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# $p$-Brane Dyons and Electric-magnetic Duality


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**Abstract**

We discuss dyons, charge quantization and electric-magnetic duality for self-interacting, abelian, $p$–form theories in the spacetime dimensions $D = 2(p+1)$ where dyons can be present. The corresponding quantization conditions and duality properties are strikingly different depending on whether $p$ is odd or even. If $p$ is odd one has the familiar $e\bar{g} - g\bar{e} = 2\pi n\hbar$, whereas for even $p$ one finds the opposite relative sign, $e\bar{g} + g\bar{e} = 2\pi n\hbar$. These conditions are obtained by introducing Dirac strings and taking due account of the multiple connectedness of the configuration space of the strings and the dyons. A two-potential formulation of the theory that treats the electric and magnetic sources on the same footing is also given. Our results hold for arbitrary gauge invariant self-interaction of the fields and are valid irrespective of their duality properties.



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# 1 Introduction

Ever since Dirac[1] introduced it in quantum mechanics in 1931, the magnetic pole has been a fascinating object. In particular, the quantization condition relating the electric and magnetic charges, its relationship with the symmetry between electricity and magnetism, and the possible generalizations of Dirac's approach, remain questions of high interest today. Among the natural extensions of Dirac's own improved formulation[2] of 1948, two will be of importance here. The first is the consideration of dyons, particles having both electric and magnetic charge[3, 4]. The second is the generalization to higher dimensions where the electromagnetic potential is replaced by a $p$–form and the electric charges become extended objects with a $p$–dimensional history instead of a worldline[5, 6]. We will establish the following results, some of which were briefly discussed or implicit in [7]:

1. The quantization condition for the electric and magnetic charges is present even when the theory is not duality invariant (this was to be expected since – as shown in [5, 6] – the quantization condition holds even when the electric and magnetic charges are extended objects of different – complementary – dimensions).

2. For spacetime dimensions $D = 2(p + 1)$ such that dyons can exist, the quantization condition takes different forms depending on the parity of $p$, namely

$$e\bar{g} - g\bar{e} = 2\pi n\hbar \ , D = 4k \ (p \text{ odd}) \ , \tag{1.1}$$
$$e\bar{g} + g\bar{e} = 2\pi n\hbar \ , D = 4k + 2 \ (p \text{ even}) \ . \tag{1.2}$$

   These conditions are obtained by using Dirac strings and analyzing the connectivity of the configuration space of the dyons and the strings. They are also shown to arise from the generalization to higher dimensions of the arguments based on: (i) the quantization of the angular momentum stored in the field of electric and magnetic poles[8] and (ii) compatibility of regular local gauge charts[9].

3. A two–potential formulation (which is not manifestly Lorentz invariant) can be given for any spacetime dimension. For, and only for, $D = 4k$ there are special self–interactions for which the source-free theory is

invariant under duality rotations; there the two–potential formulation exhibits duality invariance in a particular transparent way as a normal Noether symmetry.

Although most of the results discussed here were first obtained using the two-potential formulation which permits a more economical analysis, we will also present them in terms of the more familiar one-potential representation.

The plan of the paper is as follows. Section 2 presents an extension of Dirac's 1948 formulation[2] for electrodynamics with magnetic poles which allows for dyons and also for self–interactions of the electromagnetic field. This is the prototype of the $p$–form theory for $p$ odd and $D = 2(p+1) = 4k$. The dual formulation where a potential is introduced for the electric field rather than for the magnetic field is also discussed. Section 3 discusses the prototype theory with even $p$ and $D = 2(p+1) = 4k+2$. This is the case $p = 2$, $D = 6$. Section 4 then treats the two-potential formulation of both the odd and even $p$ cases, ending with a brief discussion of the generalization to higher dimensions. Some other aspects of interest are included in the appendices: Appendix A discusses the issues of topology of configuration space and orientation of surfaces which have bearing on the quantization condition. Appendix B discusses the general solution of the quantization conditions in the symmetric case $e\bar{g} + g\bar{e} = 2\pi n\hbar$ , Appendix C gives the analog of the angular momentum quantization conditions, and Appendix D discusses the argument based on compatibility of local gauge charts.

# 2 Electrodynamics in D=4; Dyons

## 2.1 Action

The covariant action describing the coupling of dyons to non-linear electrodynamics in four dimensions is [2]

$$I[A_\mu, z_n, y_n] = I_F + I_C + I_P \tag{2.3}$$

with

$$I_F = \int d^4x \, \mathcal{L}(F_{\mu\nu}), \tag{2.4}$$

$$\begin{aligned} I_C &= \sum_n e_n \int A_\mu(z_n) dz_n^\mu, && (2.5) \\ I_P &= -\sum_n m_n \int \sqrt{-(dz^\mu)^2}. && (2.6) \end{aligned}$$

The field strength $F_{\mu\nu}$ is defined through

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu +^* G_{\mu\nu} \tag{2.7}$$

with

$$\begin{aligned} {}^*G_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}, && (2.8) \\ G^{\mu\nu} &= \sum_n g_n \int dy_n^\mu \wedge dy_n^\nu \delta^4(x - y_n). && (2.9) \end{aligned}$$

Here we call $(e_n, g_n, m_n)$ the electric charge, magnetic charge and mass of the $n$–th particle respectively. We use signature (-,+,+,+) and take $\epsilon_{0123} = 1$. The electric and magnetic couplings are described asymmetrically: only the magnetic charge appears in the definition (2.7) of the electromagnetic strength $F_{\mu\nu}$, and only the electric charge enters the minimal coupling term (2.5).

We attach to each particle a string $y_n(\sigma_n, \tau_n)$, with $0 \le \sigma_n < \infty$ and $-\infty < \tau_n < \infty$. A particle trajectory is specified by $z_n^\mu(\tau_n) = y^\mu(\sigma_n = 0,\ \tau_n)$.

The equations of motion that follow from (2.3) are

$$\begin{aligned} \partial_\mu H^{\mu\nu} &= -j_e^\nu, && (2.10) \\ m_n \ddot{z}_n^\mu &= (e_n F^\mu_{\ \nu} + g_n {}^*H^\mu_{\ \nu}) \dot{z}_n^\nu && (2.11) \end{aligned}$$

where $j_e^\nu$ is the electric current,

$$j_e^\nu = \sum_n e_n \int dz_n^\mu \delta^4(x - z_n), \tag{2.12}$$

and where the evolution parameter $\tau_n$ in (2.11) is the particle's proper time. We have defined

$$H^{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \tag{2.13}$$

For Maxwell theory $\mathcal{L} = (-1/4) F^{\mu\nu} F_{\mu\nu}$ and $H^{\mu\nu} = F^{\mu\nu}$. Equations (2.10) and (2.11) arise from extremization of the action with respect to the vector potential $A_\mu$ and the particles coordinates $z_n^\mu$, respectively. Extremization of the action with respect to the string coordinates yields no equation provided the string attached to particle $n$ passes through no other particle ("Dirac veto"). This remains true even if each particle has both electric and magnetic charge. The original Dirac veto stated that the Dirac string of a magnetic pole cannot pass through an electric charge. One can verify that the argument remains valid for dyons in the form just stated. The analysis follows, step by step, that of [2]. The only difference is the appearance of a term of the form

$$\epsilon_{\mu\nu\alpha\beta}\int d^4x \int dy^\mu \wedge dy^\nu \int dz^\alpha \delta y^\beta \delta^4(x-y)\delta^4(x-z) = \qquad (2.14)$$
$$\epsilon_{\mu\nu\alpha\beta}\int d^4x \int d\tau d\sigma d\tau' (\dot{y}^\mu y'^\nu - \dot{y}^\nu y'^\mu)\dot{z}^\alpha \delta y^\beta \delta^4(x-y(\tau,\sigma))\delta^4(x-z(\tau'))$$

in the variation of the action with respect to the string coordinates. However (2.14) actually vanishes: the integral over $x^\mu$ forces $y^\mu = z^\mu$, *i.e.*, $\sigma = 0$ and $\tau = \tau'$, implying $\dot{y}_n^\mu = \dot{z}_n^\mu$. Thus, the whole expression is actually zero because $\epsilon_{\mu\nu\alpha\beta}$ is antisymmetric in $\mu$, $\alpha$, while the product $\dot{z}^\mu \dot{z}^\alpha$ is symmetric. Whenever necessary, we shall therefore regularize such expressions to zero [The integrand is formally singular because it involves $\delta^2(0)$]. The magnetic current appears as a "source for the Bianchi identity",

$$\partial_\mu {}^*F^{\mu\nu} = j_m^\nu \qquad (2.15)$$

with

$$j_m^\nu = \sum_n g_n \int dz_n^\mu \delta^4(x - z_n), \qquad (2.16)$$

## 2.2 Charge Quantization Condition

We now derive the quantization condition for the dyon charges, following Dirac's argument[2] for "pure" sources based on the unobservability of the strings. The extension to dyons presents no difficulty if one takes proper account of the multiple–connectedness of the configuration space of the strings and charges: We shall give it in detail here, not having seen it in the literature.

