain-energy.[2] For this purpose we introduce an analysis which in reality is the analytical equivalent of the definition of isotropy; that is, the independence of the structure of a substance of the direction in which it is measured.

Analysis of Isotropy[3]

Since the following analysis will not be limited necessarily to a rectangular Cartesian system, we suppose that our space is a Riemannian one of three dimensions. By hypothesis, the energy of deformation $\phi(e_{\alpha\beta})$ is an invariant with regard to changes in the coordinate system used and since the $e_{\alpha\beta}$'s are the components of a covariant symmetrical tensor of rank two, the coefficients $c^{\alpha\beta\gamma\delta\rho\sigma}$ of the third degree group of Eq. (1) are the components of a contravariant tensor of rank six which we shall call the "structure tensor." This tensor $c^{\alpha\beta\gamma\delta\rho\sigma}$ has in general $3^6=729$ components, but because the third degree terms form a homogeneous group and because the strain tensor $e_{\alpha\beta}$ is symmetrical in character, only 56 of these components appear in the expression (1). The additional hypothesis that the body is completely isotropic generates many relations between these 56 coefficients so that the number of independent ones is greatly reduced.

It is in order to find these latter interrelations that we introduce the following analysis. To each point of our space we attach a trihedron, each edge

[2] W. Voigt, Ann. d. Phys. und Chemie, Dritte Folge 2 (**52**) 536 (1894).

[3] The Absolute Diff. Calc.; Levi-Civita; Ch, sec. 2 (we use Einstein's notation h_α^r instead of the corresponding $\lambda_{r/\alpha}$ of Levi-Civita's); C. Kaplan and F. D. Murnaghan; Phys. Rev. **35,** 763 (1930).

of which is presented in the underlying coordinate system (x) by a contravariant tensor of rank one h_r^α, the Greek label α denoting tensor character and the Latin letter r the particular edge of the trihedron. Since our space is a Riemannian one it possesses a metric denoted by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta.$$

Hence, corresponding to the contravariant components h_r^α are the covariant components $h_r{}^\alpha$ of the rth edge, with the relations

$$\begin{aligned} h_r{}^\alpha h_\alpha{}^s &= \epsilon_r{}^s \quad (= 1 \text{ if } r = s \text{ and } 0 \text{ if } r \neq s) \\ h_r{}^\alpha h_\beta{}^r &= \delta_\beta{}^\alpha \quad (= 1 \text{ if } \alpha = \beta \text{ and } 0 \text{ if } \alpha \neq \beta) \end{aligned} \tag{5}$$

defining the orthogonality of the trihedron at P and also informing us that the tensor h is a unit one.

By means of the covariant components $h_\alpha{}^r (= g_{\alpha\beta} h_r{}^\beta)$ the contravariant tensor $c^{\alpha\beta\gamma\delta\rho\sigma}$ may be defined by its 3^6 projections along the edges of the trihedron at P. Thus by definition, these projections are given by the equations

$$c^{pqrsmn} = c^{\alpha\beta\gamma\delta\rho\sigma} h_\alpha{}^p h_\beta{}^q h_\gamma{}^r h_\delta{}^s h_\rho{}^m h_\sigma{}^n$$

or

$$c_{pqrsmn} = c_{\alpha\beta\gamma\delta\rho\sigma} h_p{}^\alpha h_q{}^\beta h_r{}^\gamma h_s{}^\delta h_m{}^\rho h_n{}^\sigma \tag{6}$$

where $c^{pqrsmn} = c_{pqrsmn}$ and so for the invariants of the structure-tensor the upper and lower labelling has no meaning except for purposes of notation.

