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ULTRA–PLANCK SCATTERING IN D=3 GRAVITY THEORIES

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ABSTRACT

We obtain the high energy, small angle, 2-particle gravitational scattering amplitudes in topologically massive gravity (TMG) and its two non-dynamical constituents, Einstein and Chern–Simons gravity. We use 't Hooft's approach, formally equivalent to a leading order eikonal approximation: one of the particles is taken to scatter through the classical spacetime generated by the other, which is idealized to be lightlike. The required geometries are derived in all three models; in particular, we thereby provide the first explicit asymptotically flat solution generated by a localized source in TMG. In contrast to D=4, the metrics are not uniquely specified, at least by naive asymptotic requirements – an indeterminacy mirrored in the scattering amplitudes. The eikonal approach does provide a unique choice, however. We also discuss the discontinuities that arise upon taking the limits, at the level of the solutions, from TMG to its constituents, and compare with the analogous topologically massive vector gauge field models.

Introduction

The scattering of two gravitationally interacting Planck energy particles has been studied in recent years from several, quite different, points of view [1, 2, 3]. To leading order in the large s, small t expansion these calculations agree, and also coincide with those of the usual eikonal approximation obtained by summing a "leading" subset of Feynman diagrams (see, *e.g.*, [4] and references therein). The essence of the approximation is graphically expressed in 't Hooft's approach [2]: choose a frame in which one of the particles is essentially light-like and so generates an impulsive plane-fronted shock wave spacetime through which the other scatters as a (quantum mechanical) test body. It is quite straightforward to apply this method to other models, at least in principle: first calculate the spacetime generated by the rapid mover and then determine the evolution of the initial free state of the other particle in this (background) geometry. The scattering amplitude is of course just the overlap of the outgoing state with a given final-state free wave function. Because the geometry is impulsive, its field is "piece-wise" pure gauge – or equivalently a pure gauge but singular metric – and the scattering is essentially of Aharonov–Bohm type. More recently, the Verlindes [3] suggested a Lagrangian-based derivation of this approximation in D=4 gravity, using a scaling argument that reduces the strong-coupling sector to a topological field theory in which the above semi-classical dynamics is the lowest order effect. Although it is not clear how these arguments can be extended beyond the lowest order, they do show heuristically, at least, how to freeze out the gravitational quantum modes at the level of the action. Their ideas have been applied to D=3 Einstein gravity [5], and compared [6] with other approaches there. In all cases, the question of whether one thereby obtains the truly *dominant* contributions is still under discussion [1, 7].

It is clearly of interest, in order to further assess their validity, to test these ideas on other gravitational systems, including those in which there are no gravitons to be frozen out in the first place, but that are limiting cases of a dynamical theory. In this respect, D=3 models are especially useful, comprising as they do both of the nondynamical examples – Einstein gravity and conformally invariant pure Chern–Simons gravity (CSG) – in which there are no gravitons at all [8], as well as their sum, topologically massive gravity (TMG), which does have a (single massive helicity 2) dynamical excitation [8]. For each theory, we will first obtain the geometries generated by a null source (which we will call a photon for brevity) by showing that, just as in D=4, they are of shock wave form. In particular, we will obtain the metric for the full nonlinear TMG field equations. This is of interest in its own right, as it provides the first known asymptotically flat solution corresponding to a localized source in TMG; even the "Schwarzschild" solution is only known to linearized order there [9]. [In [10] it was shown that this complete plane wave geometry may in fact be obtained by an infinite boost from that linearized metric.] We will then exhibit the required scattering amplitudes and show that they contain intrinsic ambiguities peculiar to the global aspects of D=3 gravities; we propose a (heuristic) choice based on the eikonal prescription. It also turns out that the limits of TMG that yield its constituent actions can lead to singularities. We will analyze this question and compare with the analogous topologically massive vector gauge models.

Plane Wave Geometries

For definiteness, take the photon source to be right-moving along x, with energy E. The distinctive property of its energy-momentum tensor (in any D) is that it has only one non-vanishing component, $T_{uu} = E\delta(\boldsymbol{y})\delta(u)$, in the usual lightcone and transverse coordinates $(u, v, \boldsymbol{y}) \equiv (t - x,\ t + x, \boldsymbol{y})$. This motivates the following ansatz for the metric in harmonic gauge,

$$ds^2 = ds_0^2 + F(u, \boldsymbol{y})du^2 \ , \qquad ds_0^2 = -dudv + d\boldsymbol{y}^2 \ , \tag{1}$$

and indeed the Einstein tensor of (1) reduces to the simple linear form

$$G_{\mu\nu} = -\tfrac{1}{2}\nabla_T^2 F\, l_\mu l_\nu\,, \qquad l_\mu \equiv \partial_\mu u\,, \tag{2}$$

in terms of the transverse $D-2$ flat-space Laplacian. [Our conventions are: $R_{\mu\nu} \sim +\partial_\alpha \Gamma^\alpha_{\mu\nu}$, $\epsilon^{txy} = +1$.] Henceforth we specialize primarily to $D{=}3$, where we simply denote y-differentiation by a prime. Clearly, only the component G_{uu} (or equivalently only R_{uyuy}, since Riemann and Einstein tensors are mutual double duals in $D{=}3$) fails to vanish. In addition, the more complicated third-derivative (traceless, conserved, and symmetric) Cotton–Weyl tensor, $C^{\mu\nu} \equiv (-g)^{-1/2}\epsilon^{\mu\alpha\beta}\, D_\alpha(R^\nu_\beta - \frac{1}{4}\,\delta^\nu_\beta\, R)$, also simplifies nicely under this ansatz, becoming

