Inequivalence of First and Second Order Formulations in D=2 Gravity Models[1]

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The usual equivalence between the Palatini and metric (or affinity and vielbein) formulations of Einstein theory fails in two spacetime dimensions for its "Kaluza–Klein" reduced (as well as for its standard) version. Among the differences is the necessary vanishing of the cosmological constant in the first order forms. The purely affine Eddington formulation of Einstein theory also fails here.

Most general constructions in physical theories are formally valid in all dimensions, even though many properties can be quite different in special D. In this note we present a simple but striking deviation from this rule in the context of some gravity models at D=2, a dimension that is always "more equal" than others.

We shall see that the standard Palatini formulations, in which metric and affinity or vielbein and connection are independently varied, no longer coincide with their purely metric or zweibein second order expressions. While this is known [1] for the usual Einstein Lagrangian, it is also true for the more interesting, "Kaluza–Klein reduced", version involving a Lagrange multiplier [2]. The source of the difference is the Weyl invariance enjoyed (only) by the first order Palatini forms. As a result metricity of the initial, affine, space is no longer recovered from the field equations; the cosmological constant must also vanish. We shall also see that even purely affine expression of Einstein given by Eddington [3] theory is different in 2D.

[1]The present results were reported in the Proceedings of the Markov Memorial Quantum Gravity Seminar.

We first review the purely Einstein version [1]. The Einstein–Palatini Lagrangian in any dimension is

$$\mathcal{L} = h^{\mu\nu} R_{\mu\nu}(\Gamma) \ , \qquad R_{\mu\nu}(\Gamma) \ \equiv \ \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma_{\mu,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma_{\alpha} - \Gamma^{\nu}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \tag{1}$$

where $h^{\mu\nu} \ \equiv \ \sqrt{-g} \ g^{\mu\nu}$, $\Gamma^{\alpha}_{\mu\nu} \ = \ \Gamma^{\alpha}_{\nu\mu}$, $\Gamma_{\mu} \ \equiv \ \Gamma^{\alpha}_{\mu\alpha}$; here the metric and affinity are to be varied independently. Commas denote ordinary differentiation and (since $h^{\mu\nu}$ is symmetric) only the symmetric part of the Ricci tensor enters in $\mathcal{L}$. [In the second order, purely metric, form (where $\Gamma^{a}_{\mu\nu}$ is the Christoffel symbol) the 2D $\mathcal{L}$ is a total divergence – the Euler density, but that is not our point here.] In, and only in, 2D, our $\mathcal{L}$ is manifestly Weyl invariant (if we take $\Gamma^{a}_{\mu\nu}$ to be inert) since the contravariant density $h^{\mu\nu}$ is unimodular. [One could alternately take $h^{\mu\nu}$ to be generic [4], but such a theory is then not a metric one at all.] This new local gauge freedom will imply that Γ_{μ}, whose usual role is to ensure covariant constancy of $\sqrt{-g}$, is undetermined, *i.e.*, that Γ-variation of the action will not fix the affinity completely to be the metric one. On the other hand, the action's metric variation, although it does not vanish identically here, will still turn out to be vacuous just as in second order form. The field equations, then, are

$$G_{\mu\nu} \equiv R_{\mu\nu}(\Gamma) - \tfrac{1}{2} \, g_{\mu\nu} \, g^{\alpha\beta} R_{\alpha\beta}(\Gamma) = 0 \ , \ \ G^{\mu}_{\mu} \equiv 0 \tag{2}$$

$$D_{\alpha} \, h^{\mu\nu} - \tfrac{1}{2}(\delta^{\mu}_{\alpha} \, D_{\lambda} \, h^{\lambda\nu} + \delta^{\nu}_{\alpha} \, D_{\lambda} \, h^{\lambda\mu}) = 0 \tag{3}$$

where the covariant derivatives on the contravariant tensor density $h^{\mu\nu}$ are with respect to Γ, and the symmetrized part of the $R_{\mu\nu}$ is understood in (2). We also note that Weyl invariance of the Einstein action forbids a cosmological term $\lambda\sqrt{-g}$ since the latter does depends on the conformal factor, whose variation implies that $\lambda = 0$. This property is, however, common to second order form where the trace $G^{\mu}_{\mu}(g)$ vanishes identically as well.