Suppose now that at P we attach a second trihedron whose edges are presented by the covariant or contravariant components $*h_\alpha{}^r$ or $*h_r{}^\alpha$ respectively, and furthermore let us suppose that the contravariant components of the two tensors h and $*h$ are connected by the linear relations

$$*h_r{}^\alpha = s_r{}^s h_s{}^\alpha \tag{7}$$

summing over the s's with $r, s, \alpha = 1, 2, 3$. Then, since the $*h$ trihedron is supposed to be orthogonal, the scalar coefficients $s_r{}^s$ are the elements of an orthogonal matrix. Indeed if the covariant presentions of the two tensors $*h$ and h are connected by the relations

$$*h_\alpha{}^r = s_s{}'^r h_\alpha{}^s \tag{8}$$

then multiplying Eqs. (7) and (8) together and summing over the α's we have that

$$*h_s{}^\alpha *h_\alpha{}^r = s_s{}^p s_q{}'^r h_p{}^\alpha h_\alpha{}^q$$

or

$$s_s{}^p s_q{}'^r \epsilon_p{}^q = \epsilon_s{}^r \quad \text{from} \quad (5)$$

and so

$$s_s{}^p s_p{}'^r = \epsilon_s{}^r. \tag{9}$$

Therefore the product of the s and s' matrices yields the unit matrix. We say that the $*h$ configuration is obtained from the h one by a rotation if the determinant of the matrix s is $+1$ and by a reflexion if this determinant is -1.

The relations between the invariants of the structure-tensor relative to the two trihedrons $^*h$ and h are readily obained; thus

$$^*c_{pqrsmn} = c_{\alpha\beta\gamma\delta\rho\sigma}{}^*h_p{}^{\alpha}{}^*h_q{}^{\beta}{}^*h_r{}^{\gamma}{}^*h_s{}^{\delta}{}^*h_m{}^{\rho}{}^*h_n{}^{\sigma}$$

and

$$c_{pqrsmn} = c_{\alpha\beta\gamma\delta\rho\sigma}h_p{}^{\alpha}h_q{}^{\beta}h_r{}^{\gamma}h_s{}^{\delta}h_m{}^{\rho}h_n{}^{\sigma}$$

and using the relations (7) between the contravariant components of $^*h$ and h, we find that

$$^*c_{pqrsmn} = c_{abcdef}s_p{}^a s_q{}^b s_r{}^c s_s{}^d s_m{}^e s_n{}^f \tag{10}$$

where we sum over the repeated Latin letters, each of which takes all the values 1, 2, 3.

We now define isotropy by saying that a medium is completely isotropic in structure in the neighborhood of a point P, if corresponding components of the invariants $^*c$ and c of the structure-tensor relative to the $^*h$ and h trihedrons are equal; that is

$$^*c_{pqrsmn} = c_{pqrsmn}. \tag{11}$$

These relations, taken together with those of (10) yield the fundamental equations defining isotropy. Thus

$$c_{pqrsmn} = c_{abcdef}s_p{}^a s_q{}^b s_r{}^c s_s{}^d s_m{}^e s_n{}^f. \tag{12}$$

Isotropy with Regard to Simple Rotations

We first consider the effect on the invariants c_{pqrsmn} of a rotation of the $^*h$ configuration through a right-angle in the h_2-h_3 plane, $^*h_1$ and h_1 coinciding. In order to fix our ideas we suppose that the $^*h$ and h systems are right-handed ones and that all rotations take place in a counterclockwise way. The rotation just suggested may then be defined by the following table

	h_1	h_2	h_3
$^*h_1$	1	0	0
$^*h_2$	0	0	1
$^*h_3$	0	-1	0

so that $s_1{}^1 = s_2{}^3 = 1$; $s_3{}^2 = -1$, the other being zero.