It is convenient to choose as time coordinate on the strings the zeroth coordinate $y^0$ itself, $\tau = y^0$. The momenta conjugate to the string spatial coordinates are then constrained by

$$\pi_i = -g^* H_{ik} \frac{\partial y^k}{\partial \sigma}. \tag{2.17}$$

The dependence of the wave functional $\Psi[A_i, z_n, y_n]$ on the string coordinates is entirely determined by the constraints. This reflects, in the quantum theory, the fact that the strings carry no degree of freedom of their own. The dependence of $\Psi$ on $y_n$ follows from integrating the quantum constraints,

$$\frac{\hbar}{i} \frac{\delta \Psi}{\delta y_n^i} = - \left[ g_n^* H_{ik} \frac{\partial y_n^k}{\partial \sigma} \right] \Psi \tag{2.18}$$

in the configuration space of the string coordinates, the particle coordinates and the vector potential. This space is not simply connected, because of the Dirac veto, and the general requirement is: Circling a loop in configuration space which is contractible to a point, the wave function must return to itself. If the loop is not contractible the wave functions need not be single–valued, but must just form a representation of the fundamental homotopy group $\pi_1$[10].

Consider the double-pass motion shown in figure 1, and discussed in detail in Appendix A. The string attached to dyon 1, with charges $(e, g)$, rotates around dyon 2, with charges $(\bar{e}, \bar{g})$, while the string attached to 2 rotates at the same time around 1. The two strings rotate out of the sheet and the first string is behind the second string, so that they never touch. The simultaneous $2\pi$ turn of both strings ("double-pass") is a closed contractible loop in the configuration space of the strings *and* the particles. By contrast, the turn of a single string is *not* contractible in the space defined by the vetos.

Since the double-pass is contractible, the phase picked up by the wave function should be a multiple of $2\pi$. In that motion, the rotation of the string attached to the first dyon brings in the phase $(1/\hbar) g\bar{e}$ because of (2.18) and the Gauss constraint $\partial_j E^j = j_e^0$. Similarly, the rotation of the string attached to the second dyon brings in the phase $-(1/\hbar) e\bar{g}$, with a minus sign because the orientations of the two-surfaces swept out by the strings in their turning are opposite. Hence, the total phase is equal to $(1/\hbar)(\bar{e}g - e\bar{g})$. Single-valuedness of the wave function for this contractible motion implies the Dirac quantization condition [1],

$$\bar{e}g - e\bar{g} = 2\pi n \, \hbar \ . \tag{2.19}$$

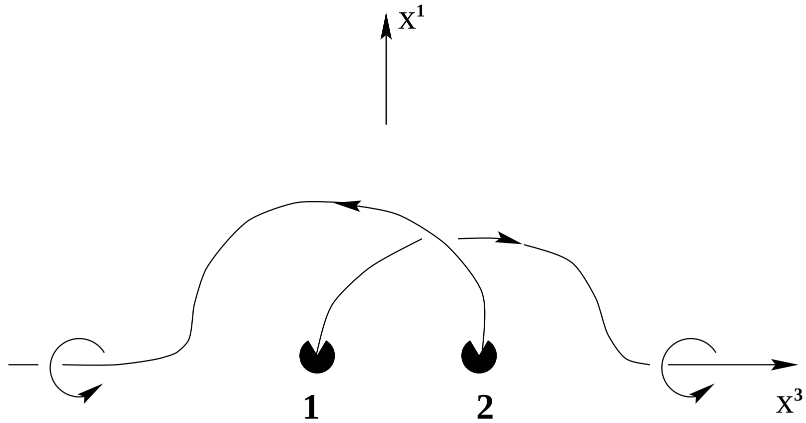


Figure 1: Double pass. The string of dyon 1 goes towards dyon 2 without touching it, whereas that of dyon 2 goes towards dyon 1 again without touching it. The strings themselves do not touch either. Rather, string 1 is behind (more into the page than string 2). We now imagine a simultaneous full turn of both strings going out of the page in the $(x^1, x^2)$ plane orthogonal to the $x^3$ axis. The dyons are kept fixed. This "motion" describes a path in the configuration space of the dyons and the strings which we call "the double pass". The fact that string 1 does not touch dyon 2, and vice versa, is mandatory (Dirac veto). However, in the double pass we also require the strings not to touch each other at any "instant" during the motion that generates it. This is not mandatory in general, but it is what makes the double pass contractible to a point.

Note that if the first dyon is purely magnetic ($e = 0$) and the second purely electric ($\bar{g} = 0$) – the case considered by Dirac –, the phase is entirely accounted for by the motion of the string attached to the magnetic pole. In fact, the constraint (2.18), which involves only the magnetic couplings, obviously implies that the wave function should not depend on the coordinates of strings attached to purely electric particles since it reduces to $\delta\Psi/\delta y^i_n = 0$ (for particle $n$ purely electric). It is thus clear that no non-trivial phase can be generated by a motion of these strings. One can thus drop them altogether, in agreement with the original treatment by Dirac in which no string was ever attached to purely electrically charged particles. In the configuration space where the electric poles carry no string, the motion of figure 1 – in which there is now only one string – becomes contractible, because one can now move the magnetic pole without restriction [the restriction that it must avoid the string attached to the electric particle is absent]. Thus it is legitimate – and actually mandatory – to require that the phase $(1/\hbar)g\bar{e}$ associated with that single motion should be a multiple of $2\pi$, in agreement with what (2.19) reduces to in this case.

By contrast, it would be wrong to require that the phase picked up by the wave function in the rotation of a single string is a multiple of $2\pi$ for generic dyons because that motion is not homotopic to the trivial motion: only the double-pass is. The phase around a non-contractible loop depends in fact on the chosen representation and is equal, as we have seen, to $(1/\hbar)g\bar{e}$ or $-(1/\hbar)e\bar{g}$, which are in general not integer multiples of $2\pi$ (only (2.19) holds).

## 2.3 Dual formulation

In the above treatment, there is a vector potential for $F_{\mu\nu}$ but none for ${}^*H_{\mu\nu}$. One may go to an alternative representation in which the roles of $F_{\mu\nu}$ and ${}^*H_{\mu\nu}$ are exchanged, as well as the roles of magnetic and electric charges. This may be done for all values of the rank of the $p$-form and the spacetime dimension $D$. We describe here the procedure for $p = 1$, $D = 4$. A Lagrange multiplier $S_{\alpha\beta}$ is introduced for the Bianchi identity, i.e. for the definition (2.7) of $F_{\mu\nu}$, so that the action becomes

$$I[A, F, S, z, y] = I_F + I_C + I_P - \frac{1}{4}\int d^4x \epsilon^{\alpha\beta\mu\nu} S_{\alpha\beta}(F_{\mu\nu} - \partial_\mu A_\nu + \partial_\nu A_\mu - {}^*G_{\mu\nu}). \tag{2.20}$$

We have adjusted the coefficient of the Lagrange multiplier term so that $S_{\alpha\beta} =^* H_{\alpha\beta}$ follows from the $F$-equation of motion. If one expresses $F$ in terms of $S$ from this equation (we assume this to be possible), one gets the action

$$I[A,S,z,y] = I_S + \frac{1}{2}\int d^4x[\epsilon^{\alpha\beta\mu\nu}S_{\alpha\beta}\partial_\mu A_\nu - S^{\mu\nu}G_{\mu\nu}] + I_C + I_P \tag{2.21}$$

where

$$I_S = \int d^4x[\mathcal{L}(F(S)) - \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}S_{\alpha\beta}F_{\mu\nu}(S)] \equiv \int d^4x\bar{\mathcal{L}}(S). \tag{2.22}$$

The original vector potential $A_\mu$ appears linearly in (2.21) and may be viewed as a Lagrange multiplier for the Bianchi identity for $S_{\mu\nu}$,

$$\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\partial_\beta S_{\mu\nu} + j_e^\alpha = 0. \tag{2.23}$$