$$C_{\mu\nu} = \tfrac{1}{2}F'''\, l_\mu l_\nu\ . \tag{3}$$

Consequently, the (parity violating) field equations of TMG,

$$G_{\mu\nu} + \tfrac{1}{\mu}\, C_{\mu\nu} = -\kappa^2\, T_{\mu\nu}\,, \tag{4}$$

reduce to a single third-order linear equation for the metric F:

$$F'' - \mu^{-1}\, F''' = 2\kappa^2 E\, \delta(y)\delta(u)\ . \tag{5}$$

Here κ^2 is the Einstein constant (with dimensions of inverse mass in $D{=}3$) and $|\mu|$ is the mass of the TMG excitation (our convention is $\mu > 0$). [Note the parity violation of TMG, as reflected in the appearance of the orientation $\epsilon_{\mu\nu\rho}$ in the equations of motion; changing the sign of μ is equivalent to performing a parity transformation.] We recall for future reference that the sign of κ^2 here is necessarily opposite to that usually taken in Einstein theory in order that the TMG excitations be non-ghost [8].

The other two gravity models discussed in the introduction are contained as limits of TMG, whose field equations (4) reduce to the (ghost sign) Einstein equations as $\mu \to \infty$, while those of pure Chern–Simons gravity ($C_{\mu\nu} = -\mu\kappa^2 T_{\mu\nu}$) are reached as $\mu \to 0$, $\kappa^2 \to \infty$, keeping $\tilde{\mu} \equiv \mu\kappa^2$ fixed. We will see, however, that taking these limits in the explicit solutions of TMG can be more delicate.

The curvature is easily obtained by one integration of (5),

$$G_{uu} = -\tfrac{1}{2} F'' = -\mu\kappa^2 E e^{\mu y}\theta(-y)\delta(u) , \tag{6}$$

upon choosing the homogeneous solution for F'' so as to obtain asymptotic flatness in y, *i.e.*, excluding any source-free graviton part of the curvature. Integrating (6) then yields the metric F, which we write as

$$F = 2E\Delta_T(y)\delta(u) , \tag{7}$$

in terms of a Green's function Δ_T for the transverse kinetic operator of (5),

$$\Delta_T(y) = \kappa^2(D^2 - \mu^{-1}D^3)^{-1}\delta(y) = \frac{\kappa^2}{\mu}[\frac{1}{\mu - D} + \frac{\mu}{D^2} + \frac{1}{D}]\delta(y) , \qquad D \equiv d/dy . \tag{8}$$

Having specified (6) by imposing asymptotic flatness, the most general homogeneous solution H that we may add to Δ_T is that of $H'' = 0$, namely $H = Ay + B$. In the following we shall take

$$\Delta_T(y) = \Delta_T^{\rm TMG}(y) = \kappa^2[\theta(-y)\mu^{-1}e^{\mu y} + \epsilon(y)\tfrac{1}{2}(y + \mu^{-1})] , \tag{9}$$

but we will return to the significance of this particular choice, and to the role of the H ambiguity. The geometry given by (7) and (9) is the advertised solution of the full nonlinear TMG field equations due to a localized (photon) source; as we saw, it is correspondingly flat (though the metric is not manifestly cartesian) at infinity and nonsingular away from the photon.

Let us consider the properties of the curvature (6) and the relevant part, (7), of the metric in our various models, with the given choice of homogeneous solutions. For TMG itself, the curvature fails to vanish only on the $y < 0$ half of the $u = 0$ null plane, while the metric has been permitted to be non-cartesian also at $y > 0$. In Einstein theory, which is (5) at $\mu = \infty$, flatness reigns everywhere outside the source. If we take the $\mu \to \infty$ limit of (6), it agrees since $\mu e^{\mu y}\theta(-y) \to \delta(y)$ (for smooth enough test functions); likewise (9) limits to the y-symmetric form $\Delta_T = \kappa^2|y|/2$. Although F is neither in pure gauge nor in asymptotically Cartesian form, we shall see that it can be set to zero locally by (singular) coordinates choices; globally a conical space structure is unavoidably present and will be quite relevant to our scattering problem.