When we substitute these values of the s's in equations (12) we find that many of the invariants are zero and that many of them are related. Thus, if $p=2$, $q=3$, $r=1$, $s=1$, $m=3$ and $n=1$, the only term of the sextuple summation of Eq. (12) different from zero is that one for which $a=3$, $b=2$, $c=1$, $d=1$, $e=2$ and $f=1$, so that

$$c_{231131} = c_{321121}.$$

Similarly we find that $c_{321121} = -c_{231131}$ and hence for the two relations to be compatible it is necessary that

$$c_{231131} = c_{321121} = 0.$$

In what follows we shall not make use of the symmetry relations between the invariants c_{pqrsmn} allowed by the fact that in the energy function ϕ the tensor $c^{\alpha\beta\gamma\delta\rho\sigma}$ is summed with the symmetrical strain tensor $e_{\alpha\beta}$, but consider the tensor (and hence its invariants) to be a perfectly general one. Also because the invariants are cumbersome to handle in their present form we introduce a notation defined by the following matrix forms:

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix} \equiv \begin{pmatrix} a & h & g' \\ h' & b & f \\ g & f' & c \end{pmatrix}.$$

Thus instead of writing c_{213233}, say, we write simply $h'f'c$ and so on. When we examine each of the 3^6 invariants with regard to the rotation above, we find that 364 are zero whilst of the remaining 365 only 182 are independent.

If in addition to symmetry of structure with regard to a right-angle rotation in the h_2-h_3 plane, we demand symmetry of structure for a similar rotation in the h_3-h_1 plane, say, we find the conditions on the invariants by simple permutation of the results of the former rotation. Thus the relation $gag = h'ah'$ due to the rotation in the h_2-h_3 plane becomes $gag = g'cg'$ in the h_3-h_1 plane; the combined effect of the two rotations leading to the relations

$$gag = h'ah' = g'cg' = fcf = hbh = f'bf'.$$

Proceeding in this way we find that of the 729 invariants, 546 are zero and of the remaining 183 only 31 are independent. Furthermore, we need no longer distinguish between f, f';g, g';h, h', the reason for doing so in the preceding analysis being one of convenience only. This latter simplification reduces the number of independent invariants to twelve. The following table presents the surviving invariants for combined rotations through right-angles in the h_2-h_3 and h_3-h_1 planes; and also from its symmetry we see that no new conditions are imposed on the 12 independent invariants by including a 90°-rotation in the remaining h_1-h_2 plane.

We consider now a rotation of the $*h$ trihedron through an angle $\theta(\neq 90°)$ in, say, the h_2-h_3 plane. This rotation is conveniently presented by the following table:

	h_1	h_2	h_3
$*h_1$	1	0	0
$*h_2$	0	$\cos\theta$	$\sin\theta$
$*h_3$	0	$-\sin\theta$	$\cos\theta$

so that $s_1^1 = 1$; $s_2^2 = s_3^3 = \cos\theta$; $s_2^3 = \sin\theta$; $s_3^2 = -\sin\theta$, the other s's being zero.

On substitution of thes values in the fundamental Eqs. (12), taking into consideration the results embodied in Table I, we obtain the following relations:

$$\begin{aligned} 4bff &= aaa - aab - bab + baa \\ 4ffb &= aaa + aab - bab - baa \\ 4gag &= aaa - aab + bab - baa \end{aligned} \tag{13}$$

and

$$\begin{aligned} baa &= bac + 2aff \\ bab &= bac + 2faf \\ aab &= bac + 2ffa. \end{aligned} \tag{14}$$

This rotation therefore reduces the number of independent invariants from 12 to 6; namely, *aaa*, *aab*, *bab*, *baa*, *bac* and *gfh* (which thus far has not been involved in these simple rotations). In order to find further relations between the invariants, we must consider the most general rotation of the $^*h$-frame relative to the h-frame; namely,

$$\begin{array}{c|ccc} & h_1 & h_2 & h_3 \\ \hline {}^*h_1 & s_1^1 & s_1^2 & s_1^3 \\ {}^*h_2 & s_2^1 & s_2^2 & s_2^3 \\ {}^*h_3 & s_3^1 & s_3^2 & s_3^3 \end{array}$$

where

$$\sum_{t=1}^{3} s_r^t s_s^t = \epsilon_s^r \left(= {}^{1\ if\ r=s}_{0\ if\ r\neq s}\right) \text{ and } \Delta = \|s_s^r\| = +1.$$

Considering Eqs. (12) in conjunction with Table I and Eqs. (13) and (14) we obtain the following pair of relations:

$$\begin{aligned} 3bac + 2(faf + ffa + aff) + 24gfh &= 0 \\ aaa + 2bac + 16gfh &= 0 \end{aligned} \tag{15}$$

thus leaving only 4 distinct invariants, *faf*, *aff*, *ffa* and *gfh*.