If one solves (2.23) by expressing $S_{\mu\nu}$ as the exterior derivative of a vector potential $Z_\mu$ plus a string term,

$$S_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu +^* M_{\mu\nu}, \tag{2.24}$$

with

$$M^{\mu\nu} = \sum_n e_n \int dy_n^\mu \wedge dy_n^\nu \delta^4(x-y_n), \tag{2.25}$$

one finds the action

$$I[Z,z,y] = I_S + I_C^{magn.} + I_P - \frac{1}{2}\int d^4x^* M_{\mu\nu}G^{\mu\nu} \tag{2.26}$$

where $S_{\mu\nu}$ is defined through (2.24) and the magnetic minimal coupling term is

$$I_C^{magn.} = -\sum_n g_n \int dz_n^\mu Z_\mu(z_n)\ . \tag{2.27}$$

The last term in the action (2.26) can be written as the integral of a total time derivative,

$$\int d^4x^* M_{\mu\nu}G^{\mu\nu} = \int dt \frac{\delta V}{\delta y_n^\alpha}\dot{y}_n^\alpha + \int dt \frac{\delta V}{\delta z_n^\alpha}\dot{z}_n^\alpha \tag{2.28}$$

where $V$ is a functional of the coordinates of the strings and the particles whose precise form will not be needed here. That $V$ exists may be verified by observing that the variational derivatives, keeping the endpoints fixed, of $\int d^4x^* M_{\mu\nu} G^{\mu\nu}$ all identically vanish, implying the form (2.28). [To check that these functional derivatives vanish, one must use the fact that expressions like (2.14) above are zero.]. However, since the configuration space of the particles and the strings is multiply connected because of the veto, the functional $V$, which exists locally, may be multiple-valued. That is, the integral of its (functional) gradient along a non-contractible loop may be non-zero. Below, we will actually exhibit a loop for which this is the case. One may add to the action the integral of the time derivative of a multiply-valued functional. This does not modify the classical equations of motion, and is also admissible quantum-mechanically, provided one changes the representation of the fundamental group in which the wave functions transform accordingly (see below). If one drops the last term in the action (2.26) one gets the magnetic description of the interaction,

$$I[Z, z, y] = I_S + I_C^{magn.} + I_P \tag{2.29}$$

in which the electric minimal coupling to $A_\mu$ has been replaced by a magnetic minimal coupling to $Z_\mu$, and magnetic string terms in $F$ have been replaced by electric string terms in the definition of the dual field $S$.

When the theory is duality invariant, which happens when the Lagrangian fulfills the conditions analysed in [11], the action $I_S$ is the same functional of $S$ as $I_F$ is of $F$. A particular example is given by the Born-Infeld theory [12], whose Hamiltonian has been studied from the duality point of view in [13]-[16] (the dual Lagrangian was explicitly worked out by Born and Infeld, equation (4.12) of [12]).

The purpose of the rather direct exercise of deriving (2.29) from the original action is to show that the phase picked up by the quantum wave functional when integrating the constraints around non contractible loops in configuration space does depend on the (multiple-valued) functional $V$. Only the phase picked up around contractible loops has an invariant meaning.

The quantum constraints read now

$$\frac{\hbar}{i}\frac{\delta\Psi}{\delta y^i_n} = \left[e_n F_{ik}\frac{\partial y^k_n}{\partial\sigma}\right]\Psi = -e_n\epsilon_{ijk}B^j y'^k_n \; \Psi \tag{2.30}$$

where $B^i$ is the magnetic field,

$$\begin{aligned} B^i &= \frac{1}{2}\epsilon^{ijk}F_{jk} \\ &= \epsilon^{ijk}\partial_j A_k + \sum_n g_n \int dy_n^0 \wedge dy_n^i \delta^{(4)}(x-y_n). \end{aligned} \quad (2.31)$$

Here, we have used

$$\frac{\partial\bar{\mathcal{L}}}{S_{\mu\nu}} = -\frac{1}{2}\ {}^*F^{\mu\nu}. \quad (2.32)$$

Consider the double-pass analyzed above. When the first string, attached to the dyon $(e,g)$ turns around the second dyon $(\bar{e},\bar{g})$, this time the integration of the constraint (2.30) brings in the phase $-(1/\hbar)e\bar{g}$ rather than $(1/\hbar)g\bar{e}$. One gets a phase different from that of above because the loop is not contractible, indicating that different representations of the fundamental group are involved [Conversely, the existence of a closed one–form in configuration space whose integral around the loop is not zero proves that the loop is not contractible]. However, if one considers the combined motions involved in the double-pass, which is contractible, one gets the same total phase $(1/\hbar)(g\bar{e}-e\bar{g})$ because the rotation of the second string brings in now $(1/\hbar)g\bar{e}$. The quantization condition is, properly, unchanged.

# 3 2-Forms in D=6; Dyons

## 3.1 Action

Dyonic sources for $p$-forms can exist in spacetime dimension $2(p+1)$. However, the quantization condition for the dyon charges of even $p$ is strikingly different from that of odd $p$.

When one introduces $p$–branes which are the higher dimensional analogs of the Dirac string one finds that the simplest contractible loop for two dyons and their branes is the higher-dimensional analog of the double-pass. The new feature for $p=2k$-forms is that the orientations of the surfaces swept out by the two membranes are now the same, instead of being opposite as in the $p=(2k+1)$–case. Thus the total phase for the double-pass acquires a relative plus sign instead of a minus sign and one obtains $e\bar{g}+g\bar{e}=2\pi n\hbar$ as the quantization condition instead of (2.19). This change of orientation is explained in detail for $D = 6$ dimensions in Appendix A. There is an easier way to verify this, based on the symmetric representation of the theory, which we describe in the next section.

The covariant action for the 2-form coupled to dyons is,

$$I[A_{\mu\nu}, z^\mu, \bar{z}^\mu, \bar{y}^\mu] = I_F + I_C + I_P \tag{3.1}$$

with

$$\begin{aligned} I_F &= \int d^6x \mathcal{L}(F^{\mu\nu\lambda}), & (3.2)\\ I_C &= \sum_n \frac{e_n}{2} \int A_{\mu\nu}(z_n) dz_n^\mu \wedge dz_n^\nu, & (3.3)\\ I_P &= -\sum_n m_n \int (-{}^{(2)}g_n)^{\frac{1}{2}} d^2\sigma_n. & (3.4) \end{aligned}$$

In (3.4), $g_n$ is the determinant of the metric induced on the two-dimensional spacetime history of the strings $z_n^\mu(\sigma_n^1, \sigma_n^2)$. We attach a membrane with history $y_n^\mu(\sigma_n^1, \sigma_n^2, \tau_n)$ to each string, $y_n^\mu(\sigma_n^1, \sigma_n^2, 0) = z_n^\mu(\sigma_n^1, \sigma_n^2)$. The field strength appearing in (3.2) is,

$$F_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda} - \partial_\lambda A_{\nu\mu} +^* G_{\mu\nu\lambda} \tag{3.5}$$

where ${}^*G_{\mu\nu\lambda}$ is the contribution of the membranes,

$$G^{\mu\nu\lambda} = \sum_n g_n \int dy_n^\mu \wedge dy_n^\nu \wedge dy_n^\lambda \delta^6(x - y_n). \tag{3.6}$$

The equations of motion following from the action (3.1) consistently describe the coupled system of the abelian 2-form with the dyons, provided the membrane attached to string $n$ does not pass through the other strings (Dirac veto) [5]. In particular, the 2-form field equation and the Bianchi identity are respectively

$$\begin{aligned} \partial_\mu H^{\mu\nu\lambda} &= -j_e^{\nu\lambda}, & (3.7)\\ \partial_\mu^* F^{\mu\nu\lambda} &= j_m^{\nu\lambda} & (3.8) \end{aligned}$$

with

$$\begin{aligned} j_e^{\nu\lambda} &= \sum_n e_n \int dz^\nu \wedge dz^\lambda \delta^6(x - z_n) & (3.9)\\ j_m^{\nu\lambda} &= \sum_n g_n \int dz^\nu \wedge dz^\lambda \delta^6(x - z_n), & (3.10) \end{aligned}$$

where

$$H^{\mu\nu\lambda} = -(3)!\frac{\partial \mathcal{L}}{\partial F_{\mu\nu\lambda}} \tag{3.11}$$

so that $H^{\mu\nu\lambda} = F^{\mu\nu\lambda}$ in the linear case.

### 3.2 Charge Quantization

One can now derive the quantization condition as in 4 dimensions. Consider two dyons with respective strengths $(e, g)$ and $(\bar{e}, \bar{g})$. In the higher-dimensional generalization of the double-pass described above, the membrane attached to the first dyon describes a three-dimensional surface which links, in five-dimensional space, the other dyonic string. Simultaneously, the membrane attached to the second dyon performs an analogous motion around the first string. Because, for each membrane, the quantum constraints which are the analog of (2.18) read

$$\frac{\hbar}{i}\frac{\delta\Psi}{\delta y^i} = \left[g\ {}^*H_{ijk}\frac{\partial y^j}{\partial\sigma^1}\frac{\partial y^k}{\partial\sigma^2}\right]\Psi\ , \tag{3.12}$$

one finds that the motion of the first membrane brings in the phase $(1/\hbar)g\bar{e}$. The motion of the second string brings in the phase $+(1/\hbar)\bar{g}e$ because the orientations are the same. Therefore it is the plus sign which is realized, so that the quantization condition is (1.2) as announced.

As for D=4 electrodynamics, there is a formulation dual to that of section 3.1. However, as explained in [7], the hyperbolic duality rotations that leave the equations of motion invariant do not respect the action. There is only a residual, discrete, duality that interchanges the electric and magnetic fields, with plus signs. The theory is invariant under this discrete duality if the dual Lagrangian $\bar{\mathcal{L}}(S)$ is the same function of $S$ as $\mathcal{L}$ is of $F$.

## 4 Two-potential Formulation

There exists a more symmetric Hamiltonian formulation, in which electric and magnetic sources are treated on the same footing. It was given in [7] for Maxwell theory but is also valid in $D = 4k$. The two-potential formulation for $D \neq 2(p+1)$ is not particularly useful.