The other limit discussed above gives pure Chern-Simons gravity (CSG), to which only a traceless $T^{\mu\nu}$ – such as that of our photon – may couple, since $C^{\mu\nu}$ is identically traceless. Also, since $C^{\mu\nu}$ is in fact the Weyl tensor in D=3, the CSG metric is only determined up to a conformal factor and spacetime is now only conformally flat outside the sources. The field equation now reduces to

$$F''' = -2\tilde{\mu}E\delta(y)\delta(u)\,, \tag{10}$$

which integrates to

$$F'' = -\tilde{\mu}E\epsilon(y)\delta(u)\ , \quad F = -\tfrac{1}{2}\tilde{\mu}Ey^2\epsilon(y)\delta(u) \tag{11}$$

up to a constant in F'' and a homogeneous $Cy^2 + Ay + B$ solution (here $H''' = 0$) in F. Note that no choice of H can make F'' asymptotically flat (let alone flat for $y \neq 0$) in y; we have taken the "most symmetric" option in (11), but F cannot in any case be turned into a pure gauge. Note also that the indeterminacy of the curvature is just the conformal factor ambiguity; the resulting Cy^2 term in F is an ambiguity over and above the effect of the homogeneous solution $Ay + B$ common to all three models. If we take the CSG limits of (6) and (9) in TMG, we find that they diverge, differing from (11) by homogeneous solutions to be sure, but with infinite coefficients. Quite clearly this could have been avoided by choosing for (9) the form $\Delta_T = \theta(-y)\kappa^2\mu^{-1}(e^{\mu y} - 1 - \mu y)$, *i.e.*, by adding $H = -\frac{1}{2}\kappa^2(y + \mu^{-1})$ to (9) (incidentally, the Einstein limit would then still be finite and in fact would be cartesian at positive y; in particular, it is possible to choose a solution with finite limit in both directions!). It may seem perverse to make the choice (9) rather than the non-singular one; we have done so because it corresponds to using the Feynman propagator for the exchanged "graviton" in the eikonal perturbation theory calculation. [The boundary condition in Δ_T is determined by the "$i\epsilon$" prescription chosen for the covariant propagator $D(x)$ integrated against the photon stress tensor. Thus, in Einstein gravity, use of D_{ret} leads to $\Delta_T \sim y\theta(-y)/2$ instead of the D_F value $y\epsilon(y)/2 = |y|/2$.] We will return to this point after demonstrating the physical consequences of the ambiguity. Suffice it to say here that (9) represents a uniform choice for all three models treated separately; this will make any H-dependence in physical quantities even more striking.

Scattering

Just like its D=4 Einstein counterpart [11], the TMG solution (9) takes the form of an impulsive plane-fronted wave, so we can apply the analysis of [2] to study the small angle scattering of a particle in the background generated by the photon. To make this shock-wave character more explicit, we first perform the coordinate transformation $v \to v + 2E\Delta_T(y)\theta(u)$, after which the interval takes the form

$$ds^2 = ds_0^2 - 2E\,\theta(u)\Delta_T'(y)dy\,du\,. \tag{12}$$

That the metric is Minkowski for $u \neq 0$ is already manifest for $u < 0$ in the coordinates of (12); for $u > 0$ we write (12) as

$$\begin{aligned} ds^2 &= -du\,d(v + 2\,E\,\Delta_T(y)) + dy^2 \\ &\equiv -du\,dv_> + dy^2\,. \end{aligned} \tag{13}$$

Thus the effect of the impulse is entirely summarized in the relation between the $x^\mu_<$ and $x^\mu_>$

coordinates on the null $u-$plane, namely by

$$u_< = 0 = u_> \ , \quad y_< = y_> \ , \quad v_< = v_> - 2E\,\Delta_T(y_>) \ . \tag{14}$$

Before the impulse, then, the test particle sees no field and the incident wavefunction can be taken to be a plane wave with (on-shell) momentum p_μ,

$$\psi_< = \frac{1}{(2\pi)^{3/2}} e^{i\,p\cdot x_<} \ . \tag{15}$$

After the impulse we have simply (still at $u = 0$)

$$\psi_> = \frac{1}{(2\pi)^{3/2}} \exp i[p_v(v - 2E\Delta_T(y)) + p_y y] \ . \tag{16}$$

Decomposing (16) into plane waves in the out-region, we find for the scattering amplitude, $< k, \text{out}|p, \text{in} >= \delta(p_v - k_v)T(q_y = p_y - k_y)$, the expression

$$T(q_y) = \int \frac{dy}{(2\pi)^2} \exp i[q_y y - 2p_v\,E\Delta_T(y)] \ . \tag{17}$$

For the kinematics discussed here, the Mandelstam variables are $s = 4|p_v|E$ and $t = -q_y^2$ since the momentum transfer is $q_y = p_y - k_y$. Thus the scattering amplitude is the usual "leading" eikonal expression [4, 7], *i.e.*, the transverse Fourier transform of the exponentiated "Coulomb" potential. The amplitude (17) can also be understood in terms of an equivalent completeness argument. Specifically, let $G(x_1; x_2)$ be the propagator in flat spacetime (x_1^μ and x_2^μ are spacetime $3-$vectors). By completeness, the propagator from $(v_1,\, y_1)$ on some initial $u < 0$ slice to $(v_2,\, y_2)$ on some final $u > 0$ slice is given by

$$G(x_1; x_2) \quad = \quad \int_{u=0} dv dy\, G(x_1; u_<, v_<, y_<) G(u_>, v_>, y_>; x_2) \ , \tag{18}$$

where the integral over v and y is on the intermediate $u_< = u_> = u = 0$ slice. Taking the Fourier transform with incoming and outgoing momenta p_μ and k_μ, yields

$$G(p, k) \quad = \quad \frac{1}{(2\pi)^3} \int dx_1\, e^{ip\cdot x_1} \int dx_2\, e^{-ik\cdot x_2} \int_{u=0} dv dy G(x_1; x_<) G(x_>; x_2)$$

