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Electromagnetic Effects in Two-Nucleon Systems*

STANLEY DESER†
Physics Department, Harvard University, Cambridge, Massachusetts
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Corrections to the electromagnetic properties of two-nucleon systems are examined by means of the relativistic two-body equation. An effective addition to the potential is derived, which describes the properties of the system in an external electromagnetic field. In particular, the correction to the deuteron's quadrupole moment is evaluated to lowest order in the meson-nucleon coupling. The result, for pseudoscalar (pseudoscalar) symmetric theory, is $\Delta Q=-1.3\times 10^{-28}(g^2/4\pi)$ cm^2.

I. INTRODUCTION

IN this paper, we shall employ the relativistic two-body equation[1-3] to consider the nucleon-recoil corrections to the electromagnetic properties of a two-nucleon system coupled by a pseudoscalar (pseudoscalar) symmetric meson field. Assuming the nucleons to be bound by the instantaneous interaction, and treating retardation effects as perturbations, we shall derive an operator whose matrix elements will describe the altered properties of the system in the presence of an external electromagnetic field.

As is well known, the magnetic moment of the deuteron is not additive, that is, $\mu_D=\mu_P+1+\mu_N-0.022$ nuclear magnetons, where μ_D, μ_P+1, μ_N are the deuteron, proton, and neutron moments, respectively. Besides the admixture of D state due to the noncentral character of the binding forces, two other sources of this nonadditivity exist. On the basis of meson theory, a nucleon's magnetic moment is (apart from the proton's spin moment) attributed to its interaction with its self-mesonic field. It is, therefore, to be expected, according to this theory, that the overlapping of the self-fields of the neutron and proton in the deuteron should produce a deviation from additivity. At present, of course, it is not even possible to account for the individual moments field-theoretically. However, one can argue that in any case these overlap corrections are not as important as those coming from that part of the currents of the bound nucleons themselves, which arises from the nucleon recoils upon exchange of virtual mesons. This second source of nonadditivity is not strictly independent of the first, nor is the degree of their interaction known. Presumably, however, since the deuteron is a loose structure, the nucleons spending most of their time outside one another's range, it may be expected that the recoil effects dominate. These, as well as relativistic kinematical corrections to the moments are included in the proper relativistic equation. In any case, as we have mentioned, the theory is at present incapable of predicting either overlap[4] or individual moments, and we shall insert the latter phenomenologically. That is, we assume that the effect of the mass operators of the particles in an external electromagnetic field, $F_{\mu\nu}$, is to yield the experimental anomalous moments, in accordance with the goals of meson theory; we shall thus be able to bypass this unresolved difficulty. Further, if as has been conjectured, the self effects are of a "strong coupling" nature, while the interaction with other particles is more legitimately to be expanded in the coupling constant, this procedure will have summed up the "strong coupling" effects into the phenomenological terms, leaving only interaction terms.

II. THE EFFECTIVE ADDITION TO THE POTENTIAL

We write the two-body equation in the presence of the field $F_{\mu\nu}$, as

$$\Bigg\{\Bigg[\gamma^{(1)}\bigg(p_1-e\frac{1+\tau_3^{(1)}}{2}A(1)\bigg)+M-\bigg(\mu_P\frac{1+\tau_3^{(1)}}{2}+\mu_N\frac{1-\tau_3^{(1)}}{2}\bigg)\frac{\sigma_{\mu\nu}^{(1)}F_{\mu\nu}^{(1)}}{2}\Bigg]\times\Bigg[\gamma^{(2)}\bigg(p_2-e\frac{1+\tau_3^{(2)}}{2}A(2)\bigg)+M-\bigg(\mu_P\frac{1+\tau_3^{(2)}}{2}+\mu_N\frac{1-\tau_3^{(2)}}{2}\bigg)\frac{\sigma_{\mu\nu}^{(2)}F_{\mu\nu}^{(2)}}{2}\Bigg]-I_{12}[F_{\mu\nu}]\Bigg\}G_{12}=1_{12},\quad(1)$$

where the isotopic spin dependence of the electromagnetic terms insures that each nucleon has the appropriate coupling to the field. We assume the unperturbed problem for zero field to have been solved, and treat the electromagnetic terms as small corrections, consistent with the smallness of e. In this fashion, we shall derive an effective addition to the two-particle potential whose matrix elements will correspond to various electromagnetic effects such as quadrupole and

* Part of a doctoral thesis submitted to Harvard University.
† National Science Foundation predoctoral fellow, now at the Institute for Advanced Study, Princeton, New Jersey.
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[2] E. Saltpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[3] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[4] See, however, H. Miyazawa, Prog. Theoret. Phys. (Japan) **7**, 207 (1952).

photodisintegration as well as magnetic moment corrections. The choice of the form of $F_{\mu\nu}$ will determine the process involved. Thus, $F_{\mu\nu}=\mathbf{H}$ will correspond to the magnetic moment problem, $F_{\mu\nu}$ an inhomogeneous electric field to quadrupole questions, while an incoming plane wave would be used for photoprocesses. In Sec. III, we shall specifically evaluate the correction to the deuteron's quadrupole moment due to the nucleon recoil currents. Only the lowest order interaction, $I_{12}[0]=-ig^2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\gamma_5{}^{(1)}\gamma_5{}^{(2)}\Delta_+$, will be included here. Further, only terms linear in $F_{\mu\nu}$ and A_μ will be kept, as we are not interested in induced moments.