## 4.1 1–forms in D=4

We start from the covariant action (2.3) and perform the standard Legendre transformation on the spatial components of the vector potential $A_i$ to reach first order form. Denoting by $\pi^i$ the momenta conjugate to $A_i$, we get

$$I = \int d^4x[\pi^i \dot{A}_i + \pi^i\, {}^*G_{0i} - \mathcal{H}(\pi, B) + A_0(\pi^i{}_{,i} + j^0)] + I'_C + I_P \tag{4.1}$$

with

$$I'_C = \sum_n e_n \int A_k(z_n) dz^k_n. \tag{4.2}$$

In (4.1), we have set

$$\mathcal{H}(\pi, B) = \pi^i F_{0i} - \mathcal{L}(F), \tag{4.3}$$

where $F_{0i}$ is expressed in terms of $\pi^i$ through $\pi^i = -H^{0i}$. We then solve the Gauss law $\pi^i{}_{,i} + j^0 = 0$ by using the strings attached to the particles[20], writing the momentum $\pi$ as the curl of a second vector potential plus a string contribution

$$-\pi^i = \epsilon^{ijk}\partial_j Z_k + \sum_n e_n \int dy^0_n \wedge dy^i_n \delta^4(x - y_n). \tag{4.4}$$

The minus sign choice establishes symmetry between electric ($-\boldsymbol{\pi}$) and magnetic fields.

The action then reduces to (4.1) without the $A_0$-term; $\pi^i$ now stands for the function of the dynamical variables defined in (4.4). We introduce the symmetric notation

$$q^a_n = (g_n, e_n), \tag{4.5}$$
$$\mathbf{A}^a = (\mathbf{A}, \mathbf{Z}), \tag{4.6}$$
$$\mathbf{B}^a = (\mathbf{B}, -\boldsymbol{\pi}), \tag{4.7}$$

with $a$=1 (magnetic), 2 (electric); now

$$\mathbf{B}^a = \boldsymbol{\nabla} \times \mathbf{A}^a + \boldsymbol{\beta}^{\boldsymbol{a}}, \tag{4.8}$$

with

$$\boldsymbol{\beta}^{\boldsymbol{a}} = \sum_n q^a_n \int dy^0_n \wedge d\mathbf{y}_n\, \delta^4(x - y_n) \tag{4.9}$$

and

$$\mathcal{H} = \mathcal{H}(\mathbf{B}^a). \tag{4.10}$$

If we also define

$$\alpha_i^a = \frac{1}{2}\epsilon_{ijk} \sum_n q_n^a \int dy_n^j \wedge dy_n^k \tag{4.11}$$

we get[1]

$$I[A^a, z_n, y_n] = \int d^4x[-\mathbf{B}^2 \cdot (\dot{\mathbf{A}}^1 + \boldsymbol{\alpha}^{\mathbf{1}}) - \mathcal{H}] + I'_C + I_P\ . \tag{4.12}$$

To derive the manifestly symmetric formulation, let us first notice the (symmetric) identity

$$\frac{1}{2}\int d^4x k_{ab}\mathbf{B}^a(\dot{\mathbf{A}}^b + \boldsymbol{\alpha}^{\boldsymbol{b}}) - \frac{1}{2}\sum_n k_{ab} q_n^a \int \mathbf{A}^b(z_n)\cdot d\mathbf{z}_n = \int d^4x \left(\partial_\mu V^\mu + \frac{1}{2}\, k_{ab}\boldsymbol{\alpha}^{\boldsymbol{a}}\boldsymbol{\beta}^{\boldsymbol{b}}\right)\ . \tag{4.13}$$

Here $k_{ab}$ is the *symmetric* matrix

$$k_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.14}$$

The form of $V^\mu$ is easily worked out; the variational derivatives of the second term vanish (as in (2.28)), which implies that

$$\frac{1}{2}\int d^4x\, k_{ab}\boldsymbol{\alpha}^{\boldsymbol{a}}\boldsymbol{\beta}^{\boldsymbol{b}} = \int dt\ d\sigma \frac{\delta W}{\delta y_n^\alpha}\dot{y}_n^\alpha + \int dt \frac{\partial W}{\partial z_n^\alpha}\dot{z}_n^\alpha\ . \tag{4.15}$$

Again, since the configuration space of the particles and the strings is multiply connected because of the vetos, the functional $W$, which exists locally, turns out to be multiple-valued.

If one adds to the action (4.12) the left-hand side of (4.13), which has vanishing variation, one obtains the manifestly symmetric form of [7]

$$I = \int d^4x \left[\frac{1}{2}\epsilon_{ab}\mathbf{B}^a \cdot (\dot{\mathbf{A}}^b + \boldsymbol{\alpha}^{\boldsymbol{b}}) - \mathcal{H}\right] + I_C + I_P \tag{4.16}$$

[1] The sum $\dot{\mathbf{A}}^a + \boldsymbol{\alpha}^{\boldsymbol{a}}$ was called $E_i^a$ in [7]. However, since $\mathbf{B}^2$ is the actual electric field (equal to $\dot{\mathbf{A}}^a + \boldsymbol{\alpha}^{\boldsymbol{a}}$ up to a gauge only on shell ), we shall not use this notation here.

with

$$I_C = \frac{1}{2}\sum_n \epsilon_{ab} q_n^b \int \mathbf{A}^a(z_n)\cdot d\mathbf{z}_n. \tag{4.17}$$

Note that the addition of (4.13) to the action has had the effect of bringing in a magnetic minimal coupling term, on a par with the standard electric minimal coupling term, each with a factor 1/2. [The importance of this factor 1/2 has been recently stressed in the string context in [17]].

Since we have derived the action (4.16) from the original Dirac action, they are equivalent. The two-potential action was previously derived in [18] for sourcefree Maxwell theory (see [19] for a further discussion of the equivalence of the two-potential formulation and the Maxwell action in the absence of sources).

It is of interest to rederive the quantization condition in the symmetric formulation. The constraints for the strings momenta are now

$$\frac{\hbar}{i}\frac{\delta\Psi}{\delta y_n^i} = \frac{q^a}{2}\epsilon_{ijk}\epsilon_{ab}B^{(b)j}y_n'^k\ \Psi \tag{4.18}$$

as follows from the kinetic term in the action (4.17) (only the kinetic term $(1/2)\epsilon_{ab}\mathbf{B}^a(\dot{\mathbf{A}}^b+\boldsymbol{\alpha}^b)$ contributes to the momenta conjugate to the spatial coordinates of the strings, since in the gauge $y_n^0=\tau_n$, the time derivatives $\dot{y}_n^k$ occur only in $\boldsymbol{\alpha}^b$ and not in $\boldsymbol{\beta}^b$).

The phase picked up by the wave function in the double-pass gets contributions $(1/2)\epsilon_{ab}q^a\bar{q}^b$ from turning string 1 around $\bar{q}^b$, and $-(1/2)\epsilon_{ab}\bar{q}^a q^b$ from turning string 2 around the dyon $q^b$. There is a minus sign because the orientations of the two-surfaces swept out by the strings in their rotation are opposite. The two contributions are equal and add up to $\epsilon_{ab}q^a\bar{q}^b$, leading to the duality-invariant quantization condition for dyons found above (equation (2.19), which reads, in the symmetric notation

$$\epsilon_{ab}q^a\bar{q}^b = nh. \tag{4.19}$$

Note again that the total phase for the double-pass is the same as in the previous representations, as it should since the double-pass is contractible. The phases from the individual rotations are different, however, reflecting the fact that different representations of the fundamental group are involved. In the symmetric description, each rotation brings in half of the total phase.

## 4.2 Duality Invariant Theories

The previous discussion including the quantization condition holds independently of the form of the Lagrangian $\mathcal{L}(\mathcal{F}_{\mu\nu})$: This is because the momenta conjugate to the string coordinates always take the same form, leading always to the same quantum constraints (2.30), (2.18), (4.18).

When the Lagrangian is duality invariant, the symmetric action (4.16) is manifestly invariant under $SO(2)$-duality rotations. This is because the kinetic term, which involves the invariant tensor $\epsilon_{ab}$, is duality invariant [7]. The Hamiltonian is also invariant, as it depends on the fields only through the invariant combination $\delta_{ab}\mathbf{B}^a \cdot \mathbf{B}^b$ and $(\epsilon_{ab}\mathbf{B}^a \times \mathbf{B}^b)$ [16].

Contrary to what is sometimes asserted, duality invariance is a standard symmetry in the sense that it is not just a symmetry of the equations of motion but also a symmetry of the action, with associated "chiral" charge [13]. Its canonical generator is simply the Chern–Simons spatial integral[7, 20]

$$G = -\frac{1}{2}\int d^3x\, \mathbf{A}^a \cdot \mathbf{B}^b \delta_{ab}\ . \tag{4.20}$$

The invariance of the Hamiltonian is equivalent to $[H, G] = 0$. [In the one-potential formulation [20], $G$ is the same, with $\mathbf{A}^2$ replaced by $\nabla^{-2}(\nabla \times \boldsymbol{\pi})$.]