$$= \quad G(p)G(k) \int dvdy\, e^{(ip\cdot x_< - ik\cdot x_>)} \ , \tag{19}$$

Amputating the external propagators $G(p)G(k)$ then gives (by the reduction formula) the scattering amplitude. The result is

$$\begin{aligned} < k, \mathrm{out}|p, \mathrm{in} > \quad &= \quad \frac{1}{(2\pi)^3} \int dvdy \exp\left(ip_v(v - 2E\Delta_T(y)) + ip_y y - ik_v v - ik_y y\right) \\ &= \quad \delta(p_v - k_v) \int \frac{dy}{(2\pi)^2} e^{(iq_y y - 2iEp_v \Delta_T(y))} \ , \end{aligned} \tag{20}$$

thereby reproducing (17).

The (TMG) Green's function Δ_T^{TMG} is, as we have seen, only determined up to homogeneous solutions $H = Ay + B$. Examining the amplitude (17), it is clear that for real s the ambiguity generated by B simply amounts to an irrelevant constant phase in the amplitude. One also sees that adding Ay to the Green's function is equivalent to transforming the incoming momentum, p_μ, via $p_y \to p_y - \tilde{A}p_v$, $p_v \to p_v$ where $\tilde{A} \equiv 2EA$. This is a Lorentz transformation provided p_u is transformed as well, according to

$$p_u = \frac{p_y^2}{4p_v} \to \frac{(p_y - \tilde{A}p_v)^2}{4p_v} = p_u - \tfrac{1}{2}\tilde{A}p_y + \tfrac{1}{4}\tilde{A}^2 p_v \ . \tag{21}$$

The photon's momentum is obviously left invariant under this transformation. Hence, the ambiguity associated with A corresponds to applying a Lorentz transformation in the entire in-region (*i.e.,* on both incoming particles). [This result can also be seen directly from the original metric (7) and (9), where adding a linear term to the Green's function is equivalent to the coordinate transformation

$$u \to u \ , \qquad v \to v + \theta(-u)(\tilde{A}y + \tfrac{1}{4}\tilde{A}^2 u + B) \ , \qquad y \to y + \tfrac{1}{2}\tilde{A}u\theta(-u) \ , \tag{22}$$

corresponding to a Lorentz transformation in the $u < 0$ halfspace.] Clearly, performing a Lorentz transformation only in the in-region does *not* in general leave the S-matrix invariant. Therefore, different choices of A yield different scattering amplitudes. This indeterminacy is special to D=3, in contrast to D=4, where a unique choice is picked out by demanding that the metric be asymptotically Cartesian. There, in the impulsive plane wave metric [11], $\Delta_T(\boldsymbol{y})$ is the two-dimensional transverse Coulomb Green's function:

$$\Delta_T(\boldsymbol{y}) \sim \log \boldsymbol{y}^2 + H(\boldsymbol{y}) \ , \ \ \nabla_T^2 \, H = 0 \, . \tag{23}$$

Performing a coordinate transformation, we obtain the analog of (12):

$$ds^2 = ds_0^2 + 2\kappa^2 E\theta(u) du \, d\Delta_T \ , \tag{24}$$

and $(u, v, \boldsymbol{y})$ are asymptotically Cartesian coordinates provided $\boldsymbol{\nabla} H$ vanishes at large distances, which requires that H be asymptotically constant. Being harmonic, it must then be a constant everywhere, and so merely corresponds to an irrelevant overall phase. In contrast, for D=3 we have seen that no choice of homogeneous solution will yield an asymptotically Cartesian metric; this clearly derives from the conical nature of the exterior spatial geometry and the absence of the corresponding Killing vectors even asymptotically [12]. [We can restate the above geometric discussion in terms of holonomy: the photon's worldline divides the $u = 0$ null hyperplane into the $y < 0$ and $y > 0$ halfplanes. The pure gravity solution is obtained geometrically by making a cut along $u = 0$ and then reidentifying $u = 0^-$ and $u = 0^+$. Points with $y < 0$ ($y > 0$) are reidentified with a Lorentz transformation denoted $\mathcal{L}_<$ ($\mathcal{L}_>$). The holonomy matrix associated with parallel transport around the source is then given by $\mathcal{L} = \mathcal{L}_< \mathcal{L}_>^{-1}$, where the first (second) factor is associated with crossing from (to) $u < 0$ to (from) $u > 0$ on the $y > 0$ ($y < 0$) side. The coordinate transformation (22) corresponds in the geometric picture to the residual freedom associated with the holonomy-preserving transformation $\mathcal{L}_< \to \mathcal{L}_< \tilde{\mathcal{L}}$, $\mathcal{L}_> \to \mathcal{L}_> \tilde{\mathcal{L}}$.]

