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Hall viscosity and angular momentum in gapless holographic models

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We use the holographic approach to compare the Hall viscosity η_H and the angular momentum density $\mathcal{J}$ in gapless systems in $2+1$ dimensions at finite temperature. We start with a conformal fixed point and turn on a perturbation which breaks the parity and time-reversal symmetries via gauge and gravitational Chern-Simons couplings in the bulk. While the ratio of η_H and $\mathcal{J}$ shows some universal properties when the perturbation is slightly relevant, we find that the two quantities behave differently in general. In particular, η_H depends only on infrared physics, while $\mathcal{J}$ receives contributions from degrees of freedom at all scales.

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I. INTRODUCTION

When parity and time-reversal symmetries are broken, new macroscopic phenomena can emerge. For example, static systems can have nonzero angular momenta [1–4], and the viscosity of energy-momentum transport can have an "odd" part (Hall viscosity) analogous to Hall conductivity [5]. In $(2+1)$ dimensions, these phenomena are of particular interest, as they can occur in rotationally invariant systems.

The generation of angular momentum and of Hall viscosity are in principle controlled by very different physics, as we discuss in detail below; the former is concerned with equilibrium thermodynamics, while the latter with transport. For *gapped* systems at zero temperature, however, there exists a general argument that the two are closely related [6,7] (see also Refs. [8–14]). In this case, the linear response of the stress tensor to an external metric perturbation can be described by a Berry phase, which in turn can be related to angular momentum. For gapless systems such an argument does not apply, and it is of interest to explore whether relations could exist between the two quantities. Here we can take advantage of the holographic duality, which provides a large number of strongly coupled, yet solvable, gapless systems which are otherwise hard to come by.

In our previous papers [15,16] (see also Ref. [17]) we examined a most general class of $(2+1)$-dimensional relativistic field theories whose gravity description in AdS_4 contains axionic couplings of scalar fields to gauge or gravitational degrees of freedom, i.e.

$$\int \vartheta_1 F \wedge F \tag{1.1}$$

or

$$\int \vartheta_2 R \wedge R, \tag{1.2}$$

where R is the Riemann curvature two-form, and F is the field strength for a bulk gauge field dual to a boundary $U(1)$ global current. Such terms were originally introduced in Refs. [18–20]. In Ref. [16], we considered more general parity-violating terms involving multiple scalar fields (see Sec. II), but these are sufficient for illustrational purposes here.

For convenience we will take $\vartheta_{1,2}$ to be odd under parity and time-reversal transformations along boundary directions so that both (1.1) and (1.2) are invariant under such transformations.[1] $\vartheta_{1,2}$ are then dual to scalar operators $\mathcal{O}_{1,2}$ in the boundary theory, which are odd under these transformations, and can be either marginal [15] or relevant [16]. The parity and time-reversal symmetries are broken when either of $\vartheta_{1,2}$ is nonvanishing. In this paper we will consider models where this is achieved by turning on a scalar source or by introducing a dilatonic coupling between the scalar and gauge fields. For both types of models, and for both (1.1) and (1.2), a remarkably concise and universal formula for the expectation value of the angular momentum density was found in terms of bulk solutions (see Sec. II for explicit expressions).

The Hall viscosity for holographic systems was discussed in Ref. [21], and specific examples were presented in Refs. [22,23]. The result of Ref. [21] can be readily extended to the general class of models of Refs. [15,16] (see Sec. II). Thus, the time is ripe for a systematic exploration of the relations between angular momentum

[1]In top-down string theory constructions, this is of course not a choice and should be determined by the fundamental theory.

density and Hall viscosity in holographic gapless systems, which will be the main goal of the current paper.[2]

Another motivation of the paper is that the results of Refs. [15,16] and Ref. [21] are expressed in terms of abstract bulk gravity solutions, which do not always have immediate boundary interpretations. It should be instructive to obtain explicit expressions/values in some simple models.

An immediate result is that, while the angular momentum density receives contributions from both the gauge and gravitational Chern-Simons terms, the Hall viscosity is only induced by the gravitational Chern-Simons term (1.2). That is, with $\vartheta_1 \neq 0$, $\vartheta_2 = 0$, while the angular momentum density is nonzero, the Hall viscosity vanishes. Moreover, even when $\vartheta_2 \neq 0$, the Hall viscosity vanishes when the operator dual to ϑ_2 is marginal. This is because the holographic expression for the Hall viscosity is proportional to the normal derivative of ϑ_2 at the horizon. In order for it to be nonzero, some energy scale must be generated.

Thus, the totalitarian principle, "everything not forbidden is compulsory," appears to not be at work for the Hall viscosity. It should be emphasized that the results of Refs. [15,16] and Ref. [21] were obtained in the classical gravity limit, which corresponds to the large-N and strong-coupling limit of the boundary theory. The vanishing of Hall viscosity is likely a consequence of the large-N limit, i.e. a nonvanishing answer may emerge by taking into account loop effects on the gravity side. In any case, it appears safe to conclude that at least for a certain class of holographic gapless systems, the angular momentum density and Hall viscosity appear not to be correlated at all.

The holographic expressions for the angular momentum density and Hall viscosity also suggest that they are controlled by different physics. The angular momentum density receives contributions from the full bulk spacetime, which translates in the boundary theory to the angular momentum density involving physics at all scales. That is, as an equilibrium thermodynamic quantity, the angular momentum behaves more like the free energy or energy, rather than entropy, which depends only on IR physics. In contrast, the Hall viscosity is expressed in terms of the values of the bulk fields at the horizon, and thus depends only on IR physics.

Nevertheless, it is of interest to explore whether there exist some gapless systems or kinematic regimes where the angular momentum and Hall viscosity are correlated. In particular, as reviewed in Sec. II, the expression for holographic angular momentum separates naturally into a sum of a contribution from the horizon $\mathcal{J}_{\text{horizon}}$ and a contribution $\mathcal{J}_{\text{integral}}$ from integrating over the bulk spacetime. Such a separation suggests that these contributions may have different physical origins. Indeed, in Ref. [16] we showed that, when the operators dual to ϑ_1 and ϑ_2 are marginal, $\mathcal{J}_{\text{horizon}}$ is related to anomalies ($\mathcal{J}_{\text{integral}}$ vanishes in the marginal case). Given that both $\mathcal{J}_{\text{horizon}}$ and the Hall viscosity η_H only involve horizon quantities, it is then natural to ask whether we could find some connection between them. Interestingly, in various classes of models, we do find the two are related in a rather simple way in the limit that the symmetry-breaking effects are small, suggesting a possible common physical mechanism underlying both.

After completing this project and while preparing this manuscript, we have received the paper [30], which disagrees with some of the results in this paper, as well as those in our earlier papers [15,16]. In particular, Ref. [30] claims that the gauge Chern-Simons term does not contribute to the angular momentum density, in disagreement with Eqs. (28) and (30) of Ref. [15] and Eqs. (2.32) and (2.50) of Ref. [16], as well as with Eqs. (6.15) and (7.5) of Ref. [17]. The difference can be traced to boundary conditions at the horizon. In Refs. [15,16] and in this paper, we chose the time-space components of the metric to vanish at the horizon,

$$h_t^i = 0, \tag{1.3}$$

where i is in the spatial direction along the boundary. On the other hand, Ref. [30] left h_t^i to be arbitrary at the horizon. We chose to impose the condition (1.3) to avoid a conical singularity when we analytically continue t to Euclidean time.

The plan of the paper is as follows: In Sec. II, we will summarize the holographic formulas of the angular momentum [15,16] and of the Hall viscosity [21]. In Sec. III, we will apply these formulas to discuss a class of holographic RG flows at a finite temperature and a chemical potential, where the parity and time-reversal symmetries are broken by a source for the scalar field. We will start with analytical results in the limit where the symmetry-breaking perturbation is small and then present numerical results. In Sec. IV, we will discuss models where these symmetries are broken by the dilaton coupling to the gauge field. We will summarize our result in Sec. V. In the Appendix, we will describe an analytic solution for the scalar field in the bulk near criticality.

II. REVIEW OF HOLOGRAPHIC ANGULAR MOMENTUM AND HALL VISCOSITY

Here we first review the results of Refs. [16] and [21] on angular momentum and Hall viscosity.