## 4.3 Two-potential Formulation in D=6

The symmetric magnetic/electric description proceeds as in D=4: We introduce the symmetric notation

$$q^a_n = (g_n, e_n), \tag{4.21}$$
$$A^a_{ij} = (A_{ij}, Z_{ij}), \tag{4.22}$$
$$B^a_{ij} = (B_{ij}, E_{ij}), \tag{4.23}$$

again with $a$=1 (magnetic), 2 (electric). Here $B^{ij}$ is the magnetic field,

$$B^{ij} = \frac{1}{3!}\epsilon^{ijklm}F_{klm}, \tag{4.24}$$

and $Z_{ij}$ the potential for the electric field. In this unified form,

$$B^{a\,ij} = \frac{1}{2}\epsilon^{ijklm}\partial_k A^a_{mn} - \beta^{a\,ij}\ , \tag{4.25}$$

where the membrane contribution $\beta^{a\,ij}$ is

$$\beta^{a\,ij} = \sum_n q_n^a \int d^6x dy_n^0 \wedge dy_n^i \wedge dy_n^j \delta^6(x - y_n) \ . \tag{4.26}$$

Similarly, we introduce

$$\alpha_{ij}^a = \frac{1}{3!}\epsilon_{ijklm} \sum_n q_n^a \int d^6x dy_n^k \wedge dy_n^l \wedge dy_n^m \delta^6(x - y_n) \ . \tag{4.27}$$

Repeating the steps that led to the action (4.12) and observing that the momentum conjugate to $A_{ij}$ is now $(-1/2)E^{ij}$, one finds for the action,

$$I[A^a, z_n, y_n] = \int d^6x[-\mathbf{B}^2 \cdot (\dot{\mathbf{A}}^1 + \boldsymbol{\alpha}^{\mathbf{1}}) - \mathcal{H}] + I_C' + I_P \tag{4.28}$$

where $\mathcal{H}(B_{ij}^a)$ is the 2-form Hamiltonian density, introducing the obvious convention

$$\mathbf{N} \cdot \mathbf{M} \equiv \frac{1}{2} N^{ij} M_{ij} \ . \tag{4.29}$$

In (4.28), $I_C'$ is the spatial part of the electric minimal coupling term. If (as before) one now adds the integral of a suitable total divergence, the action becomes symmetrized:

$$I = \int d^6x[-\frac{1}{2}k_{ab}\mathbf{B}^a \cdot (\dot{\mathbf{A}}^b + \boldsymbol{\alpha}^{\boldsymbol{b}}) - \mathcal{H}] + I_C + I_P \tag{4.30}$$

$$I_C \equiv \frac{1}{4} \sum_n k_{ab} q_n^b \int A_{ij}^a(z_n) dz_n^i \wedge dz_n^j \ . \tag{4.31}$$

The one difference from the 1-form case is simple but far-reaching: $\epsilon_{ab}$ is replaced by $k_{ab}$. Its symmetry ensures that of the kinetic term under exchange of $\mathbf{A}$ and $\mathbf{Z}$, due to the symmetry of $\epsilon^{ijklm}$ under the permutation of any two pairs of indices. As pointed out in [7], the fact that it is the non-duality invariant $k_{ab}$ that appears destroys the invariance of the kinetic term – and hence also of the action – under $SO(2)$-rotations of the $B_{ij}^a$, even when the Hamiltonian is invariant.

## 4.4 Charge Quantization

As a consequence of the modification of the kinetic term, the quantum constraints in the symmetric formulation read

$$\frac{\hbar}{i}\frac{\delta\Psi}{\delta y^k_n} = -\frac{q^a_n}{4}k_{ab}E^{b\,ij}\epsilon_{ijklm}\frac{\partial y^l_n}{\partial\sigma^1_n}\frac{\partial y^m_n}{\partial\sigma^2_n}\Psi \ . \tag{4.32}$$

Thus, the phase acquired in the double pass through the motion of the first membrane is $(1/2\hbar)k_{ab}q^a\bar{q}^b$, while the phase acquired through the motion of the second membrane is again $(1/2\hbar)k_{ab}q^a\bar{q}^b$. Demanding that the total phase is a multiple of $2\pi$ leads then to the symmetric quantization condition

$$k_{ab}q^a\bar{q}^b = nh. \tag{4.33}$$

Note that this treatment gives a way to check the orientations. Had they been opposite one would have obtained zero, in contradiction with what one obtains for single electric and magnetic poles, which are particular cases of dyons.

The condition (4.33) is not empty for a single dyon and there reads

$$k_{ab}q^aq^b = 2eg = nh. \tag{4.34}$$

For a self-dual source ($e = g$), one gets $2(e)^2 = nh$, as in [7] (the factor 2 is absent there because of different normalization conventions).

## 4.5 Chiral/Anti-chiral Decomposition

We saw that $SO(2)$-rotations of the fields are not invariances of the action here, due to the non-invariance of $k_{ab}$ in the kinetic term: Duality rotations are not even canonical transformations [7]. There can only be a residual discrete $Z_2$-symmetry,

$$\mathbf{E}\rightarrow\mathbf{B} \quad , \quad \mathbf{B}\rightarrow\mathbf{E} \tag{4.35}$$

$$e\rightarrow g \quad , \quad g\rightarrow e \tag{4.36}$$

under which the quantization condition is clearly invariant [7]. This motivates a decomposition into chiral and anti-chiral components, defined through

$$A_{ij}\pm Z_{ij} = \sqrt{2}\,U^{\pm}_{ij}, \tag{4.37}$$

and consequently also

$$B_{ij} \pm E_{ij} = \sqrt{2}\, V^{\pm}_{ij} , \tag{4.38}$$

where $V^{\pm}_{ij}$ are respectively defined in terms of $U^{\pm}_{ij}$ as $(B_{ij}, E_{ij})$ are defined in terms of $(A_{ij}, Z_{ij})$

$$V^{\pm}_{ij} = \frac{1}{2}\epsilon^{ijklm}\partial_k U^{\pm}_{mn} - \frac{1}{\sqrt{2}}[\beta^{1\,ij} \pm \beta^{2\,ij}] \; . \tag{4.39}$$

The factor $\sqrt{2}$ has been inserted to match the normalization conventions of [7] for the kinetic term. We also define $e_\pm = (g \pm e)/\sqrt{2}$. If $g = e$ (self-dual source) only $e_+ \neq 0$; if $g = -e$ (anti-self-dual source), only $e_- \neq 0$. Note that for anti-self-dual sources, $e_- = \sqrt{2}e$ is the strength of the anti-self-dual source used in [7] (and called there self-dual).

The kinetic term of the symmetric action (4.30) splits into the sum of two non-interacting pieces. In the free case this is also true of the Hamiltonian, and one gets, for the anti-self-dual part,

$$I^- = \frac{1}{4}\int d^6x[\mathbf{V}^- \cdot (\dot{\mathbf{U}}^- + \boldsymbol{\alpha}_- - \mathbf{V}^-)] + I_C + I_P \tag{4.40}$$

which is the action (22) of [7]. Here,

$$I_C = -\frac{1}{4}\int d^6x A_{ij}J^{ij}, \quad J^{ij} = \sum_n e_n \int dz^i_n \wedge dz^j_n \delta^6(x - z_n) \; . \tag{4.41}$$

The quantization condition becomes

$$\eta_{ab} e^a \bar{e}^b = nh \tag{4.42}$$

with

$$\eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.43}$$

and $e^a = (e_+, e_-)$. This implies, for purely self-dual (anti-self-dual) sources, that $e_+\bar{e}_+ = nh$ ($e_-\bar{e}_- = nh$ ). Furthermore, there is no condition on the relative strengths of self-dual and anti-self-dual pure sources (if there are only pure sources). Conversely, one can reassemble the quantization for the individual self-dual or anti-self dual pieces obtained in [7] to get (4.42).

The extension of our results to higher dimensions is a straightforward addition of indices. For $D = 4k$, $p = 2k-1$, one follows the pattern of ordinary electrodynamics (Section 2), whereas for $D = 4k + 2$, $p = 2k$ (including 0 forms in D=2 [21]), one follows the discussion of section 3 ($D = 6,\ p = 2$). The results remain unchanged.

*Note added:* While this paper was being completed, we received the preprint [22], in which the plus sign in the quantization condition for dyons in $4k + 2$ dimensions is also stressed. Interestingly it is derived there by embedding the theory in a supergravity model with a continuous, rigid $SO(5, 5)$, duality invariance group [23]. Acting with this group on a system of purely electric and purely magnetic membranes, for which the quantization condition was derived in [5, 6], one obtains dyons which fulfill $e\bar{g} + \bar{e}g = nh$. Our derivation shows that this quantization condition is actually independent both of any particular embedding of the abelian $(2k)$-form theory in a bigger model, and also of the existence of a rigid duality invariance group for the embedding theory.