A (pragmatic) resolution of this ambiguity follows if we are able (on other grounds) to find and justify appropriate boundary conditions to fix the transverse Green's function. We have just seen that there is no compelling choice within the framework of [2]. However the boundary conditions *are* determined if we go back to the derivation of these same results from the eikonal approximation. Indeed, as we will now show, the latter suggests that the "correct" transverse Green's function is to be determined from the Feynman propagator for the exchanged graviton. Recall that the eikonal approximation in field theory involves summing the contributions of generalized ladder (of exchanged gauge boson) diagrams to the $2 \to 2$ particle amplitude, keeping only the leading "hard momentum" terms in the intermediate particle propagators. This is neatly done using functional techniques: take two copies of the linearized $1 \to 1$ particle amplitude in an external field (graphically just the sum of all numbers of graviton lines emitted from a single particle line), and sew together the exchanged gravitons in all possible ways using their Feynman propagator, $(\Delta_F)_{\mu\alpha,\nu\beta}$. For the corresponding connected particle Green's functions this can be summarized as

$$G_4(x_2', x_2; x_1', x_1) = \int [dh_1][dh_2] G_2(x_2', x_2|h_2) G_2(x_1', x_1|h_1) \exp \tfrac{1}{2} \textstyle\int h_1^{\mu\nu} \Delta^{-1}_{\mu\alpha,\nu\beta} h_2^{\alpha\beta} \, . \qquad (25)$$

Just the linearized theory is required [4], both in the functional integral over h and in calculating the particle Green's functions; the former since we sew with the free propagator, and the latter because – consistent with keeping only hard internal momenta – ladders are built with the 3-point vertex alone. [Strictly speaking, this would seem to conflict with (linearized) gauge invariance, but in our approximation, matter is essentially a fixed external conserved source, so the problem is avoided.] The problem is now to determine the 1-particle propagator in a background linearized gravitational field, and thus the remaining analysis reduces to that given for QED in [13, 14] (and our presentation will be essentially equivalent to that in [4]). Following Schwinger ([15], see also [13, 14]) we introduce formal "position and momentum observables", (X, P), for the fast particle; then the single particle sector is conveniently described by the Hamiltonian $H(X, P) = H_0 + V$; where $H_0 = P^2$ and the potential is given (in harmonic gauge) by $V = h_{\mu\nu}(x) P^\mu P^\nu$. Thus $G_2 = (H_0 + V)^{-1}$, and the 2-particle T-matrix is obtained as the on-shell limit of $\mathcal{T}_2 = H_0(G_2 - H_0^{-1})H_0$. Proceeding under the approximation that the high energy particles' momenta are essentially unchanged in the collision process, we may replace (working with particle 1 say) $P \to p = (p_1 + p_1')/2$ in the resulting T-matrix. The usual exponentiation of the Born series then yields the eikonal approximation

$$< p_1'|\mathcal{T}^{(E)}|p_1 >= \int \frac{d^3x}{(2\pi)^3} e^{i(p_1-p_1')\cdot x}\, i\frac{d}{d\alpha}\, \exp\left[-\tfrac{i}{E}\int_{\alpha E}^{\infty} d\tau V(x + 2\tfrac{p}{E}\tau, p)\right]|_{\alpha=0}\,. \tag{26}$$

At this point it is useful to introduce new coordinates. Consider the same kinematics as before: the fast particle massless, with energy E. We can write, in the eikonal approximation, the null vector p as $p = E(1, \hat{\boldsymbol{p}})$, and introduce the vector $n = E(1, -\hat{\boldsymbol{p}})$ such that $n \cdot p = -2E^2$. Define $x^\mu = z^\mu + \frac{p^\mu}{E}\sigma$, where $n \cdot z = 0$. In particular, then, $(p_1 - p_1') \cdot x = q \cdot z$. Moreover, σ can be shifted into the domain of the integral in the exponent, so that the α-derivative may be exchanged for a σ-derivative. Putting this together, one easily finds that (26) simplifies to

$$< p_1'|\mathcal{T}^{(E)}|p_1 >= 2iE \int \frac{d^2z}{(2\pi)^3} e^{iq\cdot z} e^{-2\frac{i}{E}\int_{-\infty}^{\infty} d\tau h_{\mu\nu}(z + 2\frac{p}{E}\tau)p^\mu p^\nu}\,. \tag{27}$$

Having made the eikonal approximation for particle 1, we may simply substitute (27) into (25) (reduced on-shell), to obtain the result for the $2 \to 2$ particle scattering amplitude. In fact, the h-integrals apply a functional Taylor series expansion, and one finds

$$\mathcal{T}_4(p_1', p_1; p_2', p_2) = 2iE \int \frac{d^2z}{(2\pi)^3} e^{iq\cdot z} \mathcal{T}_2(p_2', p_2|h^C)\,, \tag{28}$$

where

$$h^C_{\mu\nu} = \int d^3y (\Delta_F)_{\mu\alpha,\nu\beta}(x-y) T^{\alpha\beta}(y) \tag{29}$$

is the classical solution for the fast particle's energy-momentum source $T_{\mu\nu}(y) = \frac{1}{2E}p_\mu p_\nu \int d\tau \delta^3(y - z - \frac{p}{E}\tau)$. This is precisely of the expected form; in particular the appropriate boundary conditions are clearly fixed (through the choice of Δ_F) *ab initio*. [Note the additional integral over particle 1's “initial” point, which simply incorporates translational invariance and allows energy-momentum conserving delta-functions to be factored out.]