We consider the most general bulk Lagrangian for the Einstein gravity with gauge and gravitational Chern-Simons terms (axionic couplings), and allow any number

[2] See also Ref. [24] on the Hall viscosity and angular momentum in the holographic $p_x + i p_y$ model. For discussions of Hall viscosity and of other parity-violating physics in holographic as well as in field theoretic settings, see Refs. [25–29].

of Abelian gauge fields A_a^P ($a = 0, 1, 2, 3;\ P = 1, \ldots, N$) and scalar fields ϑ^I ($I = 1, \ldots, M$),

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{-g}[\mathcal{L}_0 + \mathcal{L}_{\rm CS}], \tag{2.1}$$

where $\mathcal{L}_0$ contains the Einstein-Hilbert term, the kinetic and potential terms for the scalar fields, and the Maxwell term for the gauge fields,

$$\mathcal{L}_0 = R - \frac{1}{2}G_{IJ}(\vartheta^K)\partial_a\vartheta^I\partial^a\vartheta^J - V(\vartheta^K) \\ - \ell^2 Z_{PQ}(\vartheta^K)F^P_{ab}F^{Qab}, \tag{2.2}$$

and $\mathcal{L}_{\rm CS}$ contains the axionic couplings,

$$\mathcal{L}_{\rm CS} = -C_{PQ}(\vartheta^K)\,{}^*F^{Pab}F^Q_{ab} - \frac{1}{4}C(\vartheta^K)\,{}^*RR. \tag{2.3}$$

${}^*F$ denotes the dual of F, and similarly for ${}^*R$. The parameter κ is related to the bulk Newton constant G_4 as $\kappa^2 = 8\pi G_4$ and ℓ is the radius of the anti–de Sitter (AdS) space.

We consider a most general bulk solution consistent with translational and rotational symmetries along boundary directions,

$$ds^2 = \frac{\ell^2}{z^2}(-f(z)dt^2 + h(z)dz^2 + (dx^i)^2), \tag{2.4}$$

with $z = 0$ at the boundary and a horizon at $z = z_0$. The scalar fields ϑ^I are functions of z only, and the only nonzero component of A_a^P is A_t^P, which again depends only on z. We denote the temperature as T and the chemical potential associated with the boundary conserved current dual to the bulk gauge field A_a^P as μ^P. Thus $A_t^P(z=0) = \mu^P$, and at the horizon regularity requires $A_t^P(z = z_0) = 0$.

The angular momentum density $\mathcal{J}$ computed in Ref. [16] can be expressed as a sum of two terms,

$$\mathcal{J} = \mathcal{J}_{\rm horizon} + \mathcal{J}_{\rm integral}. \tag{2.5}$$

The first term on the right-hand side depends only on bulk fields at the horizon ($z = z_0$),

$$\mathcal{J}_{\rm horizon} = -\frac{2\ell^2}{\kappa^2}[C_{PQ}\mu^P\mu^Q + 2\pi^2 CT^2]\Big|_{z=z_0}, \tag{2.6}$$

and the second term is an integral from the horizon to the boundary,

$$\mathcal{J}_{\rm integral} = \frac{2\ell^2}{\kappa^2}\int_0^{z_0} dz\left[C'_{PQ}(A_t^P - \mu^P)(A_t^Q - \mu^Q) + \frac{C' f'^2}{8fh}\right], \tag{2.7}$$

where the prime ($'$) indicates a derivative with respect to the bulk coordinate z. Of course, by adding a total derivative term to the integrand of the bulk integral (2.7), one can change the horizon piece and may also generate a boundary contribution. Other than that the split in (2.6) and (2.7) appears most naturally in the calculation of Ref. [16], there is a sense in which the split is canonical, as follows: In Ref. [15], we found that when the scalar fields ϑ^I are dual to marginal perturbations on the boundary, the angular momentum can be expressed solely in terms of quantities at the horizon. The split in (2.6) and (2.7) has the property that in the marginal case the integral part $\mathcal{J}_{\rm integral}$ vanishes identically (as ϑ^I are z independent in this case). Thus, it appears meaningful to interpret (2.6) as a contribution from the IR physics and (2.7) as contributions from other scales.

The Hall viscosity for Einstein gravity coupled to a single scalar field with gravitational Chern-Simons coupling (1.2) was first derived in Ref. [21], and explicit computations for some specific models have been done in Refs. [22] and [23]. It can be readily generalized to the most general Lagrangian (2.1)–(2.3), and remarkably the same formula still applies, which in our notation can be written as

$$\eta_H = \frac{\ell^2}{4\kappa^2}\frac{C'f'}{fh}\Big|_{z=z_0}. \tag{2.8}$$

In particular, the gauge Chern-Simons term (1.1) does not give a contribution [17]. The reason is as follows: The Hall viscosity can be obtained from the linear response of the tensor sector, for instance, by turning on a time-dependent source in $h_{xx} - h_{yy}$ and measuring the linear response in h_{xy}. Since the linearized equations of motion for the gauge fields decouple from the tensor modes, the gauge Chern-Simons term does not contribute to the Hall viscosity.

We note an intriguing connection between (2.8) and the second term of (2.7). Denoting

$$A = \frac{\ell^2}{4\kappa^2}\frac{C'f'}{fh}, \tag{2.9}$$

we can write (2.8) as

$$\eta_H = A|_{\rm horizon}, \tag{2.10}$$

while the second term of (2.7) can be written as

$$-\int_0^1 df A, \tag{2.11}$$

where we have changed the integration variable to the redshift factor f.[3]

[3]The change of variable is legitimate, as the redshift factor $f(z)$ should be a monotonic function of z from the IR/UV connection.

III. HOLOGRAPHIC RG FLOWS: BREAKING BY A SCALAR SOURCE ON THE BOUNDARY

The expressions (2.5)–(2.7) and (2.8) are somewhat formal, as they are expressed in terms of abstract bulk gravity solutions, which do not always have immediate boundary interpretations. To gain intuition on their physical behavior, it is instructive to examine the explicit values of these expressions in some simple models.

In this section we consider a class of holographic RG flows at a finite temperature/chemical potential, where the parity and time-reversal symmetries are broken by introducing a source for the scalar field ϑ. This corresponds to turning on a perturbation on the boundary by the operator dual to ϑ. In the next section we consider a class of models where the symmetries are broken by a dilaton coupling.

A. Outline of the model

The simplest model with both nonvanishing angular momentum and Hall viscosity consists of one scalar field ϑ with the gravitational Chern-Simons coupling,

$$\mathcal{L}_1 = \frac{1}{2\kappa^2}\sqrt{-g}\left[R - \frac{1}{2}\partial_a\vartheta\partial^a\vartheta - V(\vartheta) - \frac{\alpha_{\mathrm{CS}}}{4}\ell^2\vartheta^*RR\right], \tag{3.1}$$

where α_{CS} is a constant. In order for ϑ to have a nontrivial radial profile as is required for the nonvanishing of Hall viscosity (2.8), we consider a potential $V(\vartheta)$ for which ϑ is dual to a relevant boundary operator $\mathcal{O}$. Recall that the mass m of a scalar field is related to the conformal dimension Δ of the dual operator on the boundary by $m^2\ell^2 = \Delta(\Delta-3)$, and near the AdS boundary we should have

$$\vartheta(z) \to \vartheta_0 z^{\Delta_-} + v z^{\Delta_+} z \to 0, \tag{3.2}$$

where

$$\Delta_\pm = \frac{3 \pm \sqrt{4m^2\ell^2+9}}{2}, \qquad \Delta_+ = \Delta. \tag{3.3}$$

We turn on a uniform non-normalizable mode ϑ_0, which corresponds to turning on a relevant perturbation $\vartheta_0\int d^3x\mathcal{O}$ in the boundary theory. Since $\mathcal{O}$ is odd under parity and time reversal, these symmetries are broken explicitly. At zero temperature, the system is described by a Lorentz invariant vacuum flow, and of course both angular momentum and Hall viscosity are zero. A nonzero angular momentum density and Hall viscosity can be generated by putting the system at a finite temperature T which then cuts off the flow at scale T. The bulk gravity solution (at a finite T) is described by a black brane of the form (2.4) with a nontrivial scalar profile.

For (3.1), equations (2.6)–(2.8) become

$$\mathcal{J}_{\mathrm{horizon}} = -\frac{4\pi^2\alpha_{\mathrm{CS}}\ell^2}{\kappa^2}T^2\vartheta(z_0), \tag{3.4}$$

$$\mathcal{J}_{\mathrm{integral}} = \frac{\alpha_{\mathrm{CS}}\ell^2}{4\kappa^2}\int_0^{z_0} dz\frac{f'^2}{fh}\vartheta', \tag{3.5}$$

$$\eta_H = \frac{\alpha_{\mathrm{CS}}\ell^2}{4\kappa^2}\frac{\vartheta'(z_0)f'(z_0)}{f(z_0)h(z_0)}. \tag{3.6}$$

The bulk gravity solution depends on the specific form of the potential $V(\vartheta)$, and as we will see explicitly below, so do (3.4)–(3.6). From the boundary perspective, different $V(\vartheta)$ correspond to different flows, which implies that the behavior of the angular momentum and Hall viscosity in general depends on specific flows.

