SPHERICAL ACCRETION

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ABSTRACT

We compare different examples of spherical accretion onto a gravitating mass. Limiting cases include the accretion of a collisionally dominated fluid and the accretion of collisionless particles. We derive expressions for the accretion rate and density profile for semicollisional accretion, which bridges the gap between these limiting cases. Particle crossing of the Hill sphere during the formation of the outer planets is likely to have taken place in the semicollisional regime.

Subject headings: accretion, accretion disks — planets and satellites: formation

1. INTRODUCTION

We consider accretion onto a gravitating mass M with an absorbing boundary at radius R. The mass is embedded in a medium whose density and velocity dispersion approach ρ_∞ and c_∞ as $r \to \infty$. The attraction of the central mass is strongly felt inside the gravitational radius at $r_g \equiv GM/c_\infty^2$. We neglect the particles' self-gravity, which is justified provided $\rho_\infty r_g^3 \ll M$.

We begin with a review of two examples, the accretion of a collision dominated fluid in § 2 and the accretion of collisionless particles in § 3. A convenient measure of collisionality is the optical depth across distance r_g with $\rho = \rho_\infty$. Given a scattering cross section per unit mass, or opacity,[3] κ, then

$$\tau_g \equiv \kappa\rho_\infty r_g. \qquad (1)$$

Semicollisional accretion occurs for $\tau_g < 1$.

We calculate new accretion rates and atmospheric density profiles for steady state, semicollisional accretion of elastic and inelastic particles in §§ 4 and 5, respectively. Astrophysical scenarios in which semicollisional accretion may apply are discussed in § 6.

2. HIGHLY COLLISIONAL (BONDI) ACCRETION

For $r > r_g$, $\rho(r) \sim \rho_\infty$, and for $r < r_g$, $v(r) \sim (GM/r)^{1/2}$. Therefore,

$$\dot{M}_B = 4\pi\lambda r_g^2 \rho_\infty c_\infty. \qquad (2)$$

Note that $\dot{M}_B$ is independent of R. The dimensionless coefficient λ is of order unity. With a γ-law equation of state, $\lambda = e^{3/2}/4$ for $\gamma = 1$ and $\lambda = 1/4$ for $\gamma = 5/3$ (Bondi 1952).

For $r \ll r_g$, Bondi accretion is characterized by $v \propto r^{-1/2}$, which implies $\rho \propto r^{-3/2}$ and hence $c \propto r^{-3(\gamma-1)/4}$. Provided $\gamma < 5/3$, the Mach number $\mathcal{M} \equiv v/c \to \infty$ as $r/r_g \to 0$. Pressure gradients stymie Bondi accretion for $\gamma > 5/3$. Monatomic gas, $\gamma = 5/3$, is a special limiting case.

In Bondi accretion $\tau/\tau_g \approx (r_g/r)^{1/2}$ for $r < r_g$, so collisionality increases inward. Thus $\tau_g \gg 1$ is a sufficient condition for Bondi accretion provided that $\gamma \lesssim 5/3$.

3. COLLISIONLESS ACCRETION

A direct estimate of the accretion rate follows from the application of the conservation laws for energy and angular momentum:

$$\dot{M} \approx 2\pi c_\infty \rho_\infty r_g R \approx (R/r_g)\dot{M}_B. \qquad (3)$$

Contrary to the collisional Bondi case, the collisionless accretion rate depends on R.

At $r \ll r_g$, the density $\rho/\rho_\infty \sim (r_g/r)^{1/2}$, the rms velocity dispersion is comparable to the free-fall velocity, and the net radial velocity is smaller by $\sim R/r$.

4. SEMICOLLISIONAL ACCRETION OF ELASTIC PARTICLES

Consider the steady accretion of particles that interact by elastic scatterings under the condition that $\tau \equiv \kappa\rho r < 1$ at all $r_g > r > R$. A quasi-static atmosphere of bound particles with $c^2 \sim GM/r$ is present at $R < r < r_g$. It conducts constant luminosity[4] $L \sim \tau r^2 \rho c^3$. The mass accretion rate $\dot{M} \sim r^2 \rho v$ must also be constant. $\dot{M}$ is determined by setting $v \sim c$ at $r = R$, since particles that cross the inner boundary are absorbed. Thus $v/(\tau c) \sim \dot{M}c^2/L \sim R/r$; although energy is transported outward at speed $\sim\tau c$, the speed at which mass flows inward is slower by a factor of $R/r \lesssim 1$. It follows that

$$\frac{\dot{M}}{\dot{M}_B} \sim \frac{R}{r_g}\tau_g. \qquad (4)$$

Atmospheric profiles of density, $\rho(r)$, and entropy, $s(r)$, depend on the opacity law, which we parameterize as $\kappa/\kappa_\infty = (c/c_\infty)^{-2\beta}$ with β a constant:

$$\rho/\rho_\infty \sim (r_g/r)^{3/4+\beta/2}, \quad s \propto \ln(c^3/\rho) \propto (\beta - 3/2)\ln r. \qquad (5)$$

The entropy lost (for $\beta > 3/2$) or gained (for $\beta < 3/2$) by the material as it moves toward smaller radius causes a minor variation of the conductive luminosity:

$$\frac{r}{L}\frac{dL}{dr} \sim (\beta - 3/2)\frac{R}{r} \ll 1, \qquad (6)$$

where the strong inequality applies far outside the absorbing boundary, that is, for $r \gg R$.