The relations between the dependent and independent invariants are:

Table I.

	aa	ab	ac	af	ag	ah	ba	bb	bc	bf	bg	bh
a	aaa	aab	aab				bab	baa	bac			
b	baa	bab	bac				aab	aaa	aab			
c	baa	bac	bab				bac	baa	bab			
f				faf						gag		
g					gag						faf	
h						gag						gag

	ca	cb	cc	cf	cg	ch	fa	fb	fc	ff	fg	fh
a	bab	bac	baa							aff		
b	bac	bab	baa							bff		
c	aab	aab	aaa							bff		
f				gag			ffa	ffb	ffb			
g					gag							gfh
h						faf					gfh	

	ga	gb	gc	gf	gg	gh	ha	hb	hc	hf	hg	hh
a					bff							bff
b					aff							bff
c					bff							aff
f						gfh					gfh	
g	ffb	ffa	ffb							gfh		
h				gfh			ffb	ffb	ffa			

The empty cells denote zero values.

$$
\begin{aligned}
3aaa &= 4(faf + ffa + aff) \\
3baa &= 2(faf + ffa - 2aff) - 24gfh \\
3bab &= 2(aff + ffa - 2faf) - 24gfh \\
3aab &= 2(faf + aff - 2ffa) - 24gfh \\
3bac &= -\,2(faf + aff + ffa) - 24gfh \\
bff &= aff + 2gfh \\
ffb &= ffa + 2gfh \\
gag &= faf + 2gfh.
\end{aligned} \tag{16}
$$

We are now in a position to form the group of 3rd degree terms $\phi_3 = c^{\alpha\beta\gamma\delta\rho\sigma} e_{\alpha\beta}e_{\gamma\delta}e_{\rho\sigma}$ in the expression for ϕ, the strain-energy function, for the case of a completely isotropic body. Thus, since ϕ_3 is invariant with regard to transformations of the coordinate system used, we may write

$$\phi_3 = c_{pqrsmn}e_{pq}e_{rs}e_{mn},$$

where

$$e_{pq} = e_{\alpha\beta}h_p{}^\alpha h_q{}^\beta$$

and using the results embodied in Table I in addition to Eqs. (16), we obtain the following expression for ϕ_3:

$$3\phi_3 = 2(faf + aff + ffa)[3\bar{I}_1\bar{I}_2 - \bar{I}_1{}^3] + 24gfh[\bar{I}_3 - \bar{I}_1{}^3]$$

or, without any loss in generality,

$$\phi_3 = 2faf[3\bar{I}_1\bar{I}_2 - \bar{I}_1{}^3] + 8gfh[\bar{I}_3 - \bar{I}_1{}^3] \tag{17}$$

where

$$\bar{I}_1 = g_{\alpha\beta}e^{\alpha\beta};\ \bar{I}_2 = g_{\alpha\beta}g_{\gamma\delta}e^{\alpha\gamma}e^{\beta\delta};\ \bar{I}_3 = g_{\alpha\beta}g_{\gamma\delta}g_{\rho\sigma}e^{\rho\alpha}e^{\beta\gamma}e^{\delta\sigma}.$$