The gravity description suggests, however, that in the limit $\vartheta_0 \to 0$, the behavior of these quantities should be "universal," i.e. independent of the specific form of $V(\vartheta)$. More explicitly, in this limit, throughout the flow, i.e. from the boundary to the horizon, ϑ is small. At leading order, the nonlinear terms in $V(\vartheta)$ can be neglected, and we can simply replace it with the Gaussian potential

$$V(\vartheta) = -\frac{6}{\ell^2} + \frac{m^2\vartheta^2}{2}. \tag{3.7}$$

Note that this argument for universality works not only for (3.1), but also for the general models of (2.2) and (2.3) (for the moment, let us assume the gauge fields are not turned on). Note that since ϑ_0 has dimension $\Delta_- = d - \Delta$, the appropriate dimensionless parameter is

$$\epsilon \equiv \vartheta_0 T^{-\Delta_-} \to 0. \tag{3.8}$$

This universal limit also has a natural interpretation from the boundary side; when ϵ is small, we expect that the effect of parity and time-reversal breaking can be captured by conformal perturbation theory near the UV fixed point, i.e. the angular momentum and Hall viscosity may be controlled by properties of $\mathcal{O}$ at the UV fixed point, and not by details of the RG trajectories. It would be interesting to calculate angular momentum and Hall viscosity using conformal perturbation theory, which we defer to later work.

B. Leading results in the small-ϑ_0 limit

In the small-ϑ_0 limit, to leading order we can approximate the potential $V(\vartheta)$ by (3.7) and neglect the back-reaction of the scalar field to the background geometry. Thus, we use the standard black brane metric with

$$f = \frac{1}{h} = 1 - \frac{z^3}{z_0^3} \tag{3.9}$$

and treat the scalar field as a probe. Since the scalar equation from (3.7) is linear, and in this limit the metric is independent of ϑ, it follows from (3.4)–(3.6) that, at leading order, these expressions must be linear in ϑ_0. This is, of course, consistent with the expectation from conformal perturbation theory, as ϑ_0 is a relevant boundary coupling. Given that both $\mathcal{J}$ and η_H have dimension 2, and the dimensionless expansion parameter is ϵ (3.8), we conclude on dimensional grounds that at leading order

$$\mathcal{J} \to c_J \epsilon T^2 = c_J \vartheta_0 T^{2-\Delta_-}, \tag{3.10}$$

$$\eta_H \to c_\eta \epsilon T^2 = c_\eta \vartheta_0 T^{2-\Delta_-}, \tag{3.11}$$

where c_J and c_η are some constants, and c_J can further be separated into

$$c_J = c_{\text{horizon}} + c_{\text{integral}}. \tag{3.12}$$

Since m^2 (or dimension Δ) is the only parameter in the Gaussian limit (3.7), $c_{\text{horizon}}, c_{\text{integral}}$ and c_η are functions of Δ only. These functions can be worked out analytically (see the Appendix for details and their explicit expressions). Although they are all very complicated functions of Δ, it turns out that the ratio of $\mathcal{J}_{\text{horizon}}$ and η_H (both of which only receive contributions at the horizon) is remarkably simple, given by

$$\frac{J_{\text{horizon}}}{\eta_H} = -\frac{9}{m^2\ell^2} = \frac{9}{\Delta(3-\Delta)}. \tag{3.13}$$

Note that the above expression diverges in the marginal limit $m^2 \to 0$, where ϑ becomes a z-independent constant so that η_H vanishes. The simplicity of the ratio is intriguing and suggests a possible common physical origin for both quantities.

The ratio of the integral contribution $\mathcal{J}_{\text{integral}}$ to η_H is a rather complicated function of m^2 (see Appendix), but it can be expanded around $m^2 = 0$ as

$$\frac{\mathcal{J}_{\text{integral}}}{\eta_H} = -\frac{3}{4} - 0.0383997 m^2\ell^2 + O(m^4). \tag{3.14}$$

The ratio has a finite $m^2 \to 0$ limit, as by design $\mathcal{J}_{\text{integral}}$ also approaches zero in the marginal limit. Note that due to the coefficient of the second term being rather small, and since $m^2 \in [-9/4, 0]$, equation (3.14) in fact gives a good global fit for the whole range[4] of m^2.

[4]We take $m^2 \geq -9/4$ so that the Breitenlohner-Freedman bound is satisfied and $m^2 \leq 0$ so that the scalar field is either relevant or marginal. For $m^2 > 0$, the dual operator is irrelevant, in which case the system requires a UV completion.

Combining with (3.13), we thus find that

$$\begin{aligned}\frac{\mathcal{J}}{\eta_H} &= -\frac{9}{m^2} - \frac{3}{4} + \mathcal{O}(m^2)\\ &= \frac{3}{\Delta_-} + \frac{1}{4} + \mathcal{O}(\Delta_-)\end{aligned} \tag{3.15}$$

for $\epsilon = \vartheta_0 T^{-\Delta_-} \to 0$.

C. Generic ϑ_0: Numerical results

Away from the regime of small symmetry breaking, the results will depend on the explicit form of $V(\vartheta)$. We now consider two classes of examples for illustration. As a first example, we consider the quadratic potential given by (3.7), but now treated as a full toy potential. The other class we consider was introduced in Refs. [31,32] (the same potential also arose from a superpotential in the faked supergravity construction [33], see also Ref. [34]), which we refer to as the Gao-Zhang potential after the authors of the paper:

$$\begin{aligned}V(\vartheta) = -\frac{2}{\ell^2}\frac{1}{(1+\alpha^2)^2}[\alpha^2(3\alpha^2-1)e^{-\frac{\vartheta}{\alpha}}\\ + (3-\alpha^2)e^{\alpha\vartheta} + 8\alpha^2 e^{\frac{\alpha^2-1}{2\alpha}\vartheta}].\end{aligned} \tag{3.16}$$

We will essentially use it as a proxy to a generic potential parameterized by some constant α. It should be noted that the quadratic part of the Gao-Zhang potential around $\vartheta = 0$ always has $m^2 = -2$. We therefore can only compare its results with those obtained by the Gaussian potential with the same $m^2 = -2$.

For general ϑ_0, the backreaction from the flow of ϑ to the metric can no longer be ignored. The gravity solution and equations (3.4)–(3.6) can now only be obtained numerically. Figures 1–3 show the Hall viscosity η_H, the angular momentum density $\mathcal{J}$, and their ratio for the two potentials. In these figures, all the curves corresponding to different

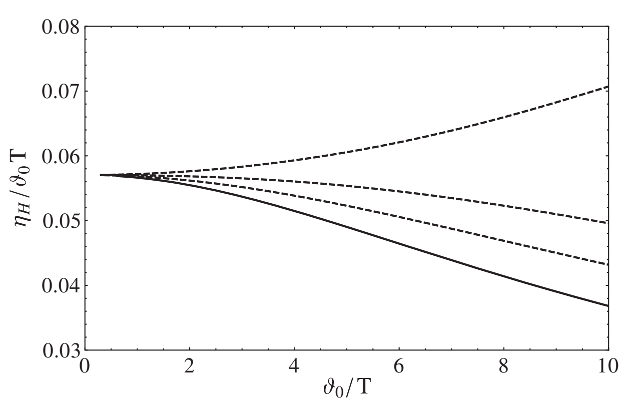

FIG. 1. Hall viscosity for the Gao-Zhang potential (3.16) with $\alpha = 1.1, 1.5, \sqrt{3}$ (dashed lines) and for a quadratic potential with $m^2 = -2$ (solid line).

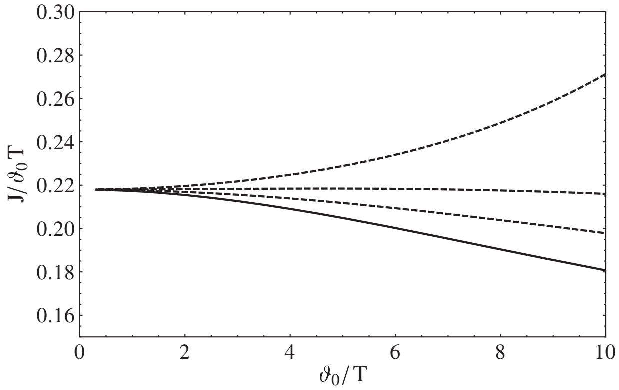

FIG. 2. Angular momentum density for the Gao-Zhang potential (3.16) with $\alpha = 1.1$, 1.5, $\sqrt{3}$ (dashed lines) and for a quadratic potential with $m^2 = -2$ (solid line).