Having “justified” the use of the Feynman propagator’s boundary conditions to define our $\Delta_T(y)$ of (9), we now evaluate the resulting scattering amplitudes, the simplest being that for Einstein gravity, where the Feynman propagator reduces to the space-symmetric form $\Delta_T^{\rm EG}(y) = \frac{1}{2}\kappa^2|y|$. The integral (17) splits into two terms which are easily calculated, and yield

$$T^{\rm EG}(q_y, s) = \frac{i}{(2\pi)^2}\left(\frac{1}{q_y + \kappa^2 s/4} - \frac{1}{q_y - \kappa^2 s/4}\right) \sim \frac{i}{2\pi^2}\,\frac{\kappa^2 s/4}{t + \kappa^4 s^2/16}\ . \tag{30}$$

In this theory, the amplitude for the scattering of a test particle in the field due to another particle has been calculated exactly [16], and at small angles is just given by the result (30) (see also [17] for a discussion of the full two-particle scattering amplitude). One may well expect this since the geometry (with the conventional sign of κ^2 used in (30)) due to a particle in this theory is flat with a conical singularity at the particle. The two terms in (30) correspond (roughly) to the test body’s passing on one side or the other of the photon’s cone. For, note that in a suitably defined center of mass system [17, 6] $\sqrt{-t}$ is proportional to the scattering angle, θ, and $\kappa\sqrt{s}/2$ to the opening angle of the effective cone. Thus $\theta_{cl} = \pm\kappa\sqrt{s}/2$ are the classical possibilities for the angle of scattering from this cone, and (30) just says that the eikonal amplitude is dominated by $\theta \sim \theta_{cl}$.

It is interesting to pursue further the apparent relation of the above result to Aharonov–Bohm (A-B) scattering. In particular, let us consider the system which certainly does represent precisely that physical situation – point particles coupled to an abelian vector Chern–Simons (CSE) action. It is well known that the constraints in this model imply that charged particles also carry a magnetic flux proportional to the charge, and that this “dressing” is essentially the only effect of the gauge field. Thus, for example, adiabatic transport of one particle around another picks up an A-B phase which contributes to the statistics; these particles are anyons. The two-particle scattering may be analyzed exactly of course, but we may also follow the procedure given above step by step. In fact for later convenience we will consider the vector system directly analogous to TMG, namely topologically massive electrodynamics (TME) [8, 18], for which

$$\mathcal{L} = -\tfrac{1}{4}\, e^2 F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}\hat{\mu}^{-1} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + J^\mu A_\mu \,, \tag{31}$$

$(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$, with field equations

$$e^2\, \partial_\nu F^{\mu\nu} - \hat{\mu}^{-1} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = J^\mu \,. \tag{32}$$

Note that e^2 has dimensions of mass in D=3, and $\hat{\mu}$ is dimensionless. The CSE theory is obtained in the limit $e^2 \to \infty$, while Maxwell electrodynamics is the limit $\hat{\mu} \to \infty$. As for gravity, we introduce the transverse Green's function

$$\left[\frac{1}{e^2} \frac{d^2}{dy^2} - \hat{\mu}^{-1} \frac{d}{dy} \right] \Delta_T^{\rm TME}(y) = -\delta(y) \,, \tag{33}$$

and the scattering amplitude is given by (17), but without the s in the exponent (due to the lower spin of the virtual exchanged boson). The Feynman propagator (in Landau gauge) is just

$$(\Delta_F^{\rm TME})_\mu^{\mu'} = \frac{e^2}{k^2 + \hat{\mu}^{-2} e^4 - i\epsilon} \left[P_\mu^{\mu'} + i\hat{\mu}^{-1} e^2 \epsilon_\mu{}^{\alpha\mu'} \frac{k_\alpha}{k^2 - i\epsilon} \right] + e^2 \frac{k_\mu k^{\mu'}}{k^4} \,, \quad P_\mu^{\mu'} \equiv \delta_\mu^{\mu'} - k_\mu k^{\mu'} k^{-2}. \tag{34}$$

Integrating against the fast particle current source we find, in coordinate space, the particular solution

$$\Delta_T^{\rm TME}(y) = \frac{\hat{\mu}}{2} \epsilon(y) + \hat{\mu} e^{\hat{\mu}^{-1} e^2 y} \theta(-y) \,. \tag{35}$$

The CSE limit of (35) is smooth and the last term vanishes; the calculation of the amplitude is trivial, yielding the usual A–B result

$$T^{\rm CSE}(q_y) \sim \frac{1}{q_y} (e^{i\hat{\mu}/2} - e^{-i\hat{\mu}/2}) \,, \tag{36}$$

where q_y is the momentum transfer. Note that $T^{\rm CSE}$ consists of two terms, which fail to cancel precisely because of the A-B phase, and has the loose interpretation that it sums the contributions from the particle's passing "on one side or the other" – it is here that the analogy goes through in (30). The homogeneous solution ambiguities are quite different in electrodynamics and in gravity, however. For CSE, the only freedom is in a constant term in Δ_T, which is of course irrelevant. In pure Maxwell theory, the homogeneous term Ay is also permitted, and does lead to a different amplitude; however, because physics is governed by the field strength, which (unlike the curvature) is a first derivative, including this term actually corresponds to a different physical situation, in which an additional source-free field is present to scatter the charge, so that there is no ambiguity in the physics here.