For the case of a rectangular Cartesian coordinate system where

$$ds^2 = dx^2 + dy^2 + dz^2$$

these invariants take the following simple form[4]

$$
\begin{aligned}
\bar{I}_1 &= e_{xx} + e_{yy} + e_{zz} \\
\bar{I}_2 &= e_{xx}{}^2 + e_{yy}{}^2 + e_{zz}{}^2 + 2(e_{yz}{}^2 + e_{zx}{}^2 + e_{xy}{}^2) \\
\bar{I}_3 &= e_{xx}{}^2 + e_{yy}{}^3 + e_{zz}{}^3 + 3(e_{xx}e_{zx}{}^2 + e_{xx}e_{xy}{}^2 + e_{yy}e_{yz}{}^2 + e_{yy}e_{xy}{}^2 \\
&\quad + e_{zz}e_{zx}{}^2 + e_{zz}e_{yz}{}^2)
\end{aligned} \tag{18}
$$

The relations between the invariants $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$ and the fundamental invariants I_1, I_2, I_3 where

$$
\begin{aligned}
I_1 &= e_{xx} + e_{yy} + e_{zz} \\
I_2 &= (e_{yy}e_{zz} - e_{yz}{}^2) + (e_{zz}e_{xx} - e_{zx}{}^2) + (e_{xx}e_{yy} - e_{xy}{}^2) \\
I_3 &= e_{xx}e_{yy}e_{zz} + 2e_{yz}e_{zx}e_{xy} - (e_{xx}e_{yz}{}^2 + e_{yy}e_{zx}{}^2 + e_{zz}e_{xy}{}^2)
\end{aligned}
$$

[4] Leon Brillouin, Ann de Physique (10) 3, 271 (1925).

are simply,

$$\begin{aligned} I_1 &= \bar{I}_1 \\ 2I_2 &= \bar{I}_1^2 - \bar{I}_2 \\ 6I_3 &= 2\bar{I}_3 - 3\bar{I}_1\bar{I}_2 + \bar{I}_1^3. \end{aligned} \tag{19}$$

Therefore ϕ_3 may be written in the form

$$\phi_3 = 4faf[I_1^3 - 3I_1I_2] + 24gfh[I_3 - I_1I_2]$$

and putting $4faf = \alpha/3$ and $24gth = \beta$, we have

$$\phi_3 = \frac{\alpha}{3}(I_1^3 - 3I_1I_2) + \beta(I_3 - I_1I_2). \tag{20}$$

The complete expression for ϕ for isotropic bodies then becomes

$$\phi = \phi_0 - P_0I_1 + \frac{\lambda + 2\mu}{2}I_1^2 - 2\mu I_2 + \frac{\alpha}{3}(I_1^3 - 3I_1I_2) + \beta(I_3 - I_1I_2) \tag{21}$$

Stress-Strain-Relations

It has been shown by Prof. F. D. Murnaghan[5] that the simple assumption that the energy of deformation is a function of the six strain components $e_{\alpha\beta}$ only, leads to the conclusion that it is a function of the three strain invariants I_1, I_2 and I_3 so that the medium is necessarily elastically isotropic. A set of equations are obtained connecting the components of the stress and strain tensors without making the assumption that the strain is infinitesimal. These equations are the following:

$$\begin{aligned} X_x &= E_{xx} - (2e_{xx}E_{xx} + e_{zx}E_{zx} + e_{xy}E_{xy}) \\ Y_y &= E_{yy} - (2e_{yy}E_{yy} + e_{xy}E_{xy} + e_{yz}E_{yz}) \\ Z_z &= E_{zz} - (2e_{zz}E_{zz} + e_{yz}E_{yz} + e_{zx}E_{zx}) \\ Y_z &= \tfrac{1}{2}E_{yz} - (2e_{yz}E_{yy} + e_{zx}E_{xy} + e_{zz}E_{yz}) \\ Z_x &= \tfrac{1}{2}E_{zx} - (2e_{zx}E_{zz} + e_{xy}E_{yz} + e_{xz}E_{zx}) \\ X_y &= \tfrac{1}{2}E_{xy} - (2e_{xy}E_{xx} + e_{yz}E_{zx} + e_{yy}E_{xy}) \end{aligned} \tag{22}$$

where we put $E\alpha\beta = \partial\phi/\partial e_{\alpha\beta}$ for brevity.