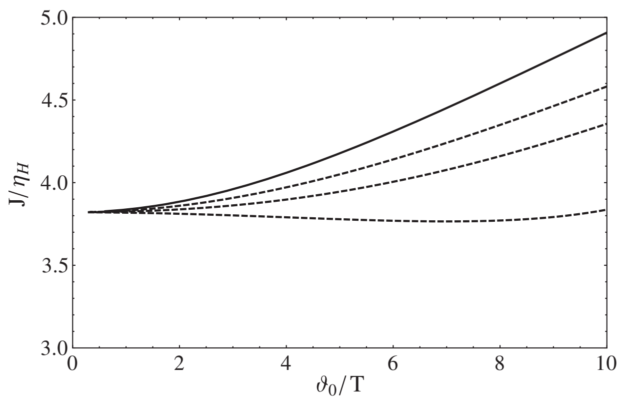

FIG. 3. Ratio of angular momentum density and Hall viscosity for the Gao-Zhang potential (3.16) with $\alpha = 1.1$, 1.5, $\sqrt{3}$ (dashed lines) and for a quadratic potential with $m^2 = -2$ (solid line).

potentials converge for $\vartheta_0 \ll T$ and approach finite values if we normalize them by $\vartheta_0 T$. Since for $m^2 = -2$ we have $2 - \Delta_- = 1$, this confirms (3.10) and (3.11). In particular, the numerical values of c_{horizon}, c_{integral} and c_η, including the ratios in (3.13) and (3.14), agree perfectly with those obtained from the analytic expressions of the Appendix for $m^2 = -2$.

For arbitrary m^2, numerical results obtained for general ϑ_0 using the Gaussian potential (3.7) also agree very well with (3.10)–(3.14) obtained in the probe limit, which serves as a good consistency check for both calculations. In Figs. 4 and 5, we show η_H and $\mathcal{J}$ as functions of m^2 obtained at a fixed $\vartheta_0/T^{\Delta_-} = 1/6$ that is sufficiently small to be in the plateau regime where $\eta_H/\vartheta_0 T^{2-\Delta_-}$ and $\mathcal{J}/\vartheta_0 T^{2-\Delta_-}$ are almost constant. The ratio $\mathcal{J}/\eta_H$ is displayed in Fig. 6, and we found it fitted well by the hyperbola

$$\frac{\mathcal{J}}{\eta_H} = -\frac{9}{m^2} + b. \tag{3.17}$$

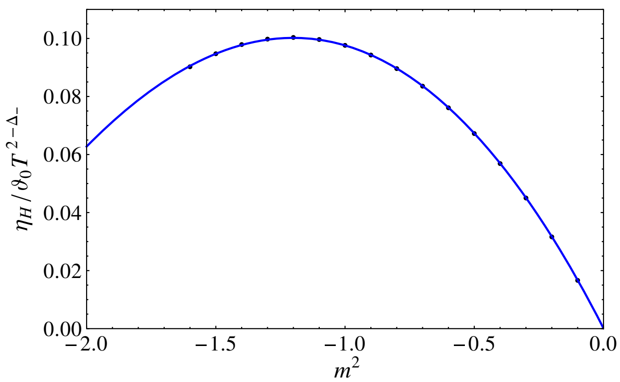

FIG. 4 (color online). $\eta_H/\vartheta_0 T^{2-\Delta_-}$ as a function of m^2 for $\vartheta_0/T^{\Delta_-} = 1/6$.

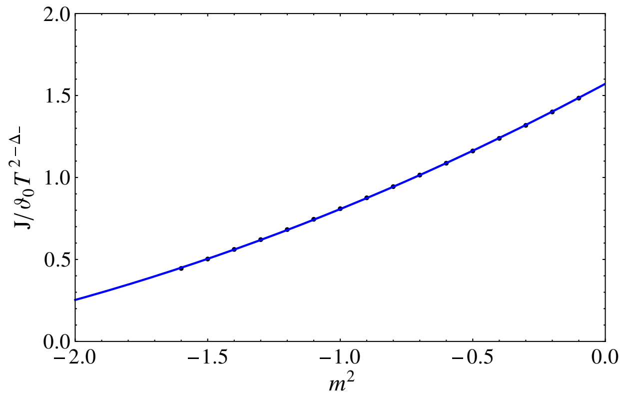

FIG. 5 (color online). $\mathcal{J}/\vartheta_0 T^{2-\Delta_-}$ as a function of m^2 for $\vartheta_0/T^{\Delta_-} = 1/6$.

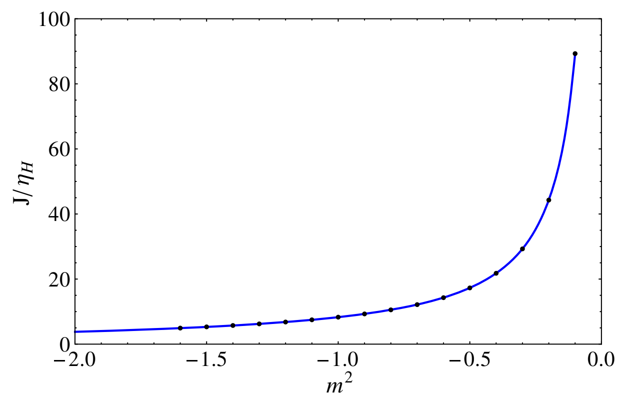

FIG. 6 (color online). Angular momentum density to Hall viscosity ratio as a function of m^2 for $\vartheta_0/T^{\Delta_-} = 1/6$.

The coefficient -9 for the $1/m^2$ is within 0.1% for the entire plateau interval from $\vartheta_0/T^{\Delta_-} = 1/12$ to $1/4$. The constant term b depends slightly on ϑ_0/T^{Δ_-}, as numerical fits show -0.63 at $\vartheta_0/T^{\Delta_-} = 1/4$ and -0.73 at $\vartheta_0/T^{\Delta_-} = 1/12$. These results indicate that as we approach the limit $\vartheta_0/T^{\Delta_-} \to 0$, b decreases monotonically and asymptotes to $-3/4$, which is consistent with (3.14).

One can fit with terms with higher orders in m^2. For example, including an additional cm^2 gives $c \approx -10^{-2}$ as $\vartheta_0/T^{\Delta_-} \to 0$, which is again consistent with value before m^2 in (3.14). Similarly, excellent agreement with (3.13) and (3.14) is found for $\mathcal{J}_{\rm horizon}/\eta_H$ and $\mathcal{J}_{\rm integral}/\eta_H$ separately.

D. Nonzero chemical potential

Let us now consider turning on a chemical potential in the holographic RG flows discussed earlier. For this purpose we add a Maxwell term to the action (3.1), i.e. the Lagrangian density becomes

$$\mathcal{L} = \mathcal{L}_1 - \frac{\ell^2}{2\kappa^2}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}, \tag{3.18}$$

and we turn on a nonzero chemical potential for the gauge field, i.e. $A_t(z=0)=\mu$. We are again interested in the leading behavior of the angular momentum and Hall viscosity in the limit $\vartheta_0 \to 0$, where we can replace the potential $V(\vartheta)$ with its quadratic order (3.7) and treat the scalar field evolution along the radial direction as a probe. Now the background geometry is replaced by that of a charged black brane with

$$f = \frac{1}{h} = 1 - \frac{z^3}{z_M^3} + \frac{z^4}{z_Q^4}, \tag{3.19}$$

where z_M and z_Q are length scales in the AdS bulk corresponding to the field theory energy and charge densities. They can be expressed in terms of temperature T and chemical potential μ as

$$z_M = \left(\frac{3}{4}\right)^{1/3}\frac{(\sqrt{3\mu^2+4\pi^2T^2}-2\pi T)^{2/3}}{\mu^{4/3}(\sqrt{3\mu^2+4\pi^2T^2}-\pi T)^{1/3}}, \tag{3.20}$$

$$z_Q = \sqrt{\frac{\sqrt{3\mu^2+4\pi^2T^2}-2\pi T}{\mu^3}}, \tag{3.21}$$

and the horizon is located at

$$z_0 = \frac{\sqrt{3\mu^2+4\pi^2T^2}-2\pi T}{\mu^2}. \tag{3.22}$$

On dimensional grounds, we again write η_H and $\mathcal{J}$ in the form (3.10)–(3.12), except that the various coefficients are now functions of μ/T, i.e. $c_J = c_J(m^2, \mu/T)$ and $c_\eta = c_\eta(m^2, \mu/T)$. Again, although $c_{\rm horizon}$ and c_η are complicated expressions, their ratio is remarkably simple and given by

$$\begin{aligned}\frac{\mathcal{J}_{\rm horizon}}{\eta_H} &= -\frac{16\pi^2T^2z_0^2}{m^2\ell^2}\\ &= -\frac{16\pi^2T^2(\sqrt{3\mu^2+4\pi^2T^2}-2\pi T)^2}{\mu^4m^2\ell^2},\end{aligned} \tag{3.23}$$

where in the second line we have expressed the horizon size z_0 in terms of μ and T. Equation (3.23) recovers (3.13) when $\mu=0$, but in the limit $T\to 0$ with a fixed μ, it behaves as

$$\frac{\mathcal{J}_{\rm horizon}}{\eta_H} = -\frac{48\pi^2T^2}{m^2\ell^2\mu^2}, \qquad T\to 0. \tag{3.24}$$

The numerical analysis suggests this happens because $J_{\rm horizon} \propto T^2$ at small T, whereas $\eta_H \propto \mu^2$. The numerical analysis also indicates $J \propto \mu^2$ in the $T\to 0$ limit, so that the ratio J/η_H tends to a constant.