In order to ascertain the effects of retardation, we shall make the assumption that the static potential,

$$I_{12}{}^{(0)}=\delta(t)\int_{-\infty}^{\infty}I_{12}dt,$$

binds the deuteron, and shall treat the retardation, $I_R\equiv I_{12}-I_{12}{}^{(0)}$ as small. Incorporating the field-dependent terms in the interaction, we may write Eq. (1) as

$$(F_1F_2-I_0-I_R-I_A-I_B)G_{12}=1_{12}, \tag{2}$$

where

$$I_0=-ig^2\gamma_5{}^{(1)}\gamma_5{}^{(2)}\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\delta(x_0)\int_{-\infty}^{\infty}\Delta_+(x)dx_0$$
$$=-ig^2\gamma_5{}^{(1)}\gamma_5{}^{(2)}\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\delta(x_0)e^{-\mu r}/4\pi r, \tag{2a}$$

$$I_A=[e_1\gamma^{(1)}A(1)+\tfrac{1}{2}\mu_1\sigma_{\mu\nu}{}^{(1)}F_{\mu\nu}(1)]F_2+[e_2\gamma^{(2)}A(2)+\tfrac{1}{2}\mu_2\sigma_{\mu\nu}{}^{(2)}F_{\mu\nu}(2)]F_1\equiv A_1F_2+A_2F_1, \tag{2b}$$

$$I_B=I_{12}[F_{\mu\nu}]-I_{12}[0], \tag{2c}$$

and

$$e_i=\tfrac{1}{2}(1+\tau_3{}^{(i)})e,\quad \mu_i=\tfrac{1}{2}(\mu_P+\mu_N)+\tfrac{1}{2}(\mu_P-\mu_N)\tau_3{}^{(i)}.$$

I_A is thus the additional interaction due to the individual particles' coupling to the field, while I_B is due to meson current effects. We have not given the explicit form of I_B, since, as is well known, it has zero expectation value in the ground state of self-mirror nuclei such as the deuteron and will, therefore, make no contribution to the specific effect here calculated, though it will have to be taken into account in transition problems.

In order to evaluate the effects of I_A and I_B, a Green's function perturbation theory will be employed. This will yield changes of energy of the states and the appropriate coefficients of the fields and their derivatives will then be the desired moment corrections. Calling $G_{12}{}^{(0)}$ the solution of the equation,

$$(F_1F_2-I_0)G_{12}{}^{(0)}=1_{12}, \tag{3}$$

we get, formally,

$$G_{12}-G_{12}{}^{(0)}=G_{12}{}^{(0)}I_AG_{12}{}^{(0)}+G_{12}{}^{(0)}I_BG_{12}{}^{(0)}+G_{12}{}^{(0)}I_AG_{12}{}^{(0)}I_RG_{12}{}^{(0)}+G_{12}{}^{(0)}I_RG_{12}{}^{(0)}I_AG_{12}{}^{(0)}+\cdots, \tag{4}$$

keeping terms to first order in the retardation and fields. Viewing $G_{12}(x_1x_2;x_1'x_2')$ as a bilinear expansion in wave functions, which is approximately true for $t_1=t_2\rightarrow+\infty$ and $t_1'=t_2'\rightarrow-\infty$, we can get the change of energy of a state n due to the perturbation by means of the Green's function perturbation theory. In virtue of the above remarks, the equation

$$G_{12}(\mathbf{r}_1\mathbf{r}_2,T_1;\mathbf{r}_1'\mathbf{r}_2',T_2)=(\mathbf{r}_1\mathbf{r}_2T_1|G_{12}{}^{(0)}+G_{12}{}^{(0)}I'G_{12}{}^{(0)}+G_{12}{}^{(0)}I'G_{12}{}^{(0)}I'G_{12}{}^{(0)}+\cdots|\mathbf{r}_1'\mathbf{r}_2'T_2), \tag{5}$$

where I' stands for the perturbing interaction, can be rewritten as

$$-\sum_n\psi_n(\mathbf{r}_{12})\bar{\psi}_n(\mathbf{r}_{12}')\exp[-iE_n(T_1-T_2)]$$
$$=-\sum_n\psi_{n0}(\mathbf{r}_{12})\bar{\psi}_{n0}(\mathbf{r}_{12}')\exp[-iE_{n0}(T_1-T_2)]$$
$$+\sum_{n,m}\psi_{n0}(\mathbf{r}_{12})\exp(-iE_{n0}T_1)\Big[\int\bar{\psi}_{n0}(x_{12}'')\times I'(x_{12}'',x_{12}''')\psi_{m0}(x_{12}''')(dx_{12}'')(dx_{12}''')$$
$$+\int\bar{\psi}_{n0}I'G_{12}{}^{(0)}I'\psi_{m0}\Big]\exp(iE_{m0}T_2)\psi_{m0}(\mathbf{r}_{12}'), \tag{6}$$

where ψ_{n0} are the unperturbed functions, ψ_n the corrected ones, and similarly for E_{n0} and E_n. The form of the left-hand side of Eq. (6) assumes I' to be time-independent, of course.