We note that the usual expressions for Hooke's law are obtained by neglecting the second degree terms on the right-hand side of Eq. (22).

We now insert in the equations above the expression of ϕ given by Eq. (21) and so obtain relations between the stress and strain components no longer linear, but involving terms of higher degrees. First we evaluate the coefficients $E_{\alpha\beta} = \partial\phi/\partial e_{\alpha\beta}$. Thus

[5] F. D. Murnaghan; National Acad. of Sciences, Proc. **14**, 890 (1928).

$$
\begin{aligned}
E_{xx} &= -P_0 + \lambda I_1 + 2\mu e_{xx} + (\alpha+\beta)I_1 e_{xx} - \alpha I_2 - \beta(I_2{}^2 + I_2) \\
&\quad + \beta(e_{yy}e_{zz} - e_{yz}{}^2) \\
E_{yy} &= -P_0 + \lambda I_1 + 2\mu e_{yy} + (\alpha+\beta)I_1 e_{yy} - \alpha I_2 - \beta(I_1{}^2 + I_2) \\
&\quad + \beta(e_{zz}e_{xx} - e_{zx}{}^2) \\
E_{zz} &- P_0 + \lambda I_1 + 2\mu e_{zz} + (\alpha+\beta)I_1 e_{zz} - \alpha I_2 - \beta(I_1{}^2 + I_2) + \beta(e_{xx}e_{yy} - e_{xy}{}^2) \\
E_{yz} &= 4\mu e_{yz} + 2(\alpha+\beta)I_1 e_{yz} + 2\beta(e_{zx}e_{xy} - e_{xx}e_{xz}) \\
E_{zx} &= 4\mu e_{zx} + 2(\alpha+\beta)I_1 e_{zx} + 2\beta(e_{xy}e_{yz} - e_{yy}e_{zx}) \\
E_{xy} &= 4\mu e_{xy} + 2(\alpha+\beta)I_1 e_{xy} + 2\beta(e_{yz}e_{zx} - e_{zz}e_{xy}).
\end{aligned} \tag{23}
$$

We now evaluate X_x. Thus,

$$
\begin{aligned}
X_x = &- P_0 + 2P_0 e_{xx} + \lambda I_1 + 2\mu e_{xx} + (\alpha + \beta - 2\lambda - 4\mu)I_1 e_{zz} \\
&- (\alpha + \beta - 4\mu)I_2 - \beta I_1{}^2 + 2(\alpha+\beta)I_2 e_{xx} - 2\alpha I_1{}^2 e_{xx} - 2\beta I_3 \\
&+ 2(\alpha+\beta)I_1 I_2 + (\beta - 4\mu)(e_{yy}e_{zz} - e_{yz}{}^2) - 2(\alpha+\beta)I_1(e_{yy}e_{zz} - e_{yz}{}^2)
\end{aligned} \tag{24}
$$

with corresponding equations for Y_y and Z_z obtained from this one by permuting the labels in the order x, y, z.

The addition of the normal stresses X_x, Y_y and Z_z leads us to a relatively simple expression; namely,

$$
\begin{aligned}
X_x + Y_y + Z_z = &- 3P_0 + 2P_0 I_1 + (3\lambda + 2\mu)I_1 + (\alpha - 2\beta - 4\mu - 2\lambda)I_1{}^2 \\
&- (3\alpha + 2\beta - 8\mu)I_2 + 6(\alpha+\beta)I_1 I_2 - 2\alpha I_1{}^3 - 6\beta I_3.
\end{aligned} \tag{25}
$$