E. Analytic Gao-Zhang solutions

As our last example of holographic RG flow, we consider a Lagrangian with a dilatonic coupling in the Maxwell term; i.e. we add the following term to the Lagrangian of (3.1),

$$\mathcal{L} = \mathcal{L}_1 - \frac{\ell^2}{2\kappa^2}\int d^4x\sqrt{-g}e^{-\alpha\vartheta}F_{\mu\nu}F^{\mu\nu}, \tag{3.25}$$

and turn on a nonzero chemical potential for the gauge field. In this case, with the potential given by (3.16), there is a family of analytic solutions [31], given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + h(r)(dx^i)^2, \tag{3.26}$$

$$f(r) = \frac{r_A}{r}\left(1+\frac{r_B}{r}\right)^{-\frac{2}{1+\alpha^2}}\left[\frac{r^3}{\ell^2 r_A}\left(1+\frac{r_B}{r}\right)^{\frac{4}{1+\alpha^2}} - 1\right], \tag{3.27}$$

$$h(r) = \frac{r^2}{\ell^2}\left(1+\frac{r_B}{r}\right)^{\frac{2}{1+\alpha^2}}, \tag{3.28}$$

in coordinates that are convenient for our purpose. The solutions are parametrized by α, with $\alpha=0$ corresponding to the Reissner-Nordström brane with scalar field turned off. The only nonvanishing component of the gauge field associated with the Maxwell tensor is

$$A_t = \frac{Q}{r_0+r_B} - \frac{Q}{r+r_B}, \tag{3.29}$$

while the scalar field is

$$\vartheta(r) = -\frac{2\alpha}{1+\alpha^2}\log\left(1+\frac{r_B}{r}\right). \tag{3.30}$$

Here r_0 is the location of the horizon, which appears in the gauge field because we impose the boundary condition $A_t(r = r_0) = 0$. Note that the solution does not depend on the Chern-Simons coupling constant α_{CS}, since ${}^*RR = 0$ for any spherically symmetric metric.

The electric charge density Q of the black brane is given by

$$Q^2 = \frac{1}{\ell^2}\frac{r_A r_B}{1+\alpha^2}. \tag{3.31}$$

The corresponding chemical potential can be read off as

$$\mu = A_t(r=\infty) = \frac{1}{\ell(r_0+r_B)}\sqrt{\frac{r_A r_B}{1+\alpha^2}}. \tag{3.32}$$

The Hawking temperature is given by

$$T = \frac{1}{4\pi}\frac{\partial f}{\partial r}\bigg|_{r=r_0} = \frac{3}{4\pi\ell}\sqrt{\frac{r_A}{r_0}}\left[1 - \frac{4r_B}{3(1+\alpha^2)(r_0+r_B)}\right]. \tag{3.33}$$

We should think of $r_{A/B}$ as given by the chemical potential in Eq. (3.32) and by the temperature in Eq. (3.33). The horizon location r_0 is obtained by solving $f(r_0) = 0$, namely,

$$\frac{r_0^3}{\ell^2 r_A}\left(1 + \frac{r_B}{r_0}\right)^{\frac{4}{1+\alpha^2}} = 1. \tag{3.34}$$

Equations (3.32), (3.33) and (3.34) can be solved analytically, and we can thus express $r_{A/B}$ and r_0 in terms of algebraic functions of μ and T. For $\alpha^2 < 1/3$, there is only one solution for given values of μ and T,

$$r_0 = \sqrt{1+\alpha^2}\,\ell^2\frac{\mu}{U}(1+U^2)^{\frac{\alpha^2-1}{\alpha^2+1}}, \tag{3.35}$$

where

$$U = \frac{\sqrt{1+\alpha^2}\left(\sqrt{4\pi^2T^2 + 3(1-3\alpha^2)\mu^2} - 2\pi T\right)}{(1-3\alpha^2)\mu}. \tag{3.36}$$

For $\alpha^2 \geq 1/3$, there are two solutions, corresponding to different values of the scalar field source ϑ_0. For simplicity, however, in this paper we will focus on the $\alpha^2 < 1/3$ case, as taking $\alpha^2 \geq 1/3$ does not offer further physical intuition.

Let us now examine the behavior of the scalar field near the boundary. Expanding ϑ as $r \to \infty$, we obtain

$$\vartheta = -\frac{2\alpha}{1+\alpha^2}\frac{r_B}{r} + \frac{\alpha}{1+\alpha^2}\left(\frac{r_B}{r}\right)^2 + \cdots. \tag{3.37}$$

The CFT operator Φ dual to ϑ carries conformal dimension 1 or 2 depending on the boundary condition for ϑ. To be more specific, let us pick the boundary condition so that the conformal dimension of Φ is 1. If one wants the conformal dimension to be 2, one can simply exchange the expectation value $\langle\Phi\rangle$ and the source ϑ in the discussion below.

According to the standard AdS/CFT dictionary, the coefficient of $1/r^2$ should be interpreted as an expectation value $\langle\Phi\rangle$ of the CFT operator Φ. The coefficient of the $1/r$ term is then interpreted as a source ϑ_0 for Φ. The nonvanishing $1/r$ term in the expansion of ϑ means that the dual CFT is deformed by turning on the relevant operator Φ. The magnitude of the deformation, ϑ_0, is proportional to r_B,

$$\vartheta_0 = -\frac{2\alpha}{1+\alpha^2}r_B = -\frac{2\alpha}{\sqrt{1+\alpha^2}}\ell^2\mu U(1+U^2)^{\frac{\alpha^2-1}{\alpha^2+1}}, \tag{3.38}$$

so that ϑ_0/T is a function of μ/T and α. Since different deformations correspond to different CFTs, in the bulk a given value of ϑ_0/T is related to a fixed value of μ/T.

Even though in the Gao-Zhang setup the scalar source is not independent of μ and T, their model can be used to gain analytic intuition into the relation between Hall viscosity and angular momentum density. In particular, Ref. [6] has argued from the field theory side that there should exist a simple proportionality relation between the two, at least in certain gapped systems. This can be readily compared with the analytic model of Gao-Zhang, and we find that for this class of models the relation between the two quantities is more complicated.

Applying Eqs. (3.4)–(3.6) to the Gao-Zhang model, we obtain for the Hall viscosity and gravitational Chern-Simons angular momentum density

$$\eta_H = \frac{\alpha_{\mathrm{CS}}}{4\kappa^2}h(r_0)f'(r_0)\vartheta'(r_0) \tag{3.39}$$

and

$$\mathcal{J} = \frac{\alpha_{\mathrm{CS}}}{4\kappa^2}\int_{r_0}^{\infty} dr\,\vartheta(r)\partial_r\left[h^2(r)\left(\partial_r\frac{f(r)}{h(r)}\right)^2\right], \tag{3.40}$$

with f, h and ϑ given in Eqs. (3.26)–(3.30). $\mathcal{J}_{\mathrm{horizon}}$ is still specified by Eq. (3.4) with the scalar field evaluated at the horizon.

The Hall viscosity can further be written in terms of T, r_0, r_A and r_B as

$$\eta_H = \frac{2\pi\alpha_{\mathrm{CS}}\ell}{\kappa^2}\frac{\alpha}{1+\alpha^2}T\sqrt{\frac{r_A}{r_0}}\frac{r_B}{r_0+r_B}, \tag{3.41}$$

which can then be recast in terms of μ and T as

$$\eta_H = \frac{2\pi\alpha_{\rm CS}\ell^2\alpha T(\sqrt{3(1-3\alpha^2)\mu^2+4\pi^2T^2}-2\pi T)}{(1-3\alpha^2)\kappa^2}. \tag{3.42}$$

The total angular momentum in Eq. (3.40) can also be integrated in closed form, but the result is long and unilluminating. In terms of μ and T, the horizon component of the angular momentum reads

$$\mathcal{J}_{\rm horizon} = \frac{8\pi^2\alpha_{\rm CS}\ell^2\alpha T^2}{(\alpha^2+1)\kappa^2}\ln(1+U^2), \tag{3.43}$$

with U defined in Eq. (3.36).