The diagonal terms yield, as usual, the energy shifts:

$$\exp\{-iE_n(T_1-T_2)\}$$
$$=\exp\{-i(E_{n0}+\Delta E_n)(T_1-T_2)\}$$
$$\approx\exp\{-iE_{n0}(T_1-T_2)\}[1-i\Delta E_n(T_1-T_2)]$$
$$=\exp\{-iE_{n0}(T_1-T_2)\}\times\Big[1-\int\bar{\psi}_{n0}I'\psi_{n0}-\int\bar{\psi}_{n0}I'G_{12}{}^{(0)}I'\psi_{n0}\Big], \tag{7}$$

so that

$$\Delta E_n=\Delta E_n{}^{(1)}+\Delta E_n{}^{(2)}=-i(T_1-T_2)^{-1}\int\bar{\psi}_{n0}I'\psi_{n0}-i(T_1-T_2)^{-1}\int\bar{\psi}_{n0}I'G_{12}{}^{(0)}I'\psi_{n0}. \tag{8}$$

In the case where the external field is time-dependent, and, therefore, the system is no longer conservative, the nondiagonal matrix elements of the right-hand side of Eq. (6) will be proportional to transition amplitudes, with the time-dependences of the wave functions and electromagnetic field amplitudes insuring that transitions take place only between energetically possible states. As we have remarked before, the expectation value of I_B vanishes in the deuteron ground state, and only contributes to transition effects. The other term in $\Delta E_n^{(1)}$ is $-i(T_1-T_2)^{-1}\int\bar{\psi}I_A\psi$. This may be shown to be approximately

$$\Delta E_n^{(1)A}=\int\bar{\psi}_{++}(\mathbf{r})(\mathbf{r}|\gamma_0^{(1)}A_2+\gamma_0^{(2)}A_1|\mathbf{r}')\psi_{++}(\mathbf{r}')d\mathbf{r}d\mathbf{r}',\quad\psi_{++}=\Lambda_+^{(1)}\Lambda_+^{(2)}\psi, \tag{9}$$

where $\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$, $\Lambda_\pm(\mathbf{p})=[E(\mathbf{p})\pm H(\mathbf{p})]/2E(\mathbf{p})$ and $\psi_{++}(\mathbf{r})$ is the positive energy part of the equal times wave function with the center-of-mass time variation removed. The approximation made is that the negative energy parts of the wave functions have negligible effect.

A_1+A_2 has been assumed time-independent, as is necessary for a ΔE to be defined.[5] This is the lowest order result, as expected; that is, to this approximation the magnetic moment is the sum of the individual ones, while the quadrupole moment also reduces to the usual expression. Equation (9) may be proven either by explicit integration in momentum space, using the relative time-dependence of solutions of problems involving only instantaneous interactions, or more simply by the equivalent coordinate space statement for $G_{12}^{(0)}$,

$$G_{12}^{(0)}(x_1x_2;x_1'x_2')=\int G_>(x_>;\mathbf{r}_>''t_<)d\mathbf{r}_>''\gamma_0^>G_{12}^{(0)}(\mathbf{r}_<\mathbf{r}_>''t_<;\mathbf{r}_<''\mathbf{r}_>'t_>')\gamma_0^<\times d\mathbf{r}_<''G_<(\mathbf{r}_<''t_>';x_<'), \tag{10}$$

and similarly for $\psi_{n0}(x_1,x_2)$,

$$\psi_{n0}(x_1,x_2)=\int G_>(x_>,\mathbf{r}_>'t_>)\gamma_0^>d\mathbf{r}_>'\psi_{n0}(\mathbf{r}_>',\mathbf{r}_<,t_<). \tag{11}$$

Such a reduction to equal times quantities can only be performed by virtue of the facts that $I_{12}^{(0)}$ is an instantaneous interaction, and that only $++$ components are kept.