To determine $\Gamma^{\alpha}_{\mu\nu}$, we first trace (3), which yields $D_{\nu}h^{\mu\nu} = 0$ and so implies that

$$D_{\alpha}h^{\mu\nu} \equiv \partial_{\alpha}h^{\mu\nu} + \Gamma^{\mu}_{\alpha\lambda}h^{\lambda\nu} + \Gamma^{\nu}_{\alpha\lambda}h^{\lambda\mu} - \Gamma_{\alpha}h^{\mu\nu} = 0 \ . \tag{4}$$

That (4) is not a complete set of equations is clear from the fact that its $(\mu\nu)$ trace vanishes identically in 2D because $g_{\mu\nu}\partial_\alpha h^{\mu\nu}$ does. Since $\delta h^{\mu\nu} \equiv \sqrt{-g}\,(\delta g^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} g_{\alpha\beta}\delta g^{\alpha\beta})$. Normally, the covariant constancy of the metric or metric density expressed in (4) does of course determine the affinity completely to be the metric one. In any D, straightforward algebraic manipulation of (4) yields

$$\Gamma^\alpha_{\mu\nu} = \{^{\;\alpha}_{\mu\nu}\} \;+\; \tfrac{1}{2}(g_{\mu\nu}X^\alpha - \delta^\alpha_\mu X_\nu - \delta^\alpha_\nu X_\mu) \;, \;\; X_\mu \equiv D_\mu \;\ln\; \sqrt{-g} \;=\; \partial_\mu \;\ln\; \sqrt{-g} - \Gamma_\mu \;\;. \qquad (5)$$

whose trace is

$$(1 - D/2)(\Gamma_\mu - \partial_\mu \;\ln\; \sqrt{-g}) = 0 \;. \qquad (6)$$

Here the dimensionality appears explicitly and hence, as advertised, spacetime is not entirely fixed to be a purely metric manifold at D=2, since the Γ_μ component of the affinity remains undetermined.

The Einstein equation (2) still remains vacuous. Inserting (5) into the Ricci tensor yields

$$\sqrt{-g}\; R_{\mu\nu}(\Gamma) = \sqrt{-g}\; R_{\mu\nu}(g) + \tfrac{1}{2}\; g_{\mu\nu}\; \partial_\alpha(h^{\alpha\beta} X_\beta) \;. \qquad (7)$$

Since the extra term is a pure trace, it will not affect the traceless $G_{\mu\nu}$, which remains identically null. Hence this Palatini model is even more undetermined than its second order form: not only is the metric left arbitrary, but so is Γ_μ. Note that the trace of (7),

$$h^{\mu\nu} R_{\mu\nu}(\Gamma) = \sqrt{-g}\; R(g) + \partial_\mu(h^{\mu\nu} X_\nu) \;, \qquad (8)$$

shows that the scalar curvature density differs from the metric Euler density by a divergence. The second term on the right is needed to cancel the Weyl dependence of the first, as is most easily seen in a conformally flat frame, $g_{\mu\nu} = e^{2\phi}\; \eta_{\mu\nu}$, in which the curvature depends only on Γ_μ:

$$h^{\mu\nu} R_{\mu\nu}(\Gamma) = -\partial^\mu \Gamma_\mu \qquad (9)$$

We now turn to the more interesting "Kaluza–Klein reduced" model [2] involving a Lagrange multiplier N. The second order, metric, theory $I = \int d^2x\ N\ h^{\mu\nu} R_{\mu\nu}(g)$ is no longer vacuous, since the metric is now determined through the $R(g) = 0$ equation, while N obeys $D_\mu \partial_\nu N = 0$. One immediate difference is that unlike in metric form, where a cosmological term is permitted in presence of N, the Palatini form still excludes both $\lambda N \sqrt{g}$ and $\lambda \sqrt{g}$ additions to $\mathcal{L}$: Weyl invariance again forces $\lambda = 0$ here just as it did in the $N = 1$ model. In our formulation, multiplying the $\mathcal{L}$ of (1) by N now implies

$$h^{\mu\nu} R_{\mu\nu}(\Gamma) = 0 \tag{10}$$

as well as (2), and (3) with $h^{\mu\nu}$ replaced by $N h^{\mu\nu}$ there. However, it is easy to see that N must be constant: The trace of the new (3) implies $D_\nu(N h^{\mu\nu}) = 0$, so (4) holds with $N h^{\mu\nu}$ replacing $h^{\mu\nu}$; its $(\mu\nu)$ trace reads

$$0 = g_{\mu\nu}\, D_\alpha(N h^{\mu\nu}) \equiv g_{\mu\nu} \partial_\alpha (N h^{\mu\nu}) \equiv N g_{\mu\nu} \partial_\alpha h^{\mu\nu} + 2\sqrt{-g}\, \partial_\alpha N \equiv 2\sqrt{-g}\, \partial_\alpha N\ . \tag{11}$$

Consequently, we may fall back on the previous results of the $N = 1$ model, except that now (10) is a field equation, *i.e.*, (8) vanishes. In conformal gauge, we see from (9) that Γ_μ is therefore divergenceless,

$$\Gamma_\mu = \epsilon_\mu{}^\nu \partial_\nu \chi\ , \tag{12}$$

but χ is still unrelated to the metric. Furthermore, we have lost any constraint on the metric, since it is only the affine curvature scalar (8) that vanishes, and that only contains Γ_μ as in (9), but not the metric: This is again the legacy of Weyl invariance, that it removes the one (conformal) variable in the metric tensor, leaving nothing else to be determined.