To gain some physical intuition, we can expand the Hall viscosity, gravitational angular momentum and horizon component of the angular momentum in a series at small μ as

$$\frac{\eta_H}{T^2} = \frac{\alpha_{\rm CS}\ell^2}{\kappa^2}\left[\frac{3\alpha\mu^2}{2T^2}+\frac{9\alpha(3\alpha^2-1)\mu^4}{32\pi^2T^4}+\mathcal{O}\left(\frac{\mu^6}{T^6}\right)\right], \tag{3.44}$$

$$\frac{\mathcal{J}}{T^2} = \frac{\alpha_{\rm CS}\ell^2}{\kappa^2}\left[\frac{18\alpha\mu^2}{5T^2}+\frac{9\alpha(57\alpha^2-49)\mu^4}{160\pi^2T^4}+\mathcal{O}\left(\frac{\mu^6}{T^6}\right)\right], \tag{3.45}$$

$$\frac{\mathcal{J}_{\rm horizon}}{T^2} = \frac{\alpha_{\rm CS}\ell^2}{\kappa^2}\left[\frac{9\alpha\mu^2}{2T^2}+\frac{27\alpha(9\alpha^2-7)\mu^4}{64\pi^2T^4}+\mathcal{O}\left(\frac{\mu^6}{T^6}\right)\right]. \tag{3.46}$$

Finally, when including an axionic term

$$\mathcal{L}_{\rm CS} = -\beta_{\rm CS}\ell^2\vartheta^*F^{ab}F_{ab}, \tag{3.47}$$

the angular momentum density for the Gao-Zhang solutions is

$$\mathcal{J} = \frac{4\beta_{\rm CS}\ell^2}{\kappa^2}\int_{r_0}^{\infty}dr(A_t(r)-\mu)A_t'(r)\vartheta(r). \tag{3.48}$$

This expression can be integrated in closed form, but it is unilluminating. In the small-μ limit, it can be expanded as

$$\frac{\mathcal{J}}{\mu^2} = \frac{\beta_{\rm CS}\ell^2}{\kappa^2}\left[\frac{3\alpha\mu^2}{2\pi^2T^2}+\frac{9\alpha(33\alpha^2-31)\mu^4}{256\pi^4T^4}+\mathcal{O}\left(\frac{\mu^6}{T^6}\right)\right]. \tag{3.49}$$

IV. HOLOGRAPHIC VEV FLOW: BREAKING BY THE DILATON COUPLING

We now consider a class of models in which the scalar ϑ is normalizable at the AdS boundary, but the parity and time-reversal symmetries are broken by the dilaton coupling. We consider the Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{-g}\left[R-\frac{1}{2}(\partial\vartheta)^2-V(\vartheta)-\ell^2e^{-\alpha\vartheta}F^2+\mathcal{L}_{\rm CS}\right] \tag{4.1}$$

and put the system at finite chemical potential. In this setup, the bulk gauge field which is needed for a nonzero chemical potential can drive a normalizable nontrivial scalar hair through the dilatonic coupling. The parity and time-reversal symmetries are broken, since the dilaton coupling $e^{-\alpha\vartheta}$ is not invariant under $\vartheta\to-\vartheta$.

Below, we will discuss the gauge Chern-Simons term and the gravitational Chern-Simons term separately. Since both the angular momentum and the Hall viscosity are linear in these couplings, we can simply add the two results to obtain the whole picture. Just as in the previous section, we use two types of potentials, the Gaussian potential (3.7) and the Gao-Zhang potential (3.16).

A. Gauge Chern-Simons coupling

For a single scalar field, let us parametrize the gauge Chern-Simons term as

$$\mathcal{L}_{\rm CS} = -\beta_{\rm CS}\ell^2\vartheta^*F^{ab}F_{ab}. \tag{4.2}$$

This term generates angular momentum density but not Hall viscosity, which was first pointed out by Ref. [17]. Specializing Eqs. (2.6) and (2.7) to Lagrangian (4.1), the angular momentum density is given by

$$\mathcal{J} = -\frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\mu^2\vartheta(z_0)+\frac{2\beta_{\rm CS}\ell^2}{\kappa^2}\int_0^{z_0}dz(A_t(z)-\mu)^2\vartheta'(z), \tag{4.3}$$

while the gravitational Chern-Simons contribution is the same as in Eqs. (3.4) and (3.5).

Figure 7 shows the angular momentum density as a function of μ^2/T^2 for a Gaussian potential with $m^2=-2$ and Gao-Zhang potentials with various α. The vertical axis is normalized by the dilatonic coupling α. We note that in the small μ/T limit, all curves scale as $\mathcal{J}\propto\alpha\mu^4/T^2$. This is to be expected, since in this limit the scalar field can be expanded in a perturbative series with the first term proportional to $\alpha\mu^2/T^2$ (by parity and dimensional analysis), and the gauge fields in Eq. (4.3) contribute a factor of μ^2.

Figure 8 shows the angular momentum as a function of μ^2/T^2 for Gaussian potentials of various m^2. We note that although for all masses $\mathcal{J}\propto\alpha\mu^4/T^2$ in the small μ/T limit, the slope depends on m^2. We also remark that, for large μ/T, the angular momentum density is proportional to μ^2 for all the potentials we have investigated. Since the vertical axes of these figures are taken to be $\mathcal{J}/\mu^2$, this can be seen in some of the curves becoming horizontal for large μ^2/T^2. We note that all curves flatten at large μ^2/T^2, even though

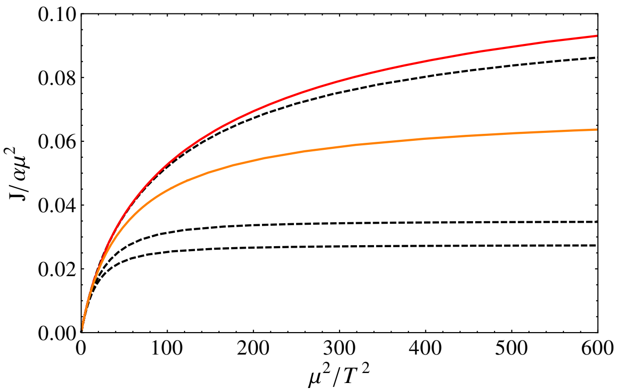

FIG. 7 (color online). The angular momentum density generated by the gauge Chern-Simons coupling as a function of μ^2/T^2 for the Gaussian potential $m^2 = -2$ and dilatonic coupling $\alpha = 0.5$ (upper solid red line) and 0.9 (lower solid orange line) and for Gao-Zhang potentials with $\alpha = 0.5$, 1.5 and $\sqrt{3}$ (dashed lines).

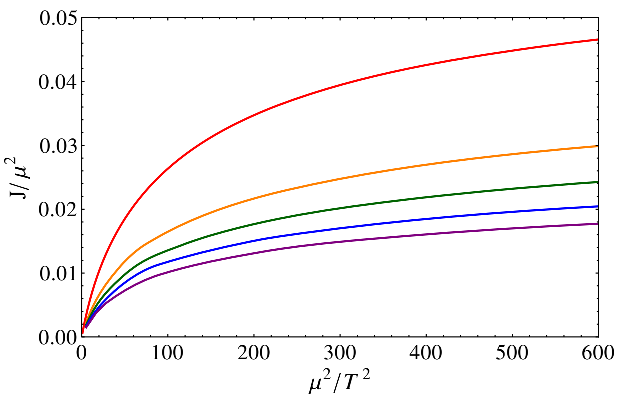

FIG. 8 (color online). The angular momentum density generated by the gauge Chern-Simons coupling as a function of μ^2/T^2 for dilatonic coupling $\alpha = 0.5$ and Gaussian potentials with $m^2 = -2$ (upper line, red), 1.4 (orange), -1 (green), -0.6 (blue) and -0.2 (lower line, purple).

this is not apparent in the range displayed in the figures. The asymptotic value of $\mathcal{J}/\mu^2$ depends on the dilatonic coupling, the scalar field mass and the details of the potential. It would be interesting to further explore this relation to better determine the type of models where it holds.

B. Gravitational Chern-Simons coupling

We now turn our attention to the gravitational Chern-Simons coupling, adding

$$\mathcal{L}_{\mathrm{CS}} = -\frac{\alpha_{\mathrm{CS}}}{4}\ell^2 \vartheta^* RR \tag{4.4}$$

to Lagrangian (4.1). This will generate both Hall viscosity and angular momentum, according to Eq. (3.6) for the Hall viscosity and Eqs. (3.4) and (3.5) for the angular momentum density.

Figures 9–11 show the Hall viscosity, the angular momentum density, and their ratio as functions of μ^2/T^2 for the Gaussian potential with $m^2 = -2$ and for Gao-Zhang potential with various α. For small μ/T, all curves converge, and we have

$$\eta_H = 0.032\alpha\mu^2, \qquad \mathcal{J} = 0.039\alpha\mu^2. \tag{4.5}$$

Figure 12 shows Hall viscosity and angular momentum density as functions of m^2 at $\mu^2/T^2 = 0.1$. This value of μ^2/T^2 is sufficiently small to be in the plateau regime, and η_H/μ^2 and $\mathcal{J}/\mu^2$ are μ/T independent. We note that both η_H and $\mathcal{J}$ are nonzero at $m^2 \to 0$, and their values in this limit are

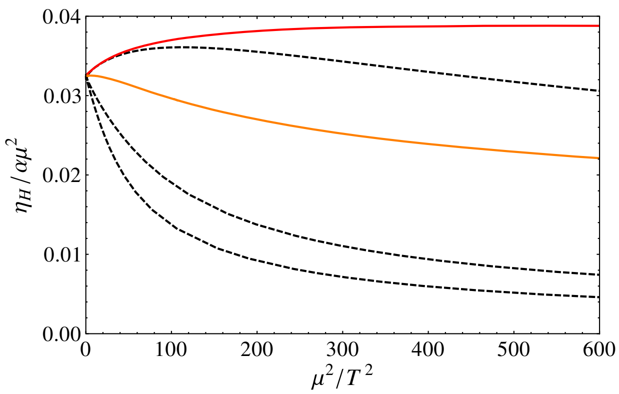

FIG. 9 (color online). Hall viscosity as a function of μ^2/T^2 for Gao-Zhang potentials with $\alpha = 0.5$, 1.5 and $\sqrt{3}$ (dashed lines) and for quadratic potentials with $m^2 = -2$ and dilatonic coupling $\alpha = 0.5$ (upper solid red line) and 0.9 (lower solid orange line).