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[3] We define opacity as a shorthand for collisional cross section per unit mass. Its use does not imply that the transfer of radiation is involved.

[4] Luminosity is defined as the rate at which energy passes through a certain radius. In this example, the luminosity is carried by conduction rather than by radiation.

TABLE 1
SUMMARY OF OUR RESULTS ON SEMICOLLISIONAL ACCRETION OF INELASTIC PARTICLES

Property	Bondi Accretion	Partially Collisional Accretion		Collisionless Accretion	
Optical depth	$\tau_g > 1$	$1 > \tau_g > (R/r_g)^{1/4}$	$(R/r_g)^{1/4} > \tau_g > R/r_g$	$R/r_g > \tau_g > M_g R/Mr_g$	$\tau_g < M_g R/Mr_g$
Accretion rate	$\dot{M}_B = 4\pi\rho_\infty r_g^2 c_\infty$	$\dot{M}_B \tau_g$		$\dot{M}_B R/r_g$	
Atmosphere	$\rho_\infty(r/r_g)^{-3/2}$	$\rho_\infty(r/r_g)^{-5/4},$ $\tau_g\rho_\infty(r/r_g)^{-3/2}$		$\rho_\infty(r/r_g)^{-5/4}$	$\rho_\infty(r/r_g)^{-1/2}$

Bahcall & Wolf (1976) solved the important case of gravitational interactions in connection with the accretion of stars by a black hole. Because the cross section for strong scatterings is proportional to c^{-4}, $\beta = 2$. Thus $\rho \propto r^{-7/4}$. An earlier attempt by Peebles (1972) yielded the incorrect result $\rho \propto r^{-9/4}$, and herein lies an interesting lesson. Peebles assumed $\dot{M} \sim \tau r^2 \rho c$, whereas the correct result follows from setting $L \sim \tau r^2 \rho c^3$ (Begelman 1977; Shapiro 1985). In steady state accretion, both $\dot{M}$ and L must be independent of r. However, as shown above, the conduction of energy occurs at the maximum rate permitted by relaxation, whereas mass is transported more slowly.

5. SEMICOLLISIONAL ACCRETION OF INELASTIC PARTICLES

5.1. Accretion Rate

Collisions of unbound particles inside r_g occur at relative velocities of order $c_\infty(r_g/r)^{1/2}$. Provided they dissipate a significant fraction of the center-of-mass kinetic energy, they produce bound particles. Comparable yields of bound particles come from collisions between two unbound particles and from collisions between an unbound and a bound particle. In either case, the addition of bound particles comes mainly from collisions that occur near r_g.[5] Subsequent collisions cause the bound particles to accrete. The total accretion rate is bounded by the sum of the collisional and collisionless accretion rates,

$$\dot{M} \sim 2\pi c_\infty \rho_\infty r_g(\tau_g r_g + R). \quad (7)$$

For $\tau_g > R/r_g$, collisions dominate the accretion rate and

$$\dot{M} \sim \tau_g \dot{M}_B. \quad (8)$$

5.2. Semicollisional Atmosphere

Next we turn our attention to the density profile in semicollisional atmospheres. Collisions involving an unbound particle serve as a source for bound particles at a rate $\dot{M} \sim 2\pi\tau_g r_g^2 \rho_\infty c_\infty$. In steady state the density of bound particles around r_g is of the same order of magnitude as ρ_∞. It is set by a balance between the rate at which bound particles are produced and the rate at which their mutual collisions cause them to drift inward.

The mass accretion rate is independent of radius for $r \ll r_g$. Together with knowledge of the average radial velocity, v, this allows us to determine the density profile for constant κ. Where $\tau \sim \kappa\rho r \lesssim 1$, $v/c_\infty \sim \tau(r_g/r)^{1/2}$, but $v/c_\infty \sim (r_g/r)^{1/2}$ where $\tau \gtrsim 1$. Thus

$$\rho \approx \begin{cases} \rho_\infty(r_g/r)^{5/4}, & \text{if } \tau \lesssim 1, \\ \rho_\infty\tau_g(r_g/r)^{3/2} & \text{if } \tau \gtrsim 1. \end{cases} \quad (9)$$

For $R/r_g < \tau_g^4$, there is a transition between the low and high optical depth regimes at $r/r_g \approx \tau_g^4$.