There is another lesson to be learned from TME: whereas the limit of the solution with Feynman propagator boundary conditions to the (lower-derivative) CSE is smooth, that in the other, QED, direction encounters a singularity. Expanding (35), we see that as $\hat{\mu} \to \infty$, Δ_T tends to $\hat{\mu}/2 + e^2 y\theta(-y)$. This is the expected QED result $-\frac{1}{2}e^2|y|$, but only up to the homogeneous solution $H = \frac{1}{2}e^2 y + \hat{\mu}/2$. As is clear from (17), the divergent constant term appears in the amplitude as an irrelevant overall phase. However the linear term is curious given the ambiguity discussed above. It is important to note that the behaviour of the amplitude in the limits is strongly dependent on the infrared structure of the model. For example, if we introduce an infrared regulating mass for the vector field, we find

$$\Delta_T^M(y) = \frac{e^2}{\lambda} e^{-\hat{\mu}^{-1}e^2 y/2} e^{-\lambda|y|/2}\,, \quad \lambda \equiv \sqrt{\hat{\mu}^{-2}e^4 + 4M^2}\,. \tag{37}$$

The infrared dependence is strikingly obvious from the fact that limit $M \to 0$ does not reproduce (35). However, by keeping $M \neq 0$ we find: $\hat{\mu} \to \infty$ limits to the correspondingly mass-regulated Green's function in QED, $e^2 e^{-M|y|}/(2M)$; but for $e^2 \to \infty$ the result differs from the pure CSE one, $\hat{\mu}\epsilon(y)/2$, albeit only again by the "infrared phase" $\hat{\mu}/2$. Thus, although the model is, strictly speaking, infrared regular – as is to be expected since the excitation of TME is massive – this discussion shows that the infrared structure may still be very subtle. One obtains the flavour of what is going on directly from the lowest order perturbative calculation of one vector exchange, using the eikonal kinematics $p_1 \sim p_1'$, $p_1^u \sim p_2^v \sim 0$, $q^u \sim q^v \sim 0$. Then $p_1.p_2 \sim s$, while $\epsilon^{\mu\alpha\nu}(p_1)_\mu q_\alpha (p_2)_\nu \sim q_y s$, and thus by contracting (34) against the conserved currents (*i.e.* $p_1^\mu (p_2)_{\mu'}$) we find that in the dominant (t-channel) contribution the odd-parity "CS" contribution is enhanced by the relative factor of $q_y/(q_y^2 - i\epsilon)$ at small angles. However, for the mass-regulated propagator precisely the opposite is true, since then the factor becomes $q_y/(q_y^2 + M^2)$ and this term is suppressed at small angles. A little more work shows that such observations follow also for the eikonal sum. Using (35) and (17), and the following representation of the incomplete gamma function

$$\int dx \exp[-\alpha x - \beta e^{-x}] = \beta^{-\alpha}\gamma(\alpha, \beta)\,, \tag{38}$$

we find the TME amplitude

$$T^{\rm TME} \sim \frac{e^{\hat{\mu}/2}}{q_y} + \frac{e^{-\hat{\mu}/2}}{q_y}\beta^{-\alpha}\alpha\gamma(\alpha, \beta)\,, \tag{39}$$

where $\alpha = iq_y\hat{\mu}/e^2$ and $\beta = i\hat{\mu}$. The small angle (small q_y) expansion is then precisely the large e^2 expansion in the second term (to make this quite obvious the identity $\alpha\gamma(\alpha, \beta) = \gamma(\alpha{+}1, \beta) + \beta^\alpha e^{-\beta}$ is useful).

Returning to our scattering problem, the amplitude (17) (with (9)) of the dynamical TMG theory has a similar complicated expression in terms of the incomplete gamma function,

$$T^{\rm TMG}(q_y) = \frac{i}{(2\pi)^2}\frac{1}{q_y - \kappa^2 s/4}e^{-\beta} + \frac{1}{(2\pi)^2}\frac{1}{\mu}\,(2\beta)^{-\alpha}\gamma(\alpha, 2\beta)e^{\beta}\,, \tag{40}$$

where

$$\alpha = \frac{i}{\mu}(q_y + \kappa^2 s/4)\,, \quad \beta = i\frac{\kappa^2}{4\mu}s\,. \tag{41}$$

For large μ, the TMG amplitude can be expanded in powers of $1/\mu$ to provide Chern–Simons corrections to pure gravity, the leading one being

$$T^{\rm TMG}(q_y,s)=\frac{i}{(2\pi)^2}\left(\frac{\eta_+}{q_y+\kappa^2 s/4}-\frac{\eta_-}{q_y-\kappa^2 s/4}\right)+\mathcal{O}(\mu^{-2}) \tag{42}$$

where $\eta_\pm = 1\pm\beta+\mathcal{O}(\frac{1}{\mu^2})$. The effect of this first correction is to modify the otherwise equal coefficients of the two A–B contributions in (30), as is to be expected from its parity-violating character; note also that the expansion brings in rising powers of s.