Henceforth we shall focus on $\Delta E_n^{(2)}$ only, keeping just the two terms linear both in I_R and $F_{\mu\nu}$. In $\Delta E_n^{(2)}$ we replace the intermediate $G_{12}^{(0)}$ by G_1G_2, the product of the free-particle propagation functions.[6] We can, therefore, write the formula for $\Delta E_n^{(2)}$ as

$$\Delta E_n^{(2)}=-i(T_1-T_2)^{-1}\int\bar{\psi}_{n0}(\mathbf{r}_>,\mathbf{r}_<'',t_>)\gamma_0^<\times d\mathbf{r}_<'G_<(\mathbf{r}_<''t_>,x_<)(x_1x_2|I_AG_1G_2I_R+I_RG_1G_2I_A|x_1'x_2')G_>(x_>',\mathbf{r}_>'''t_<')\gamma_0^>\times d\mathbf{r}_>'''\psi_{n0}(\mathbf{r}_>''',\mathbf{r}_<',t_<'), \tag{12}$$

where use has been made of Eq. (11) and the adjoint equation. Since $\psi_{n0}(\mathbf{r}_1\mathbf{r}_2t)=\psi_{n0}(\mathbf{r})\exp(iP_nX)$, we may exhibit the form of the effective addition $V_{nn}(\mathbf{r},\mathbf{r}')$ to the interparticle potential, which is defined by

$$\Delta E_n=\int\bar{\psi}_{n0}(\mathbf{r})V_{nn}(\mathbf{r},\mathbf{r}')\psi_{n0}(\mathbf{r}')d\mathbf{r}d\mathbf{r}'. \tag{13}$$

We see from Eq. (12) that

$$V_{nn}(\mathbf{r},\mathbf{r}')=\int\exp(-iP_nX)G_<(I_AG_1G_2I_R+I_RG_1G_2I_A)G_>\exp(iP_nX''), \tag{14}$$

where $X=\frac{1}{2}(x_1+x_2)$.

One can similarly define a V_{mn} for transition problems where the perturbation induces jumps between states m and n. It is to be noted that, unlike the usual potentials, V_{mn} is energy dependent, being a function of the energies of the states between which it is to be taken; this is so because in removing the center-of-mass dependence, we have partially made V into a matrix element.

Comparing Eq. (13) with Eq. (9), we see that the retardation corrections to the lowest-order operator, A_1+A_2, are given by V_{nn} for the state n. V_{nn} is in general an integral operator, as is to be expected, since the effect of the recoils is to smear out the contact between the fields. Also, in the sense that V_{nn} depends on the energy of the state, it is velocity dependent; however, in practice the binding is small and V is essentially independent of it.

We now proceed to a more explicit evaluation of $\Delta E_n^{(2)}$. In momentum space, the expression for $\Delta E_n^{(2)}$ may be written as

$$\Delta E_n^{(2)}=-i\int\bar{\psi}_{n0}(p)[I_A(p,p';P)G_1(\tfrac{1}{2}P+p')\times G_2(\tfrac{1}{2}P-p')I_R(p'-p'')]\psi_{n0}(p'')(dp)(dp')(dp'')+\text{term with } I_A \text{ and } I_R \text{ interchanged}, \tag{15}$$

where we have set:

$$\psi_n(x_1,x_2)=\exp(iP_nX)(2\pi)^{-2}\int e^{ipx}\psi_n(p)(dp), \tag{16a}$$

$$\bar{\psi}_n(x_1,x_2)=\exp(-iP_nX)(2\pi)^{-2}\times\int e^{-ipx}\bar{\psi}_n(p)(dp), \tag{16b}$$

[5] We have taken the dependence of the external field operators to be purely on the relative distance; any apparent dependence of the $F_{\mu\nu}$ on the center-of-mass displacement R does not spoil our results, since we are interested only in internal structure effects, which are essentially uncoupled to the over-all center-of-mass motion. Any operator whose expectation value will be needed can be split into a part proportional to R, which is dropped, and an R-independent one.

[6] This approximation does not lead to infrared divergences in meson theory, unlike the electrodynamic case. It may also be mentioned that the dropping of the intermediate $\bar{\psi}_{n0}\psi_{n0}$ state (n_0 represents the bound state) compensates for the omission of the $\frac{1}{2}[\Delta E^{(1)}(T_1-T_2)]^2$ term, which should properly be included in the expansion of $\exp\{-i(E_{n0}+\Delta E_n)(T_1-T_2)\}$ in Eq. (7), both terms being $=-\Sigma_n(\int\bar{\psi}_{n0}I'\psi_{n0})^2$.

$$I(p,p';P)=(2\pi)^{-4}\int\exp\{-iP_n(X-X')-ipx+ip'x'\}\times I(x_1x_2;x_1'x_2')(dx)(dx')d(X-X'),\quad(16c)$$

$$G_{(i)}(p)=(\gamma^{(i)}p+M)^{-1},\quad(16d)$$

and $P=p_1+p_2$, $p=\frac{1}{2}(p_1-p_2)$.