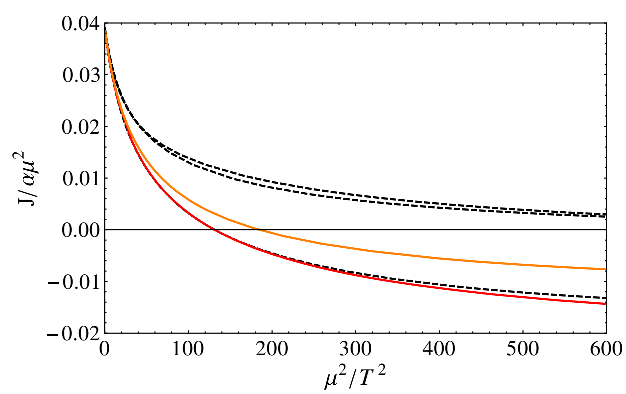

FIG. 10 (color online). Gravitational angular momentum density as a function of μ^2/T^2 for Gao-Zhang potentials with $\alpha = 0.5$, 1.5 and $\sqrt{3}$ (dashed lines) and for quadratic potentials with $m^2 = -2$ and dilatonic coupling $\alpha = 0.5$ (lower solid red line) and 0.9 (upper solid orange line).

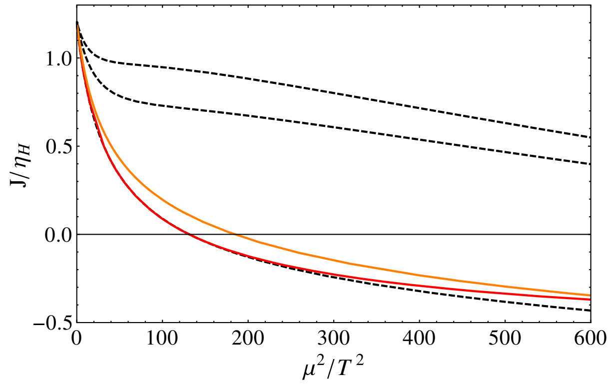

FIG. 11 (color online). Gravitational angular momentum density to Hall viscosity ratio as a function of μ^2/T^2 for Gao-Zhang potentials with $\alpha = 0.5$, 1.5 and $\sqrt{3}$ (dashed lines) and for quadratic potentials with $m^2 = -2$ and dilatonic coupling $\alpha = 0.5$ (lower solid red line) and 0.9 (upper solid orange line).

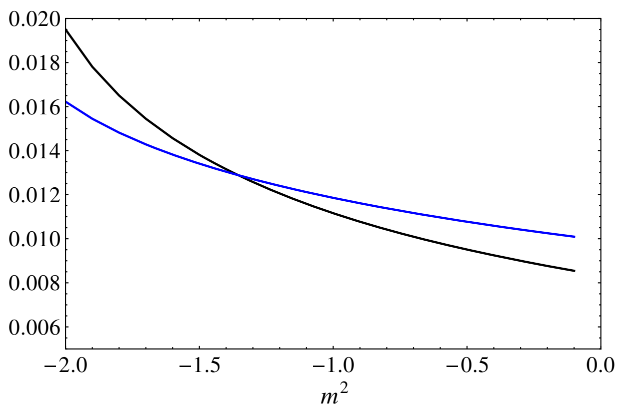

FIG. 12 (color online). η_H/μ^2 (blue, upper line on the right) and $\mathcal{J}/\mu^2$ (black, lower line on the right) as a function of m^2 for $\mu^2/T^2 = 0.1$.

$$\frac{\eta_H}{\mu^2} = 0.0099 + \mathcal{O}(m^2), \qquad \frac{\mathcal{J}}{\mu^2} = 0.0082 + \mathcal{O}(m^2). \tag{4.6}$$

The two numerical coefficients vary by less than 1% as μ^2/T^2 is increased from 0 to $\mu^2/T^2 \lesssim 0.6$. Both the horizon and the bulk terms contribute to the total angular momentum and are of the same order of magnitude, but they have opposite signs.

The ratio J/η_H is represented in Fig. 13 for $\mu^2/T^2 = 0.1$. In the small-m limit, it can be expanded as

$$\frac{\mathcal{J}}{\eta_H} = 0.84 + \mathcal{O}(m^2), \tag{4.7}$$

where again the numerical coefficient varies by less than 1% for $\mu^2/T^2 \lesssim 0.6$. For the horizon part of the angular momentum density, we have

$$\frac{\mathcal{J}_{\text{horizon}}}{\eta_H} = 1.33 + \mathcal{O}(m^2) \tag{4.8}$$

at $\mu^2/T^2 = 0.1$, and the numerical coefficient decreases by about 1% at $\mu^2/T^2 = 0.6$. It is possible to better understand the ratio $\mathcal{J}_{\text{horizon}}/\eta_H$ by going to the probe approximation and using the scalar field equation to relate ϑ and $\vartheta'(z_0)$. Doing so gives

$$\frac{\mathcal{J}_{\text{horizon}}}{\eta_H} = \frac{-144\pi^2 T^2 \vartheta(z_0)}{18\alpha\mu^2 + m^2\ell^2\vartheta(z_0)\rho}, \tag{4.9}$$

where

$$\rho \equiv 3\mu^2 + 4\pi T\left(\sqrt{3\mu^2 + 4\pi^2 T^2} + 2\pi T\right). \tag{4.10}$$

Equation (4.9) agrees well with the numerical data in the probe limit.

V. DISCUSSION

In this paper, we computed the angular momentum density and the Hall viscosity for specific classes of holographic models dual to gapless relativistic quantum systems in $(2+1)$ dimensions. Unlike gapped systems at zero temperature, no simple relation is known between the angular momentum and the Hall viscosity. We found that although the angular momentum density receives contributions both from the gauge and gravitational Chern-Simons terms, the Hall viscosity requires the gravitational Chern-Simons term. This highlights a distinction between the two quantities for gapless systems.

Moreover, when the operator dual to the scalar field ϑ is marginal, the Hall viscosity vanishes even when ϑ couples to the gravitational ${}^*RR$. This is because the holographic expression for the Hall viscosity (2.8) is proportional to $C' = \partial_z C(\vartheta(z))$ at the horizon $z = z_0$. In order for it to be nonzero, some energy scale is required. We therefore

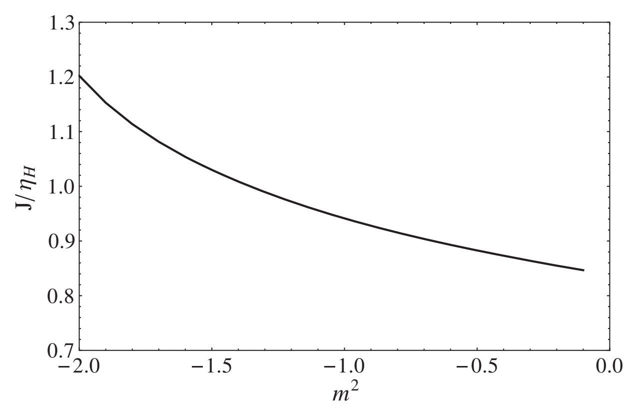

FIG. 13. Angular momentum density to Hall viscosity ratio as a function of m^2 for $\mu^2/T^2 = 0.1$.

conjecture that the Hall viscosity η_H vanishes for a conformal field theory, at least in the large-N limit.

Nevertheless, we also found that when both quantities are induced by the gravitational Chern-Simons term, their ratio shows universal properties as the systems approach their criticalities. In Sec. III, we studied a boundary system perturbed by a relevant operator of dimension Δ. The operator is odd under the parity and time-reversal symmetries, and thus we are breaking these symmetries explicitly. To the leading order in the pertubative expansion, the holographic computation shows that the ratio of the angular momentum density $\mathcal{J}$ and the Hall viscosity η_H depends only on Δ and is given by

$$\frac{\mathcal{J}}{\eta_H} = \frac{9}{\Delta(3-\Delta)} - \frac{3}{4} + O(3-\Delta). \tag{5.1}$$

The $1/(3-\Delta)$ pole is a reflection of the fact that the Hall viscosity η_H vanishes in the marginal case of $\Delta = 3$.