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# Appendix A. Geometry of the "double pass"

We first perform the analysis in three space dimensions and then extend it to higher dimensions.

Consider the situation illustrated in Fig. 1, as described in the caption.

At each instant of the turning, we may imagine each dyon moving by "eating up" its own string, so that they both end up, again on the $x^3$ axis, but with dyon 2 on the left of dyon 1. Since the two strings never touch while the double pass is generated, this displacement of the dyons does not violate the Dirac veto. The operation deforms the whole surface generated by the double pass into two single strings, which correspond to a single point in configuration space. Note that in the original formulation of Dirac, where strings are only attached to the magnetic poles, the same procedure shows that the "single pass" used by Dirac is contractible to a point also. However, when strings are attached to both charges, the single pass (in which only one string turns) is not contractible because one would violate the Dirac veto: at the instant when the two strings cross, the displacement needed to flip the dyons in the above argument would make one dyon go through the string of the other.

The displacement of string 1 around dyon 2 generates a surface with the topology of a two–sphere, which bounds a three–ball with positive orientation relative to

$$dx^1 \wedge dx^2 \wedge dx^3 \quad , \tag{A.1}$$

whereas the orientation associated with the displacement around dyon 1 is the same as

$$dx^1 \wedge dx^2 \wedge (-dx^3) = -dx^1 \wedge dx^2 \wedge dx^3 \quad . \tag{A.2}$$

The $-dx^3$ in (A.2) instead of the $+dx^3$ in (A.1) reflects the Dirac strings going from 2 to 1 being oriented opposite to that going from 1 to 2. The turning that generates the double pass takes place in the $(x^1, x^2)$ plane, and since both strings turn in the same direction, we have written $+dx^1 \wedge dx^2$ in both cases. Comparison of (A.1) and (A.2) shows that the two spheres have opposite orientation, giving rise to the antisymmetric quantization rule described in section 2.2 of the main text.

We now proceed to higher dimensions. Consider first, for definiteness, the case of a two–form potential in D=6 spacetime as discussed in section 3. The dyons are then infinite strings in D=5 space. We take them to lie along the 4- and 5-axes respectively, separated by a distance $a$ along the 3-axis (a configuration that will be generalized in Appendix C).

In the double pass, the membrane of dyon 1 generates a 3–sphere which links dyon 2 and vice versa. Without loss of generality one may take the 3–sphere linking the first dyon to lie in the subspace $x^4 = 0$ and that linking the second dyon to lie in the subspace $x^5 = 0$. The rotation that generates the double pass takes place in the $(x^1, x^2)$ plane, which is complementary to the subspace containing the two dyons and the line that joins them, just as in three space dimensions.

We are now interested in the orientation of these two 3–spheres relative to the orientation of the five dimensional space and to that of the dyons themselves (in the three dimensional case the dyons were points and had only one intrinsic orientation). As in the three dimensional case, the orientation of the sphere linking dyon 2 may be taken to be positive relative to

$$dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \quad , \tag{A.3}$$

whereas for the sphere linking dyon 1 we have

$$dx^1 \wedge dx^2 \wedge (-dx^3) \wedge dx^5 \wedge dx^4 = +dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \quad . \tag{A.4}$$

Also as before, we have replaced $dx^3$ in (A.3) by $-dx^3$ in (A.4), because exchanging the dyons reverses the orientation of the coordinate along the line that joins them and we have taken $dx^1 \wedge dx^2$ with the same sign in both cases by definition of the double pass. There is now, however, a new feature, which is the exchange of the $x^4$ and $x^5$ coming from the exchange of the dyons. Due to this, the surfaces have the same orientation, giving rise to the symmetric quantization indicated in text.

The extension to a $p$–form potential in $2p+1$ space dimensions is immediate. Each dyon contributes a $(p-1)$–form, $\omega^{(p-1)}$, to the exterior product, so that one has

$$dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{(2p+1)} = dx^1 \wedge dx^2 \wedge dx^3 \wedge \omega_1^{(p-1)} \wedge \omega_2^{(p-1)} \ . \tag{A.5}$$

Under exchange of the two dyons on the double pass, one has

$$dx^1 \wedge dx^2 \longrightarrow dx^1 \wedge dx^2 \tag{A.6}$$

$$dx^3 \longrightarrow -dx^3 \tag{A.7}$$

$$\omega_1^{(p-1)} \wedge \omega_2^{(p-1)} \longrightarrow (-1)^{(p-1)} \omega_1^{(p-1)} \wedge \omega_2^{(p-1)} \quad . \tag{A.8}$$

Hence the relative orientation of the two surfaces is $(-1)^p$, giving rise to antisymmetric quantization for odd $p$ and symmetric quantization for even $p$.

# Appendix B. Quantizations for Even p

It was observed long ago that the quantization condition in four dimensions does not force the electric charge carried by a dyon to be an integer multiple of the minimum electric charge $e_0$ carried by purely electric particles [3, 4, 24]. However, if one imposes CP conservation, the electric charge carried by dyons must be an integer, or half-integer, multiple of $e_0$ [25]. We shall show that the same result holds also for $2p$-forms, but without having to invoke CP conservation explicitly.

The general solution of the quantization condition

$$k_{ab} q^a \bar{q}^b = e\bar{g} + g\bar{e} = nh \tag{B.1}$$

can be presented in the chiral-anti-chiral basis, in which (4.42) shows that the allowed values of the charges form a Lorentzian lattice. However, for comparison with the $2p+1$-case, we shall discuss the quantization condition in its original form and shall assume that there are purely electric sources.

Let $e_0$ be the minimum value for the electric charge. Then, the quantization condition shows that the magnetic charge of dyons is an integer multiple of $(2\pi\hbar)/e_0$,

$$e_0 g = 2\pi n\hbar. \tag{B.2}$$

Consider a dyon with the minimum allowed magnetic charge $g_0 = (2\pi\hbar)/e_0$. Since the quantization condition for a single dyon is non-empty, and reads $2eg = 2\pi m\hbar$, the electric charge $e$ carried by this dyon should fulfill

$$2e = me_0, \tag{B.3}$$

as announced.

If the integer $m$ in (B.3) is even, then the electric charge carried by any dyon must be an integer multiple of $e_0$. This follows from the quantization applied to the above "reference" dyon and any other dyon. The charges of a general dyon are thus given by $(ke_0, lg_0)$ with $k$, $l$ integers.

If, on the other hand, the integer $m$ is odd, so that the above dyon has an electric charge which is a half-integer multiple of $e_0$, then, all dyons with odd magnetic charge have also an electric charge that is a half-integer multiple of $e_0$, while all dyons with even magnetic charge have an electric charge which is an integer multiple of $e_0$.

In D=4, one may shift the electric charge of dyons by adding a CP-violating term $\theta \int F \wedge F$ to the action [25]. This possibility does not exist in D=4 $p+2$ because the curvature form $F$ is then of odd degree, so that $F \wedge F$ is identically zero.

# Appendix C. Quantization From Angular Momentum

The quantization condition for the product of electric and magnetic charges has a very appealing physical basis [8] in terms of the spectrum of the angular momentum operator. We show here how this argument extends to dyons in higher dimensions and especially how it brings in the two different types of quantization for even or odd $p$ in a natural way. Recall that in D=2($p$+1), the electric and magnetic fields are spatial $p$-forms and their sources are ($p$-1) spatial extended objects.

We begin by revisiting the original argument in D=4. Consider a magnetic pole of strength $g$ located at the origin of coordinates and an electric pole of strength $e$ at the point $(0,0,a)$. The total angular momentum stored by the field points in the $x_3$ direction, and is

$$\begin{aligned} J_{12} &= \int d^3x(x_1T_{02} - x_2T_{01}) = \int \left[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})\right]_3 d^3x \\ &= -2eg \int d^3x \Phi^E \frac{\partial}{\partial x_3} \Phi^M = \frac{eg}{4\pi} \quad , \end{aligned} \tag{C.1}$$

in terms of the corresponding scalar potentials

$$\Phi^E = \frac{1}{4\pi(x_1^2 + x_2^2 + (x_3 - a)^2)^{1/2}} \tag{C.2}$$

$$\Phi^M = \frac{1}{4\pi(x_1^2 + x_2^2 + x_3^2)^{1/2}} \ . \tag{C.3}$$

The half-integer quantization of this angular momentum yields the Dirac quantization condition.

Note that the integral in (C.1) does not depend on the parameter $a$ as long as this parameter is different from zero [this can be seen before doing the integral just from dimensional analysis]; for $a$=0 the integral manifestly vanishes. Thus there is a striking difference between an electric pole and a magnetic pole which are very close to each other and a dyon where the two poles are at the same point.

The angular momentum is antisymmetric under the exchange of the two poles, which amounts to exchange of **E** and **B**; therefore if we have two dyons, the angular momentum will be given by

$$J_{12} = \frac{1}{4\pi}(\bar{e}g - e\bar{g}) \quad , \tag{C.4}$$

for dyons of charge $(e, g)$ and $(\bar{e}, \bar{g})$, located at $(0, 0, 0)$ and $(0, 0, a)$ respectively. This gives the quantization condition (1.1).