Following are the expressions for the tangential stresses:

$$
\begin{aligned}
Y_z &= \left[2\mu + 2P_0 + 2\left(\frac{\alpha+\beta}{2} + \frac{\lambda + 2\mu}{1}\right)I_1 + 2(\alpha+\beta)I_2 - 2\alpha I_1{}^2\right]e_{yz} \\
&\quad - (4\mu + 2\alpha + \beta)(e_{zx}e_{xy} - e_{xx}e_{yz}) \\
Z_x &= \left[2\mu + 2P_0 + 2\left(\frac{\alpha+\beta}{2} - \frac{\lambda + 2\mu}{1}\right)I_1 + 2(\alpha+\beta)I_2 - 2\alpha I_1{}^2\right]e_{zx} \\
&\quad - (4\mu + 2\alpha + \beta)(e_{xy}e_{yz} - e_{yy}e_{zx}) \\
X_y &= \left[2\mu + 2P_0 + 2\left(\frac{\alpha+\beta}{2} - \frac{\lambda + 2\mu}{1}\right)I_1 + 2(\alpha+\beta)I_2 - 2\alpha I_1{}^2\right]e_{xy} \\
&\quad - (4\mu + 2\alpha + \beta)(e_{yz}e_{zx} - e_{zz}e_{xy}).
\end{aligned} \tag{26}
$$

Application to the Case of Normal and Uniform Compression

Let us suppose that we have a medium in a state of equilibrium under a normal and uniform pressure P, the initial state being one of normal and uniform pressure P_0. The displacement is given in general by the following values of u, v, w:

$$v = -ax + b; \quad v = -ay + b; \quad w = -az + b$$

which lead to the relations

$$I_2 = I_1^2/3$$

and

$$I_3 = I_1^3/27.$$

The sum of the normal stresses then becomes:

$$-3P = -3P_0 + 2P_0I_1 + (3\lambda + 2\mu)I_1 + \left(\alpha - 2\beta - 4\mu - 2\lambda - \frac{3\alpha + 2\beta - 8\mu}{3}\right)I_1^2 + \frac{16}{9}\beta I_1^3.$$

Also $I_1 = -\Delta v/v_0$ where $\Delta v = v_0 - v$, v_0 being the volume at the initial pressure P_0 and v the volume at the variable pressure P.

Finally we obtain the following equation connecting P and $\Delta v/v_0$:

$$\frac{3P}{\dfrac{\Delta v}{v_0}\left(3 + 2\dfrac{\Delta v}{v_0}\right)} - \frac{P_0}{\dfrac{\Delta v}{v_0}} = \rho + \sigma\frac{\Delta v}{v_0} \tag{27}$$

where $\rho = 3\lambda + 2\mu/3$ and $\sigma = 8/9\beta$. If the initial stress P_0 equals zero, then Eq. (27) becomes

$$\frac{3P}{\dfrac{\Delta v}{v_0}\left(3 + 2\dfrac{\Delta v}{v_0}\right)} = \rho + \sigma\frac{\Delta v}{v_0}\,.$$

The energy of compression takes the simple form

$$\phi = \phi_0 + P_0\frac{\Delta v}{v_0} + \frac{1}{2}\rho\left(\frac{\Delta v}{v_0}\right)^2 + \frac{1}{3}\sigma\left(\frac{\Delta v}{v_0}\right)^3. \tag{29}$$

In order to illustrate the use of Eq. (27) we consider P. W. Bridgman's[6] experiments on the change of the volume of sodium under pressure. The following table contains the data of the experiment at 30°C, together with values of the pressure calculated from Eq. (27). The values for ρ and σ were calculated at the pressures 1000 kg/cm² and 3000 kg/cm² and were found to be respectively 63630 and 72164.

TABLE II.

Pressure kg/cm² Observed	$\Delta V/V_0$ at 30° Observed	Pressure kg/cm² Calculated
1000	0.0153	1000
2000	0.0299	2009
3000	0.0437	3000
4000	0.0570	4009
5000	0.0697	5009
6000	0.0819	6007
7000	0.0937	7009
8000	0.1050	8000
9000	0.1159	8990
10000	0.1263	9962
11000	0.1365	10944
12000	0.1465	11933

[6] P. W. Bridgman, Proc. Am. Acad. Arts and Sc. **58**, 203 (1923).