We also found that the angular momentum density can be decomposed as $\mathcal{J} = \mathcal{J}_{\text{horizon}} + \mathcal{J}_{\text{integral}}$, where $\mathcal{J}_{\text{horizon}}$ is a contribution from the horizon, and $\mathcal{J}_{\text{integral}}$ is an integral from the horizon to the boundary. On the other hand, η_H depends only on data at the horizon. From the point of view of the boundary theory, η_H and $\mathcal{J}_{\text{horizon}}$ are due to IR physics, while $\mathcal{J}_{\text{integral}}$ depends on dynamics at all scales. We found that η_H and $\mathcal{J}_{\text{horizon}}$ are related in a particularly simple way as

$$\frac{\mathcal{J}_{\text{horizon}}}{\eta_H} = \frac{9}{\Delta(3-\Delta)}, \tag{5.2}$$

with no $O(3-\Delta)$ corrections. We also found similar universal properties when the parity and time-reversal symmetries are broken by the dilatonic coupling. It would be interesting to find out if the decomposition $\mathcal{J} = \mathcal{J}_{\text{horizon}} + \mathcal{J}_{\text{integral}}$ can be explained from the point of the boundary theory and if (5.2) can be derived using its conformal perturbation.

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APPENDIX: ANALYTIC CALCULATION IN THE SMALL-ϑ_0 LIMIT

This appendix presents some exact results obtained by solving the scalar field equation in the probe limit for a Schwarzschild black brane. For the metric (3.9) and quadratic potential, the scalar field equation

$$\frac{1}{\sqrt{-g}}\partial_a(g^{ab}\sqrt{-g}\partial_b\vartheta(z)) - V'(\vartheta) = 0 \tag{A1}$$

can be rewritten as

$$z^2(z^3 - z_M^3)\vartheta'' + z(z^3 + 2z_M^3)\vartheta' + m^2 z_M^3\vartheta = 0. \tag{A2}$$

This can be solved analytically, and the solution is a sum of two hypergeometric functions. Demanding analyticity at the horizon, we obtain

$$\begin{aligned}\frac{\vartheta}{\vartheta_0} &= z^{3-\Delta}{}_2F_1\left(1-\frac{\Delta}{3}, 1-\frac{\Delta}{3}; 2-\frac{2\Delta}{3}; \frac{z^3}{z_M^3}\right)\\ &\quad - \frac{\Gamma(2-\frac{2\Delta}{3})\Gamma^2(\frac{\Delta}{3})z^\Delta z_M^{3-2\Delta}{}_2F_1(\frac{\Delta}{3}, \frac{\Delta}{3}; \frac{2\Delta}{3}; \frac{z^3}{z_M^3})}{\Gamma^2(1-\frac{\Delta}{3})\Gamma(\frac{2\Delta}{3})},\end{aligned} \tag{A3}$$

where the first term is the non-normalizable mode and the second is the normalizable response. With this expression, we find

$$c_\eta = -\frac{3^{1-\Delta}\pi^{\Delta-\frac{3}{2}}m^2\cot(\frac{\pi\Delta}{3})\Gamma(\frac{3}{2}-\frac{\Delta}{3})}{2^{5-\frac{4\Delta}{3}}\Gamma(1-\frac{\Delta}{3})}, \tag{A4}$$

$$c_{\text{horizon}} = -\frac{3^{3-\Delta}\pi^{\Delta-\frac{3}{2}}\cot(\frac{\pi\Delta}{3})\Gamma(\frac{3}{2}-\frac{\Delta}{3})}{2^{5-\frac{4\Delta}{3}}\Gamma(1-\frac{\Delta}{3})}, \tag{A5}$$

and

$$
\begin{aligned}
c_{\text{integral}} = {} & \frac{2^{2\Delta-7}\pi^{\Delta-1}(2\Delta-3)}{3^{\Delta}\Delta\Gamma^2(1-\frac{\Delta}{3})\Gamma(\frac{2\Delta}{3})}\csc\left(\frac{2\pi\Delta}{3}\right) \\
& \times\left\{3\Gamma^2\left(\frac{\Delta}{3}+1\right)\Gamma\left(\frac{\Delta+4}{3}\right)\left[\Delta(\Delta+4){}_3\tilde{F}_2\left(\frac{\Delta+3}{3},\frac{\Delta+3}{3},\frac{\Delta+7}{3};\frac{\Delta+10}{3},\frac{2\Delta}{3}+1;1\right)\right.\right. \\
& \left.+9{}_3\tilde{F}_2\left(\frac{\Delta+4}{3},\frac{\Delta}{3},\frac{\Delta}{3};\frac{\Delta+7}{3},\frac{2\Delta}{3};1\right)\right]-9\Delta\Gamma\left(1-\frac{\Delta}{3}\right)\Gamma\left(2-\frac{\Delta}{3}\right)\Gamma\left(\frac{7}{3}-\frac{\Delta}{3}\right) \\
& \left.\times{}_3\tilde{F}_2\left(1-\frac{\Delta}{3},2-\frac{\Delta}{3},\frac{7}{3}-\frac{\Delta}{3};2-\frac{2\Delta}{3},\frac{10}{3}-\frac{\Delta}{3};1\right)\right\}, \qquad \text{(A6)}
\end{aligned}
$$

where ${}_3\tilde{F}_2$ is a regularized hypergeometric function defined as

$$
{}_3\tilde{F}_2(a_1,a_2,a_3;b_1,b_2;z)\equiv\frac{{}_3F_2(a_1,a_2,a_3;b_1,b_2;z)}{\Gamma(b_1)\Gamma(b_2)}. \qquad \text{(A7)}
$$

[1] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, New York, 2003).
[2] M. Stone and R. Roy, Phys. Rev. B **69**, 184511 (2004).
[3] C. Hoyos, S. Moroz, and D. T. Son, Phys. Rev. B **89**, 174507 (2014).
[4] J. A. Sauls, Phys. Rev. B **84**, 214509 (2011).
[5] J. E. Avron, R. Seiler, and P. G. Zograf, Phys. Rev. Lett. **75**, 697 (1995).
[6] N. Read, Phys. Rev. B **79**, 045308 (2009).
[7] N. Read and E. H. Rezayi, Phys. Rev. B **84**, 085316 (2011).
[8] F. D. M. Haldane, arXiv:0906.1854.
[9] T. L. Hughes, R. G. Leigh, and E. Fradkin, Phys. Rev. Lett. **107**, 075502 (2011).
[10] B. Bradlyn, M. Goldstein, and N. Read, Phys. Rev. B **86**, 245309 (2012).
[11] C. Hoyos and D. T. Son, Phys. Rev. Lett. **108**, 066805 (2012).
[12] T. L. Hughes, R. G. Leigh, and O. Parrikar, Phys. Rev. D **88**, 025040 (2013).
[13] F. M. Haehl and M. Rangamani, J. High Energy Phys. 10 (2013) 074.
[14] M. Geracie and D. T. Son, arXiv:1402.1146.
[15] H. Liu, H. Ooguri, B. Stoica, and N. Yunes, Phys. Rev. Lett. **110**, 211601 (2013).
[16] H. Liu, H. Ooguri, and B. Stoica, Phys. Rev. D **89**, 106007 (2014).
[17] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz, and A. Yarom, J. High Energy Phys. 05 (2012) 102.
[18] R. Jackiw and S. Y. Pi, Phys. Rev. D **68**, 104012 (2003).
[19] F. Wilczek, Phys. Rev. Lett. **58**, 1799 (1987).
[20] S. M. Carroll, G. B. Field, and R. Jackiw, Phys. Rev. D **41**, 1231 (1990).
[21] O. Saremi and D. T. Son, J. High Energy Phys. 04 (2012) 091.
[22] J.-W. Chen, N.-E. Lee, D. Maity, and W.-Y. Wen, Phys. Lett. B **713**, 47 (2012).
[23] J.-W. Chen, S.-H. Dai, N.-E. Lee, and D. Maity, J. High Energy Phys. 09 (2012) 096.
[24] D. T. Son and C. Wu, J. High Energy Phys. 07 (2014) 076.
[25] A. Nicolis and D. T. Son, arXiv:1103.2137.
[26] S. Nakamura, H. Ooguri, and C.-S. Park, Phys. Rev. D **81**, 044018 (2010).
[27] H. Ooguri and C.-S. Park, Phys. Rev. D **82**, 126001 (2010).
[28] J. Erdmenger, M. Haack, M. Kaminski, and A. Yarom, J. High Energy Phys. 01 (2009) 055.
[29] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam, and P. Surowka, J. High Energy Phys. 01 (2011) 094.
[30] C. Wu, arXiv:1311.6368.
[31] C. J. Gao and S. N. Zhang, Phys. Rev. D **70**, 124019 (2004).
[32] C. J. Gao and S. N. Zhang, Phys. Lett. B **605**, 185 (2005).
[33] H. Elvang, D. Z. Freedman, and H. Liu, J. High Energy Phys. 12 (2007) 023.
[34] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).