Presence of matter does not alter things dramatically. If it does not involve the connection explicitly, the matter's stress tensor defined according to $\delta I_{MATT}/\delta g_{\mu\nu}$ will vanish since $G_{\mu\nu}$ does. Although a (second quantized) spinor field action in 2D is actually connection-independent, higher rank tensors will involve it in general. This dependence will introduce the usual matter torsion, but not affect the metric indeterminacy of Γ_μ.

Very similar considerations hold when zweibeins $e_{\mu a}$/connections $\o_{\mu ab}$ are used instead of the metric. The Einstein Lagrangian here ($e \equiv |\det e_{\mu a}|$ and $e^{\mu a}$ is the usual inverse)

$$\begin{aligned} \mathcal{L} &= e e^{\mu\alpha} e^{\nu b} R_{\mu\nu ab}(\o) \;, \quad \o_{\mu ab} = -\o_{\mu ba} \\ R_{\mu\nu ab}(\o) &\equiv (\partial_\mu \o_{\nu ab} - \o_{\mu ac}\o_{\nu cb}) - (\nu\mu) \end{aligned} \tag{13}$$

is still Weyl-invariant at D=2, since it is homogeneous of order zero in the zweibeins, and also simplifies drastically, since we may write $\o_{\mu ab} \equiv \epsilon_{ab}\,\o_\mu$, thereby reducing $R_{\mu\nu ab}$ to the abelian form $\epsilon_{ab}(\partial_\mu \o_\nu - \partial_\nu \o_\mu)$ and $\mathcal{L}$ to the minimal expression

$$\mathcal{L} = 2\epsilon^{\mu\nu}\partial_\mu \o_\nu \;, \tag{14}$$

in terms of the (constant) Levi–Civita density $\epsilon^{\mu\nu}$. Thus, the first order theory does not involve the zweibein $e_{\mu a}$ at all, let alone determine $\o_\mu$ in terms of it. Indeed, there are no field equations at all here! The "K-K reduced" theory, multiplying $\mathcal{L}$ by N only implies that $\epsilon^{\mu\nu}\partial_\mu \o_\nu = 0$ and $N =$ const. Again, Weyl invariance requires that $\lambda = 0$ for either λe or $\lambda N e$ cosmological terms.

Our final "different in D=2" model is an old formulation of Einstein gravity, due to Eddington [3]. His proposal was to consider the purely affine Lagrangian

$$\mathcal{L}_E = (-\det\, R_{\mu\nu}(\Gamma))^{1/2} \;, \tag{15}$$

in terms of the symmetrized part of the $R_{\mu\nu}(\Gamma)$ in (1). Since $R_{\mu\nu}$ is a tensor, $\mathcal{L}$ is a scalar density and the field equations are tensorial, resembling (4):

$$D_\alpha (R^{\mu\nu}\sqrt{-\det R_{\mu\nu}}) = 0 \tag{16}$$

where $R^{\mu\nu}$ is the (assumed to exist) matrix inverse of $R_{\mu\nu}$. This follows simply from the fact that for any determinant, $\delta(\det\, R_{\mu\nu}) = R^{\mu\nu}\delta R_{\mu\nu}(\det\, R_{\mu\nu})$, and from the (symmetrized) Palatini identity

$\delta R_{\mu\nu} = D_\alpha\, \delta\, \Gamma^\alpha_{\mu\nu} - \frac{1}{2}(D_\mu\, \delta\Gamma_\nu + D_\nu \delta\Gamma_\mu)$. In general dimension, (16) means that $R^{\mu\nu}(\Gamma)\sqrt{-\det R_{\mu\nu}}$, and hence also $R_{\mu\nu}(\Gamma)$, is covariantly constant, *i.e.*, if we call $R_{\mu\nu}(\Gamma)$ by the name $g_{\mu\nu}$,

$$R_{\mu\nu}(\Gamma) = \lambda g_{\mu\nu}\ , \tag{17a}$$

then $g_{\mu\nu}$ is covariantly constant

$$D_\alpha(\Gamma) g_{\mu\nu} = 0\ . \tag{17b}$$

But these are of course the Einstein equations for the metric $g_{\mu\nu}$ with a cosmological term. This reasoning fails precisely at D=2 because $R^{\mu\nu}\sqrt{-\det R_{\mu\nu}}$ is unimodular (for any 2D symmetric tensor!) and so (as we have seen in detail earlier) there are not enough variables available in (16) to specify the full metric, *i.e.*, to imply (17). We are again reminded that a seemingly generic statement like (16) can degenerate in a particular dimension.

We have not investigated whether the "ultratopological" models discussed here might have interesting quantum consequences.

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References

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