We transcribe Eq. (11) into momentum space in order to exhibit the explicit relative energy dependence of $\psi_{++}(p)$:

$$\psi_{++}(p)=(P-2E)(\tfrac{1}{2}P+p_0-E)^{-1}\times(\tfrac{1}{2}P-p_0-E)(2\pi)^{-\frac{1}{2}}i\phi_{++}(\mathbf{p}).\quad(17)$$

The Fourier transforms of I_A and I_R are easily seen to be

$$I_A(p,p';P)=A_1(p-p')F_2(\tfrac{1}{2}P-p)+A_2(p-p')F_1(\tfrac{1}{2}P+p),\quad(18)$$

where $A(p)$ is the transform of $A(x)$, and

$$I_R(p,p')=(2\pi)^{-4}\mathcal{O}_{12}(p_0-p_0')^2\omega^{-2}[\omega^2-(p_0-p_0')^2]^{-1},$$
$$\omega=[(\mathbf{p}-\mathbf{p}')^2+\mu^2]^{\frac{1}{2}},\quad \mathcal{O}_{12}=-ig^2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\gamma_5^{(1)}\gamma_5^{(2)}.\quad(19)$$

We can now perform the integrations over p_0, p_0', p_0'' corresponding to setting the times equal in coordinate language. At this stage, we have

$$\Delta E_n^{(2)}=(2\pi)^{-4}i\int\phi_{++}^*(\mathbf{p})(2\pi)^{-\frac{1}{2}}i(P-2E)$$
$$\times(\tfrac{1}{2}P+p_0-E)^{-1}(\tfrac{1}{2}P-p_0-E)^{-1}\gamma_0^{(1)}\gamma_0^{(2)}\mathcal{O}_{12}$$
$$[(\tfrac{1}{2}P+p_0''-H_1')^{-1}\gamma_0^{(1)}A_1(p'-p'')$$
$$+(\tfrac{1}{2}P-p_0''-H_2')^{-1}\gamma_0^{(2)}A_2(p'-p'')]$$
$$\times(p_0-p_0'')^2\omega_{\mathbf{p}-\mathbf{p}'}^{-2}[(p_0-p_0'')^2-\omega^2]^{-2}(2\pi)^{-\frac{1}{2}}i$$
$$\times(P-2E'')(\tfrac{1}{2}P+p_0''-E'')^{-1}(\tfrac{1}{2}P-p_0''-E'')^{-1}$$
$$\times\phi_{++}(\mathbf{p}'')(dp)(dp')(dp'')$$
$$+\text{same with } p_\mu\leftrightarrow p_\mu'',\quad A(p)\leftrightarrow A(-p),\quad\text{and}$$
$$\gamma_0^{(1)}\gamma_0^{(2)}\mathcal{O}_{12}(\tfrac{1}{2}P\pm p_0''-H_i')^{-1}\gamma_0^{(i)}A_i\rightarrow$$
$$\gamma_0^{(i)}A_i(\tfrac{1}{2}P\pm p_0-H_i')^{-1}\gamma_0^{(1)}\gamma_0^{(2)}\mathcal{O}_{12}.\quad(20)$$

To arrive at Eq. (20) we have employed the fact that $A(\Delta p)=A(\Delta\mathbf{p})\delta(\Delta p_0)$. The integrations over p_0, p_0'' are carried out with the usual prescription, assigning small negative imaginary parts to the masses. The result with $P-2E$ and $P-2E''$, the binding energies, set equal to zero is:

$$\Delta E_n^{(2)}=\tfrac{1}{2}i(2\pi)^{-3}\int\phi_{++}^*(\mathbf{p})\{\gamma_0^{(1)}\gamma_0^{(2)}\mathcal{O}_{12}\omega_{\mathbf{p}-\mathbf{p}'}^{-3}$$
$$\times[\Lambda_+^{(1)}(\mathbf{p}')\gamma_0^{(1)}A_1(\mathbf{p}'-\mathbf{p}'')$$
$$+\Lambda_+^{(2)}(\mathbf{p}')\gamma_0^{(2)}A_2(\mathbf{p}'-\mathbf{p}'')]$$
$$+[\gamma_0^{(1)}A_1(\mathbf{p}-\mathbf{p}')\Lambda_+^{(1)}(\mathbf{p}')$$
$$+\gamma_0^{(2)}A_2(\mathbf{p}-\mathbf{p}')\Lambda_+^{(2)}(\mathbf{p}')]$$
$$\times\gamma_0^{(1)}\gamma_0^{(2)}\mathcal{O}_{12}\omega_{\mathbf{p}'-\mathbf{p}''}^{-3}\}\phi_{++}(\mathbf{p}'')d\mathbf{p}d\mathbf{p}'d\mathbf{p}''.\quad(21)$$

The form of this correction in coordinate space is of interest. Recalling Eq. (13), we find that

$$V_{nn}(\mathbf{r},\mathbf{r}')=\tfrac{1}{2}i(2\pi)^{-3}\{\gamma_0^{(1)}\gamma_0^{(2)}\mathcal{O}_{12}K_0(\mu r)$$
$$\times[\Lambda_+^{(1)}(\mathbf{r}-\mathbf{r}')\gamma_0^{(1)}A_1(\mathbf{r}')+\Lambda_+^{(2)}(\mathbf{r}-\mathbf{r}')\gamma_0^{(2)}A_2(\mathbf{r}')]$$
$$+[\gamma_0^{(1)}A_1(\mathbf{r})\Lambda_+^{(1)}(\mathbf{r}-\mathbf{r}')+\gamma_0^{(2)}A_2(\mathbf{r})\Lambda_+^{(2)}(\mathbf{r}-\mathbf{r}')]$$
$$\times K_0(\mu r')\gamma_0^{(1)}\gamma_0^{(2)}\mathcal{O}_{12}\}.\quad(22)$$