Consider now the general case in $D = 2(p + 1)$ dimensions. An electric pole, which is now a $(p - 1)$–dimensional extended object, is located at $(0, 0, a; x_{a_1}, \ldots, x_{a_{p-1}}; 0, \ldots, 0)$ , with $-\infty < x_a < \infty$. At the same time, a magnetic pole is located at $(0, 0, 0; 0, \ldots, 0; x_{b_1}, \ldots, x_{b_{p-1}})$, with $-\infty < x_b < \infty$, $4 \leq a \leq p + 2$, $p + 3 \leq b \leq 2p + 1$.

The electric–magnetic fields produced by these poles are derived from two potentials $\Phi^E$–$\Phi^M$, which are spatial $p - 1$ forms that generalize the scalar potentials (C.2) and (C.3) for $p = 1$,

$$\begin{aligned} H^0{}_{i_1 \ldots i_p} &= (d\Phi^E)_{i_1 \ldots i_{p-1}, i_p} \ , & \text{(C.5)} \\ {}^*F^0{}_{i_1 \ldots i_p} &= (d\Phi^M)_{i_1 \ldots i_{p-1}, i_p} \ , & \text{(C.6)} \\ & & \text{(C.7)} \end{aligned}$$

and whose only nonvanishing component are

$$\begin{aligned} \Phi^E_{a_1 \ldots a_{p-1}} &= -\frac{e\epsilon_{a_1, \ldots a_{p-1}}}{pS_{p+1}(x_1^2 + x_2^2 + (x_3 - a)^2 + x_{b_1}^2 + \cdots + x_{b_{p-1}}^2)^{\frac{p}{2}}} \ , & \text{(C.8)} \\ \Phi^M_{b_1 \ldots b_{p-1}} &= -\frac{g\epsilon_{b_1, \ldots b_{p-1}}}{pS_{p+1}(x_1^2 + x_2^2 + x_3^2 + x_{a_1}^2 + \cdots + x_{a_{p-1}}^2)^{\frac{p}{2}}} \ , & \text{(C.9)} \end{aligned}$$

where

$$S_{p+1} = \frac{2\pi^{\frac{p+2}{2}}}{\Gamma\left(\frac{p+2}{2}\right)} \tag{C.10}$$

is the area of the $(p + 1)$–sphere.

The angular momentum stored in the field is given by

$$J_{ij} = \int d^{D-1}x(x_i T_{0j} - x_j T_{0i}) \ , \tag{C.11}$$

where $T_{0i}$ is given by

$$T_{0i} = \frac{1}{(p!)^2}\epsilon_{ij_1\cdots j_p\ k_1\cdots k_p} H^{0j_1\cdots j_p} {}^*F^{0k_1\cdots k_p}\ . \tag{C.12}$$

It is convenient to work with the spatial dual , ${}^\dagger J$ of the angular momentum, which in this case is

$$\begin{aligned} {}^\dagger J_{i_1\ldots i_{2p-1}} &= \frac{1}{2}\epsilon_{i_1\ldots i_{2p-1}lm}\int d^{D-1}x J_{lm} \\ &= \frac{1}{(p-1)!^2}\delta^{[j_1\cdots j_p\ k_1\cdots k_p]}_{[i_1\cdots\cdots\cdots i_{2p-1}l]}\int d^{D-1}x\ x^l \Phi^E_{j_1\cdots j_{p-1},j_p}\Phi^M_{k_1\cdots k_{p-1},k_p} \\ &= -\frac{2(-1)^p}{(p-1)!^2}\delta^{[j_1\cdots j_{p-1}\ k_1\cdots k_p]}_{[i_1\cdots\cdots\cdots\cdots\ i_{2p-1}]}\int d^{D-1}x\ \Phi^E_{j_1\cdots j_{p-1}}\Phi^M_{k_1\cdots k_{p-1},k_p}\ , \end{aligned} \tag{C.13}$$

where $\delta^{[123\ldots]}_{[123\ldots]} = 1$ and we have performed an integration by parts, dropping a vanishing contribution at infinity. It is easy to see from the symmetries of the integral in (C.13) that only

$$J_{12} ={}^\dagger J_{3\ 4\ \cdots\ (2p+1)} = -2(-1)^p\int d^{D-1}x\ \Phi^E_{4\cdots(p+2)}\Phi^E_{(p+3)\cdots(2p+1),3} \tag{C.14}$$

$$= -\frac{(-1)^p eg\Gamma(\frac{p}{2})^2}{8\pi^{p+2}}\int \frac{d^{D-1}x}{(x_1^2+x_2^2+(x_3-a)^2+x^bx_b)^{p/2}}\frac{\partial}{\partial x_3}\left(\frac{1}{(x_1^2+x_2^2+x_3^2+x^ax_a)^{p/2}}\right)$$

survives. The integral in (C.14) has the key property that upon integration over one of the $x^a$ and one of the $x^b$ coordinates it yields (up to a sign, which depends on the choice of orientation), the same expression with $p$ replaced by $p-1$. This follows from

$$\int_{-\infty}^{\infty}\frac{dx}{(a+x^2)^{p/2}} = \frac{\sqrt{\pi}\Gamma(\frac{p-1}{2})}{a^{\frac{p-1}{2}}\Gamma(\frac{p}{2})}\ . \tag{C.15}$$

Hence the result for the calculation of the angular momentum of an electric pole of charge $e$ and a magnetic pole of charge $g$ is (up to a sign) the same for all dimensions.

The symmetry of the angular momentum under the exchange of the two poles is now $(-1)^p$ as can be seen from (C.13). This shows that the quantization condition for dyons is symmetric for even $p$ and antisymmetric for odd $p$.

It is of interest to point out that again the integral (C.13) is independent of the distance $a$ between the poles as long as $a$ is different from zero. The integral vanishes for $a = 0$ and also for $a \neq 0$ if the two $(p-1)$–dimensional sources and the vector $a^i$ define a subspace whose dimension is less than $(2p-1)$. For the angular momentum not to vanish we need that subspace to be of maximal dimension $(2p-1)$. Then the angular momentum lies in its orthogonal plane.

# Appendix D. Local gauge charts

One may also derive the quantization conditions for $p$–brane dyons using the analysis based on twisted connections and gauge patches a la Wu and Yang[9]. This may be done both in the two–potential and one–potential formulations. We will carry out the discussion for $p = 1$, $D = 4$. The generalization to higher dimensions is straightforward[26].

## D.1 Two–potential formulation

The key point is to realize that the connection $\omega$ is given by

$$\omega_j = \frac{i}{\hbar}\epsilon_{ab}q^b A^a_j \ , \tag{D.1}$$

without the factor 1/2 one might naively expect from the explicit interaction term

$$I_{int.} = \frac{1}{2}\epsilon_{ab}q^b \int \mathbf{A}^a \cdot d\mathbf{z} \tag{D.2}$$

appearing in (4.17).

To derive the connection one needs to consider a single particle and relate the momentum $p_i$ canonically conjugate to its position $z^i$, to the mechanical momentum

$$P_i = \frac{m\dot{\mathbf{z}}}{\sqrt{1-\dot{\mathbf{z}}^2}} \tag{D.3}$$

(for simplicity we parameterize $y^0 = \tau$ for the worldsheet). This is because $ip_j/\hbar$ corresponds to the partial derivative $\partial/\partial z^j$ whereas $iP_j/\hbar$ corresponds to the covariant derivative

$$\nabla_i = \frac{\partial}{\partial z^i} + \omega_i \ . \tag{D.4}$$

Therefore we have

$$\frac{\hbar}{i}\omega_j = P_j - p_j \ . \tag{D.5}$$

Now, to evaluate the canonical momentum we need the part of the action which contains time derivatives. That part is given by the sum of (D.2) and the contribution

$$\int d^4x \frac{1}{2}\epsilon_{ab}(\boldsymbol{\alpha^b}\cdot\mathbf{B^a} - \boldsymbol{\beta^b}\cdot\partial_0\mathbf{A^a}) \ , \tag{D.6}$$

which, by using (4.9), (4.11) and performing the integral over $d^4x$ may be written as

$$\begin{aligned}
& \frac{q^b}{2}\epsilon_{ab}\int\left(\partial_j A_i^a dy^j\wedge dy^i + \partial_0 A_i^a dy^0\wedge dy^i\right) \\
= \ & \frac{q^b}{2}\epsilon_{ab}\int d[A_i(y)dy^i] \\
= \ & \frac{q^b}{2}\epsilon_{ab}\int_{\tau_1}^{\tau_2} A_i^a(z)dz^i + \frac{q^b}{2}\epsilon_{ab}\int A_i^a(y)dy^i\Big|_{\tau_1}^{\tau_2} \ .
\end{aligned} \tag{D.7}$$

This last form combined with (D.2) shows that the connection is indeed given by (D.1).