We can also obtain the amplitude in the CSG model, subject to the caveat given earlier that the spacetime of (11) is only fixed up to a conformal factor; if the scattered particle is also nearly null (in the same Lorentz frame), this ambiguity becomes irrelevant. We omit the details, but with the choice (11) and (17), the amplitude may be written in terms of the error function, ${\rm erf}(x)\equiv(2/\sqrt{\pi})\int_0^x dz e^{-z^2}$, *i.e.*

$$T^{\rm CSG}(q_y,s)=(2\pi/s\tilde\mu)^{1/2}[e^{i(\xi+\pi/4)}\{1-{\rm erf}((-i\xi)^{1/2})\}+{\rm c.c.}]\ ,\quad \xi\equiv 4q_y^2/(s\tilde\mu)\ . \tag{43}$$

[It is possibly worth noting that to lowest order (43) can be put in the form (42) but with different $\eta_\pm$, and with $\kappa^2 s\to 2\sqrt{\tilde\mu s}$, although the significance of this remark is not clear.]

We are left with the question of limits of TMG solutions. As in TME, the limit to the lower derivative (here Einstein gravity) model is smooth, as can be seen by considering the respective Δ_T's in the two theories. In fact, the full TMG Feynman propagator [8] in harmonic gauge reads

$$\begin{aligned}(\Delta_F^{\rm TMG})_{\mu\nu}^{\mu'\nu'}(k) \;=\; & \frac{\kappa^2}{2}\frac{\mu^2}{k^2+\mu^2-i\epsilon}\left[-\left(\delta_\mu^{(\mu'}\delta_\nu^{\nu')}-2\eta_{\mu\nu}\eta^{\mu'\nu'}\right)\frac{1}{k^2-i\epsilon}-\frac{1}{\mu^2}\left(P_{\mu\nu}P^{\mu'\nu'}-2\eta_{\mu\nu}P^{\mu'\nu'}\right)\right. \\ & \left.+\frac{i}{2\mu}\frac{k_\alpha}{k^2}\epsilon_{(\mu}^{\ \ \alpha(\mu'}\delta_{\nu)}^{\nu')}+\frac{1}{k^2+4\mu^2}\left(\frac{i}{2\mu}\frac{k_\alpha}{k^2}k_{(\mu}\epsilon_{\nu)}^{\ \ \alpha(\mu'}k^{\nu')}-\frac{1}{k^2}k_{(\mu}P_{\nu)}^{\ (\mu'}k^{\nu')}\right)\right]\ ,\end{aligned} \tag{44}$$

where the round brackets denote (un-normalized) symmetrization. This clearly tends smoothly to the Einstein result (in the same gauge),

$$(\Delta_F^{\rm EG})_{\mu\nu}^{\mu'\nu'}(k) = -\frac{\kappa^2}{2}\left(\delta_\mu^{(\mu'}\delta_\nu^{\nu')} - 2\eta_{\mu\nu}\eta^{\mu'\nu'}\right)\frac{1}{k^2 - i\epsilon} \tag{45}$$

as $\mu \to \infty$.

The other limit—to CSG—of TMG solutions is, however, not smooth at any level. Indeed we have already seen that in this case the classical solution only has a good limit if divergent homogeneous terms in the transverse propagator are dropped. The situation here is even worse than in the TME case, where only the constant part of the homogeneous solution diverged, and thus the divergence is absorbed in the usual exponentiated phase. Even then the limiting result differs from the QED result there due to the remainder of the homogeneous solution (clearly a drastic modification of the large y behaviour), and it is *this* which is changed by a regularization of the infrared behaviour. The perturbative arguments which illuminated what was happening there may also be applied for TMG, although in fact they do not successfully carry over to the full eikonal sum. Indeed for TMG, as is evident from the full Feynman propagator, we may trace the additional singularity to the fact that – at tree level – CSG has an additional gauge invariance corresponding to Weyl rescaling of the metric. It is precisely the fate of this invariance when coupled to interacting matter theories which makes the quantization of CSG an interesting problem. Typically we should expect no subtleties in D=3. One approach would be to maintain manifest diffeomorphism invariance by obtaining the model as the limit of TMG, and yet we have just seen some hints that this limit may be problematic. This should reward further study.

Summary

We have obtained the "leading order" two-particle gravitational scattering amplitudes for three quite different gravity models in D=3 in the high s, small t regime using the semi-classical approximation of [2], which is formally equivalent to that of the usual leading order eikonal expansion. However, it is peculiar to D=3 that asymptotic flatness is not sufficient to determine the Green's function completely. We have argued heuristically in this respect that the eikonal treatment gives a unique result, dictated by the use of the Feynman propagator. There is no such ambiguity in D=4, where *global* asymptotically Cartesian coordinates can always be introduced. The resulting amplitude for D=3 Einstein gravity was, as expected, of A–B form. For full TMG, we also obtained the scattering amplitude but found no dramatic properties; it is not A–B and indeed its corrections in an expansion about the Einstein value "dephase" the two characteristic A-B pieces of the latter. We also discussed the discontinuous aspects of the limiting process from TMG to CSG at the level of solutions, and compared it with a similar phenomenon in vector gauge theory. At the classical level, the impulsive plane wave spacetimes generated by a null source that were obtained in TMG provide the first explicit solution to full TMG due to a localized source.

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