V is a velocity-independent integral operator, owing to the presence of the positive energy projection operators. $K_0(\mu r)$ is the usual Hankel function, which behaves logarithmically at the origin and asymptotically as $(\mu r)^{-\frac{1}{2}}\exp(-\mu r)$. If we suitably change the form of $\mathcal{O}_{12}$ in Eq. (22), we obtain the results for various other types of meson theory; further, a similar treatment may be given for the electrodynamic case to ascertain corrections to the hydrogen and positronium moments. Although the corrections to be expected are very small, they may eventually prove worth carrying out in view of the alteration of the results for such experiments as the Lamb shift, which are obtained on the assumption of an additive moment for the system. In positronium only a quadratic Zeeman effect is expected, on account of the symmetry properties of the atom.

III. CORRECTIONS TO THE DEUTERON'S QUADRUPOLE MOMENT

In what follows, we shall employ Eq. (22) to evaluate the relativistic contributions to the quadrupole moment of the deuteron. This question is of particular interest, since the recoil term effectively yields a D-state admixture. This may be seen schematically as follows: the moment Q may be written essentially as $\langle(3z^2-r^2)R\rangle$, where R is an operator expressing the various recoil effects, say after reduction to nonrelativistic wave functions. (Equivalently, the effect of R is to bring in various l states for a given j.) That part of R which is the coefficient of $P_2(\theta)$ in an expansion will then yield a contribution, since $(3z^2-r^2)=3P_2(\theta)$ and the angular averaging selects out only this term for expectation values between S states. It will be found, however, on carrying out the evaluation, that the result cannot account for the observed Q, so that a further admixture of D state is still required (as is indeed provided by the static potential).

While a similar evaluation may be carried out for the magnetic moment terms, the number obtained would be of far less significance since magnetic moment effects, being essentially high frequency, depend strongly on the shape of the wave function at the origin where it is least known, whereas in the quadrupole calculation the asymptotic form predominates.

Equation (21) simplifies to some extent in our quadrupole calculation. In this case,

$$\gamma_0^{(i)}A_i(\Delta\mathbf{p})=-\tfrac{1}{2}e(1+\tau_3^{(i)})A_0(\Delta\mathbf{p}). \qquad (23)$$

Since we are interested in the coefficient of $\partial^2 A_0(R)/\partial R_i\partial R_j$, both $A_\mu(1)$ and $A_\mu(2)$ reduce to $\frac{1}{8}r_ir_j\partial^2 A_0(R)/\partial R_i\partial R_j\delta(\mathbf{p}-\mathbf{p}')$ where $\mathbf{r}$ is the operator $i\nabla_p$ acting on $\delta(p-p')$. Since, further, the $\Lambda_+^{(i)}(\mathbf{p}')$ can now act directly on either $\phi_{++}^*(\mathbf{p})$ or $\phi_{++}(\mathbf{p}'')$, we can set them equal to unity with sufficient accuracy, since $\Lambda_+(\mathbf{p})\phi_+(\mathbf{p}')=1+O(M^{-2})$. We employ the fact that the deuteron is in a charge singlet state to eliminate $\tau_3^{(1)}+\tau_3^{(2)}$ terms and to set $\langle\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\rangle=-3$. At this stage our result reads:

$$\Delta E^{(2)}=-\tfrac{3}{2}(2\pi)^{-2}(g^2/4\pi)eq \times\int\phi_{++}^*(\mathbf{p})[(3z'^2-r'^2)\omega_{\mathbf{p}-\mathbf{p}'}^{-3}]\gamma_0^{(1)}\gamma_0^{(2)} \times\gamma_5^{(1)}\gamma_5^{(2)}\phi_{++}(\mathbf{p}')d\mathbf{p}d\mathbf{p}', \qquad (24)$$

where

$$q=\tfrac{1}{6}(3\partial^2/\partial z^2-\nabla_R^2)A_0(R).$$

In the above, the z' and r' represent derivatives with respect to p' acting on ω.

Before going on with the actual evaluation, we must reduce the ϕ's and γ's to Pauli form in the usual way. This gives $\Delta E^{(2)}$ in terms of $\phi_{++}^{(++)}$, the desired nonrelativistic wave functions,

$$\Delta E^{(2)}=\tfrac{3}{2}(2\pi)^{-2}(g^2/4\pi)(2M)^{-2}eq\int\phi_{++}^{*(++)}(\mathbf{p}) \times[(3z'^2-r'^2)\omega_{\mathbf{p}-\mathbf{p}'}^{-3}]\sigma^{(1)}\cdot(\mathbf{p}-\mathbf{p}') \times\sigma^{(2)}\cdot(\mathbf{p}-\mathbf{p}')\phi_{++}^{(++)}(\mathbf{p}')d\mathbf{p}d\mathbf{p}', \qquad (25)$$

or

$$\Delta E^{(2)}=-\tfrac{3}{2}\pi^{-1}(g^2/4\pi)e(2M)^{-2}q\int\phi^*(\mathbf{r}) \times[\boldsymbol{\sigma}^{(1)}\cdot\nabla\boldsymbol{\sigma}^{(2)}\cdot\nabla(3z^2-r^2)K_0(\mu r)]\phi(\mathbf{r})d\mathbf{r}. \qquad (26)$$