We can now verify that collisions in the vicinity of r_g dominate the rate at which unbound particles become bound. Even for the steeper Bondi density profile, the rate of capture of particles increases with r for $r < r_g$:

$$\rho_{\text{ub}}(r)\rho_b(r)c\kappa r^3 = \dot{M}_B\tau_g(r/r_g)^{1/2}, \quad (10)$$

where the subscripts "b" and "ub" refer to bound and unbound particles, respectively. This approaches the semicollisional accretion rate given by equation (8) as $r \to r_g$. Thus, collisions between an unbound and a bound particle make a contribution to the accretion rate that is comparable to that made by collisions between two unbound particles.[6]

For $\tau_g < R/r_g$, the semicollisional atmosphere solution still applies. However, its contribution to the accretion rate is negligible. The formation timescale of the atmosphere is $\rho_\infty r_g^3/(\tau_g\dot{M}_B)$, whereas the accretion timescale is $M/(\dot{M}_B R/r_g)$. Thus, an atmosphere has sufficient time to form provided

$$\tau_g > \frac{M_g}{M}\frac{R}{r_g}, \quad (11)$$

where $M_g \approx \rho_\infty r_g^3$ is the mass of surrounding fluid contained within the gravitational radius r_g.

Our results for the accretion rates, as well as the bound-particle atmosphere, are summarized in Table 1.

6. SEMICOLLISIONAL PLANETESIMAL ACCRETION

Consider the accretion of planetesimals by a protoplanet. Our idealized planetesimals are identical, indestructible, inelastic spheres with radii s and density ρ_s, so $\kappa \sim 3/(s\rho_s)$. Their velocity dispersion, c_∞, is set by a balance between excitation due to viscous stirring by protoplanets and damping by mutual collisions. The notation in this section follows that in Goldreich et al. (2004).

Suppose the planetesimal velocity dispersion is set at the boundary between shear- and dispersion-dominated limits. Then the thickness of the planetesimal disk is comparable to the planet's Hill radius, R_{H}, which in turn is comparable to the gravitational radius, r_g.[7] Denoting the surface mass density of planetesimals by σ_∞, we have $\tau_g \sim \tau_{\text{disk}} \sim \sigma_\infty/(s\rho_s)$, where τ_{disk} is the vertical optical depth of the planetesimal disk. Semicollisional accretion applies for $\alpha \lesssim \tau_g \lesssim 1$. All treatments of planet formation of which we are aware are based on collisionless accretion. However, this limit is appropriate only if the planetesimals are large enough. For the fast growth of planets,

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[5] The statements in these last two sentences can be verified after the density profile inside r_g is determined.

[6] The presence of an atmosphere of bound particles is a consequence of semicollisional accretion, but it is not the reason that the accretion rate is enhanced above the collisionless value.

[7] $R_{\text{H}} \sim R/\alpha$, where α is approximately the angular size of the Sun as seen from the protoplanet's orbit.

which requires small planetesimals, semicollisional accretion may be the appropriate regime. For example, a surface density of $\sigma_\infty \approx 1$ g cm^{-2} is commonly adopted for the protoplanetary disk around 30 AU, where $\alpha \sim 10^{-4}$. With these values, the size range for semicollisional accretion is 1 cm $< s <$ 100 m.

The collision rate inside the Hill sphere (cf. eq. [8]) exceeds the collisionless accretion rate by $\sim \alpha^{-1} \tau_{\text{disk}}$. Each collision produces one or two bound particles. Spherical symmetry is likely to be a poor approximation inside the Hill sphere. The secular component of the Sun's tidal potential has a minimum in the protoplanet's orbit plane. Collisions among bound particles damp their motions perpendicular to this plane, leading to the formation of an accretion disk. It is unclear what fraction of these particles will ultimately be accreted by the growing protoplanet.

7. SUMMARY

The semicollisional accretion of inelastic particles proceeds as follows: Unbound particles that collide within the gravitational radius r_g, become bound, and are ultimately accreted. This sets the accretion rate. The central mass is surrounded by a quasi-static atmosphere of bound particles. Its density profile is determined by taking $\dot{M}$ independent of r and setting the mean inward radial velocity equal to τc. Where $\tau < 1$ inside r_g, there is an additional population of unbound particles, but its contribution to the total density is minor. A bound atmosphere exists even at such low τ_g that collisionless accretion dominates. For extremely low τ_g, a steady state bound atmosphere cannot form on the timescale in which the central body grows by collisionless accretion. These results are summarized in Table 1.

Accretion rates and density profiles for the semicollisional accretion of inelastic particles differ from those for the semicollisional accretion of elastic particles, such as gravitationally interacting particles (Bahcall & Wolf 1976). In the elastic case, the majority of the bound particles are never accreted; they are only temporarily captured. Ejections are a consequence of the outward conduction of gravitational energy released mainly near the bottom of the atmosphere.

Mutual collisions alter the nature of the accretion of small planetesimals by protoplanets. The formation of an accretion disk is a likely outcome. If its density exceeds that of the unbound particles, it would affect the formation and evolution of binaries in scenarios such as that of Goldreich et al. (2002).

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