The connection produced by a dyon of charge $Q^a$ at the origin may be taken to have only non vanishing spherical component

$$A_\phi^a = \frac{Q^a}{4\pi}(1\pm\cos\theta) \ . \tag{D.8}$$

In (D.8) the upper sign excludes the North pole $\theta = 0$, whereas the lower sign excludes the South pole $\theta = \pi$. Demanding that the two gauge patches overlap so as to give the same transport along – say – the equator yields the desired quantization condition

$$2\pi i n = \int(\omega_\phi^+ - \omega_\phi^-)d\phi = \int \frac{i}{2\pi\hbar}\epsilon_{ab}q^b Q^a d\phi = \frac{i}{\hbar}\epsilon_{ab}Q^a q^b \ . \tag{D.9}$$

## D.2 One–potential formulation

The one-potential formulation given in the text can also accommodate dyons but it only exhibits a connection for transporting either purely electric charges or purely magnetic ones. In the first case the connection is the usual $A_\mu$, whereas in the second it is the $Z_\mu$ introduced in Sec. 2.3.

However, in either case the connection, which does not appear explicitly is also present, as can be seen from the following argument. We will work for definiteness in the usual formulation with $A_\mu$. The Lagrangian depends on the velocity $\dot{z}^\mu$ of each particle both explicitly through the kinetic and minimal coupling terms, and also implicitly as the velocity of the end point of the attached Dirac string. It is through this last dependence that the "missing" connection $Z_\mu$ enters, much in the same way as the "missing" 1/2 in the two–potential formulation is found.

Indeed, the Dirac string velocities enter the action through the term

$$-g\int {}^*H = -g\int {}^*H_{\mu\nu}\frac{\partial y^\mu}{\partial\tau}\frac{\partial y^\nu}{\partial\sigma}d\tau d\sigma \ , \tag{D.10}$$

as is most easily seen by inserting $\pi_\mu$ obtained from the covariant form

$$\pi_\mu = -g{}^*H_{\mu\nu}\frac{\partial y^\nu}{\partial\sigma} \ , \tag{D.11}$$

of the constraint (2.17) into the kinetic term $\int \pi_\mu \dot{y}^\mu d\sigma d\tau$ of the Hamiltonian action.

Now, in the Wu–Yang analysis, ${}^*H$ in (D.10) is the field produced by a fixed source dyon of charge $(\bar{e},\bar{g})$ located, say, at the origin of coordinates. Thus one may write

$${}^*H = dZ +{}^* M \ , \tag{D.12}$$

where $Z$ is the dual connection and ${}^*M$ is the Dirac string contribution of the source dyon (see Sec. 2.3).

Next, one argues that by choosing the Dirac string of the source dyon not to cross the Dirac string of the test dyon $(e,g)$, $M$ may be taken equal to zero in the integral (D.10). This is not only possible, but also mandatory if one wants to have a non–singular $Z_\mu$ as is essential for the Wu–Yang analysis, which uses regular connections over local patches (${}^*H$ is assumed to be regular away from its source, so if $M$ is different from zero, $dZ$ must be singular and hence so must be $Z$ ).

Once $M$ is set equal to zero, one may convert the surface integral (D.10) into a line integral over the boundary of the worldsheet by means of the Stokes theorem to obtain

$$\begin{aligned}-g\int {}^*H &= -g\oint Z \\ &= -g\int_{\tau_1}^{\tau_2} Z_\mu\frac{\partial z^\mu}{\partial\tau}d\tau - g\int_{\sigma=0}^{\infty} Z_\mu\frac{\partial y^\mu}{\partial\sigma}d\sigma \ .\end{aligned} \tag{D.13}$$

This last relation shows that the Dirac string yields a contribution$-gZ_\mu$ to the conjugate momentum of the particle. Therefore the total connection is

$$\omega_\mu = eA_\mu - gZ_\mu \ . \tag{D.14}$$

Note that the minus sign in front of the magnetic charge agrees with that of the minimal coupling term in Eq. (2.27) as it should.

Finally, one observes that for a source dyon $(\bar{e}, \bar{g})$ at the origin of coordinates one has

$$A^{\pm} = \frac{\bar{g}}{4\pi}(1 \pm \cos\theta)d\phi \ , \tag{D.15}$$

$$Z^{\pm} = \frac{\bar{e}}{4\pi}(1 \pm \cos\theta)d\phi \ , \tag{D.16}$$

which yields the quantization condition $e\bar{g} - g\bar{e} = 2\pi n\hbar$.

The preceding conclusions apply without change for all odd $p$-forms $A$. However, when $p$ is even, the term (D.10) comes into the action with the opposite sign. This stems from the fact that the string contribution to ${}^*F$ in (2.7) is ${}^*({}^*G) = (-1)^p G$, and is what is responsible for the difference in sign in the expression for $\pi_\mu$ for odd and even $p$ (compare (2.18) with (3.12)). Thus for any $p$ the analog of (D.14)is given by

$$\omega_{\mu_1\cdots\mu_p} = eA_{\mu_1\cdots\mu_p} + (-1)^p gZ_{\mu_1\cdots\mu_p} \ . \tag{D.17}$$

Furthermore, since the form of the equations of motion for $F$ and ${}^*H$ is the same for all $p$, the dual connection $Z_\mu$ for the source dyon is always obtained from the corresponding $A_\mu$ by simply replacing $\bar{e}$ by $\bar{g}$, just as was the case in (D.15) and (D.16). Therefore (D.17) gives rise to a quantization condition of symmetry $(-1)^p$.

# References


[1] P.A.M. Dirac, *Proc. Roy. Soc. London* **A 133** (1931) 60.

[2] P.A.M. Dirac, *Phys. Rev.* **74** (1948) 817.

[3] J. Schwinger, *Phys. Rev.* **173** (1968) 1536; *Science* **165** (1969) 757.

[4] D. Zwanziger, *Phys. Rev.* **176** (1968) 1489.

[5] C. Teitelboim, *Phys. Lett* **B 167** (1986) 63, 67.

[6] R. Nepomechie *Phys. Rev.* **D 31** (1985) 1921.

[7] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, *Phys. Lett.* **B 400** (1997) 80.

[8] See for example, M. N. Saha, *Indian J. Phys.* **10** (1936) 145; M. Fierz, *Helv. Phys. Acta* **17** (1944) 17; H.A. Wilson, *Phys. Rev.* **75** (1949) 309; B. Zumino in *Strong and Weak Interactions, Present Problems* (A. Zichichi, ed; Academic Press 1966).

[9] T.T. Wu and C.N. Yang, *Phys. Rev.* D14 (1976) 437.

[10] see e.g. C. J. Isham, in *Relativity, Groups and Topology, II*, eds B.S. De Witt and R. Stora (Elsevier, N.Y. 1984) and references therein.

[11] G. W. Gibbons and D. A. Rasheed, *Nucl. Phys.* **B 454** (1995) 185.

[12] M. Born and L. Infeld, *Proc. Roy. Soc. (London)* **A 144** (1934) 425.

[13] I. Bialynicki-Birula, *Nonlinear Electrodynamics: Variations on a Theme by Born and Infeld* in *Quantization Theory of Particles and Fields*, (ed. Jancewicz and Lukierski, World Scientific, 1983).

[14] I. Bengtsson, *Int. J. Mod. Phys.* **A 12** (1997) 4869.

[15] A. Khoudeir and Y. Parra, *On duality in Born-Infeld theory*, hep-th/9708011.

[16] S. Deser and Ö. Sarıoğlu, *Electric/Magnetic Duality in Hamiltonian Form and Lorentz Invariance*, hep-th 9712067.

[17] Yeuk-Kwan E. Cheung and Zheng Yin, *Anomalies, branes, and currents*, hep-th/9710206.

[18] J.H. Schwarz and A. Sen, *Nucl. Phys.* **B 411** (1994) 35.

[19] H.O. Girotti, M. Gomes, V.O. Rivelles, A.J. da Silva, *Phys. Rev.* **D56** (1997) 6615.

[20] S. Deser and C. Teitelboim, *Phys. Rev.* **D 13** (1976) 1572; S. Deser, M. Henneaux and C. Teitelboim, *Phys. Rev.* **D 55** (1997) 268.

[21] M. Henneaux and C. Teitelboim *Consistent Quantum Mechanics of Chiral p–Forms*,in Quantum mechanics of fundamental systems 2 (Proceedings of the conference held in Santiago, Chile). Ed. C. Teitelboim and J, Zanelli. Plenum Press, New York, 1987, 79-112.

[22] M.S. Bremer, H. Lü, C.N. Pope and K.S. Stelle, *Dirac Quantisation Conditions and Kaluza-Klein Reduction*, hep-th/9710244.

[23] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities. I*, hep-th/9710119.

[24] For a recent review, see D. I. Olive, *Exact Electromagnetic Duality*, hep-th/9508089.

[25] E. Witten, *Phys. Lett.* **86 B** (1979) 283.

[26] M. Henneaux and C. Teitelboim, *Found.Phys.* **16** (1986) 593.