Calling the quantity in brackets $W(\mathbf{r})$, we may write it in the following way after carrying out the differentiations:

$$\begin{aligned}W(\mathbf{r})=&[-2K_0(x)+(4/3)xK_1(x)]+6K_0\sigma_z^{(1)}\sigma_z^{(2)}\\&+(x^2K_0-xK_1)(\cos^2\theta-\tfrac{1}{3})+(4/3)xK_1S_{12}\\&+(2xK_1+x^2K_0)(\cos^2\theta-\tfrac{1}{3})S_{12}\\&-6xK_1[\sigma_z^{(1)}\cos\theta(\sigma_z^{(2)}\cos\theta+\tfrac{1}{2}(\sigma_+^{(2)}e^{-i\varphi}\\&+\sigma_-^{(2)}e^{i\varphi})\sin\theta)]-6xK_1[\sigma^{(2)}\leftrightarrow\sigma^{(1)}], \qquad (27)\end{aligned}$$

where $x=\mu r$, $S_{12}=(3\boldsymbol{\sigma}^{(1)}\cdot\mathbf{r}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{r}-r^2)r^{-2}$, $\sigma_\pm=\sigma_x\pm i\sigma_y$. We can now proceed to take expectation values between ground-state wave functions, which are the usual sums of $^3S_1$ and $^3D_1$ states:[7]

$$r\phi(r)=u(r)\mathcal{Y}_{101}^1+w(r)\mathcal{Y}_{121}^1. \qquad (28)$$

In a spin-triplet state, $\langle\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}\rangle=1$; we also employ the following spin information:

$$-\sigma_z^{(1)}\sigma_z^{(2)}\chi_{1,m_s}=(-1)^{m_s}\chi_{1,m_s}, \qquad (29)$$

where χ_{1,m_s} are the usual spin functions, and

$$\begin{aligned}\sigma_+^{(i)}\alpha(i)=\sigma_-^{(i)}\beta(i)=0,\quad \sigma_+^{(i)}\beta(i)=2\alpha(i),\\ \sigma_-^{(i)}\alpha(i)=2\beta(i).\end{aligned} \qquad (30)$$

α, β are the spin-up and -down functions, respectively.

$$\begin{aligned}S_{12}\mathcal{Y}_{101}^1&=(8)^{\frac{1}{2}}\mathcal{Y}_{121}^1,\\ S_{12}\mathcal{Y}_{121}^1&=(8)^{\frac{1}{2}}\mathcal{Y}_{101}^1-2\mathcal{Y}_{121}^1.\end{aligned} \qquad (31)$$

With this information, we can reduce all terms to radial integrals:

$$\begin{aligned}\int\phi^*(\mathbf{r})W(\mathbf{r})\phi(\mathbf{r})d\mathbf{r}\\ =\int_0^\infty dr\{&u^2[4(1+x^2/15)K_0-(32/15)xK_1]\\&+(2\sqrt{2}/3)uw(xK_1-x^2K_0/5)\\&+w^2[(\tfrac{2}{5}+\tfrac{1}{3}x^2)K_0-(97-36(6)^{\frac{1}{2}})xK_1/105]\}.\end{aligned} \qquad (32)$$

An approximate evaluation may be made, which employs the asymptotic forms of u and w throughout; this will give a slight overestimate of the actual figures. We shall keep here only the u^2 terms, which is sufficient for our accuracy.

$$u\sim(2\alpha)^{\frac{1}{2}}e^{-\alpha r}, \qquad (33)$$

where $\alpha^{-1}=4.31\times10^{-13}$ cm.

The integrals occurring in Eq. (32) may be evaluated exactly to give[8]

$$I=\int_0^\infty e^{-xy}K_0(x)dx=(1-y^2)^{-\frac{1}{2}}\cos^{-1}y. \qquad (34)$$

Similarly,

$$\int_0^\infty x^2K_0(x)e^{-xy}dx=d^2I/dy^2 =(1-y^2)^{-2}[(2y^2+1)I-3y], \qquad (35a)$$

$$\int_0^\infty xK_1(x)e^{-xy}dx=I+ydI/dy=(1-y^2)^{-1}(I-y). \qquad (35b)$$

In our case, y is the ratio of twice the meson Compton wavelength to the radius of the deuteron, and is about 0.65. The value of ΔQ is, therefore,

$$\Delta Q=-1.3\times10^{-28}(g^2/4\pi)\ \text{cm}^2. \qquad (36)$$

Thus, the S-state contributions to the relativistic

[7] The notation of J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952) is used.

[8] G. N. Watson, *Theory of Bessel Functions* (Macmillan Company, New York, 1948), p. 388.

corrections to Q are of the order of $4.8\times g^2/4\pi$ percent of the experimental value,[9] and for $g^2/4\pi$ of the order of 10, about 50 percent of the actual moment and of the opposite sign. We may also note here that the corrections coming from the reduction to Pauli functions in $\Delta E^{(1)}$ are much smaller than the effect treated above. We have

$$\Delta E_{\text{quad}}{}^{(1)} = -\tfrac{1}{4}eq\int\phi_{++}{}^*(3z^2-r^2)\phi_{++}d\mathbf{r}$$
$$= -\tfrac{1}{4}eq\int\phi_{\text{n.r.}}{}^*(3z^2-r^2)\phi_{\text{n.r.}}d\mathbf{r}$$
$$-\tfrac{1}{2}eq(2M)^{-2}\int\phi_{\text{n.r.}}{}^*p^2(3z^2-r^2)\phi_{\text{n.r.}}d\mathbf{r}. \quad (37)$$

Any effects from the second term on the right-hand side of Eq. (37) are of the order of $(\mu/2M)^2$ of the leading term, for $p^2/(2M)^2$ represents less than the average value of $(v/c)^2$ in the deuteron, since the r^2 in Eq. (37) tends to weight the integral toward smaller p. Thus, the second term has a ratio to the first of less than 0.5 percent. The first term is the usual nonrelativistic expression for the quadrupole moment; it must now be increased to balance the negative sign of the correction term, which implies a rise in the required percentage of D state.

Although the large size of the correction obtained may be due, in part, to the particular form of the retarded interaction employed, the present considerations indicate that, in a correct treatment of the deuteron problem, the recoil effects will contribute appreciably to a calculation of the moments.

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[9] Compare the result of F. Villars, Phys. Rev. **86**, 476 (1952) that $\Delta Q = -3.7\times10^{-29}(g^2/4\pi)$ cm^2. However, the two calculations differ in several respects. First, Villars did not employ the two-body equation. Second, he used the total interaction, rather than the retarded part, to compute the recoil effects. This counts the instantaneous contribution twice, since its effect is already taken into account through use of bound-state wave functions. Third, he employed different wave functions.

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Field Theory of Equations with Many Masses

B. T. DARLING
24 West 87 Street, New York, New York

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This paper develops the field theory of many mass equations with special attention to spin $\frac{1}{2}$ and the operator (1) of the author's previous paper on the irreducible volume character of events. The field is assumed to interact with the electromagnetic field which is introduced in a gauge-invariant way. General expressions for the charge-current four-vector and the symmetrical energy-momentum tensor are derived and are shown to satisfy the appropriate conservation theorems. According to a theorem of Leichter, the general solution is shown to be a superposition of nonorthogonal mass states which we designate as the root fields. Nevertheless, the physical quantities, such as the current four-vector, the energy-momentum tensor, etc., are shown to decompose into a sum over those of individual mass states but with an alternation of sign for consecutive roots. The Lagrangian takes the form of an alternating sum over the individual free-field Lagrangians for the mass states, plus the usual term $+(1/c)j_\mu A_\mu$ for the interaction with the electromagnetic field. The matter field may be quantized by treating the root fields as independent anticommuting fields. The transformation to the interaction representation is obviously unaltered and the charge and mass renormalization may be treated following Schwinger. To the order of approximation in Schwinger II these renormalizations are not affected. It would seem that these methods of quantization, together with the usual treatment of the electromagnetic field, are at variance with the manifest nonlocal nature of the theory for the irreducible volume character of events.

INTRODUCTION

THE earliest multiple-mass equations arose in an attempt to circumvent the divergence difficulties in electromagnetic theory and consisted in introducing besides the photon of zero rest mass one additional nonvanishing rest mass.[1] The first considerations of equations of infinite order with a continuous or discrete spectrum of masses were those of Blokhinzev, who developed the theory for scalar neutral fields with the view of their possible application to mesons.[2] He used Bose quantization based on a set of operators which decomposed the wave field into free fields satisfying the Schrödinger-Klein-Gordon equation for the individual masses contained in the mass spectrum of the operator. Born next introduced fields involving exponential operators [exp$(a\Box)$, where $\Box=\sum_\lambda u_\lambda{}^2$] in connection with his method of mass quantization.[3] The present author, in connection with a theory of fundamental length, seems to be the first to propose an infinite-order differential equation for the Dirac field. In this theory

[1] F. Bopp, Ann. Physik **38**, 345 (1940); **42**, 573 (1943); A. Landé and L. H. Thomas, Phys. Rev. **60**, 121, 514 (1940); **65**, 175 (1944); B. Podolsky *et al.*, Phys. Rev. **62**, 68 (1942); **65**, 228 (1944); Revs. Modern Phys. **20**, 40 (1948); A. Green, Phys. Rev. **72**, 628 (1947); D. Montgomery, Phys. Rev. **69**, 117 (1947).

[2] D. Blokhinzev, J. Phys. (U.S.S.R.) **11**, 72 (1947).

[3] M. Born, Revs. Modern Phys. **21**, 463 (1949).