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# Non-Abelian Conifold Transitions and $N = 4$ Dualities in Three Dimensions

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**Abstract**

We show how Higgs mechanism for non-abelian $N = 2$ gauge theories in four dimensions is geometrically realized in the context of type II strings as transitions among compactifications of Calabi-Yau threefolds. We use this result and T-duality of a further compacitification on a circle to derive $N = 4$, $d = 3$ dual field theories. This reduces dualities for $N = 4$ gauge systems in three dimensions to perturbative symmetries of string theory. Moreover we find that the dual of a gauge system always exists but may or may not correspond to a lagrangian system. In particular we verify a conjecture of Intriligator and Seiberg that an ordinary gauge system is dual to compacitification of Exceptional tensionless string theory down to three dimensions.

# 1 Introduction

One of the most important lessons we have learned recently in string theory is the fact that interesting field theories can be realized by considering singular compactifications of string theory with or without D-branes present. In this setup one can translate aspects of field theories in question to facts about the geometry of the manifold. This general idea is known as *geometric engineering*.

One of the main powers of geometric engineering is the flexibility in constructing any field theories we wish to construct. This is perhaps the most important aspect of this method (for example the construction of exceptional gauge groups has not been done in a geometrically faithful way in any other approach). But in addition, and what seems to be very surprising at first sight, is that in this setup the non-trivial field theory dualities can in one way or another be reduced to *classical symmetries* of string theory. This seems quite surprising. This can be done in particular for $N = 4$ theories in $d = 4$ by considering type IIA on ALE space of ADE type times $T^2$, where T-duality of $T^2$ is a geometric realization of Olive-Montonen S-duality for the ADE group [1, 2]. Similarly exact results for $N = 2$ gauge systems can be obtained by geometric engineering of type II strings on Calabi-Yau threefolds [3, 4], and by using mirror symmetry which is a classical symmetry of type II strings. This approach has been extended to $N = 1$ theories in $d = 4$ in [5, 6, 7, 8, 9] in which the dualities are realized as classical symmetries of strings. Similarly higher dimensional critical theories (with tensionless strings) have also been constructed from this viewpoint and in particular $N = 1$ theories in five dimensions [10, 11, 12] and $N = 1$ theories in six dimensions [13, 14, 15, 16] have been engineered. In certain cases constructions can also be done using D-branes in the presence of NS 5-branes [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27], and often there is a T-duality [28] which connects the two pictures (see in particular [7]).

An interesting duality was proposed for three dimensional theories with $N = 4$ in [29]. This was further extended to a large number of non-abelian gauge theories in [30, 31]. So far, the only approach from string theory involving a derivation of $N = 4$ dualities in $d = 3$ with non-abelian gauge groups involves the use of non-perturbative string dualities [17, 18]. One of our aims in this paper is to show how duality of $N = 4$ theories in $d = 3$ can also be reduced to classical symmetries of type II strings. This is done by constructing local models of $N = 4$ gauge systems involving a non-compact Calabi-Yau threefold times a circle and using the T-duality of the circle to exchange type IIA and type IIB strings. The main ingredient needed in this description is a precise understanding of how the Coulomb/Higgs phases of the gauge system are realized geometrically. Realization of Coulomb branches have been understood in the type IIA [32, 33, 34, 35, 36], and the type IIB setup [37, 38, 3, 4]. However much less is known about the Higgs branch. In this paper we will develop techniques to describe the Higgs branch in a geometrical way.

This construction not only allows us to rederive the $N = 4$ dualities in $d = 3$ from perturbative symmetries of strings, but it also allows us to see why in some cases the dual of a gauge system is *not* a lagrangian quantum field theory. A special case of this was already conjectured in [29], which we shall verify in this paper. We believe this is actually an important lesson, far more general than the example being studied here. In particular if we wish to find dual pairs for all field theories we should broaden the class of field theories under study to include non-lagrangian quantum field theories, which have been encountered in string theory (and the higher dimensional versions of which are distinguished by the appearance of tensionless strings). This may also explain why the search for dual pairs of gauge theories in four dimensions have been incomplete so far. In fact based on the three dimensional theories we study in this paper it is natural to conjecture that *for every quantum field theory in any dimension there are dual descriptions, which may or may not involve lagrangian systems.* We can verify this conjecture for the $N = 4$ theories in $d = 3$ which can be geometrically engineered. In this case the existence of a dual description is an automatic consequence of our setup.

The organization of this paper is as follows: In section 2 we introduce the basic idea and review some facts about $N = 4$ dualities in $d = 3$. In section 3, in anticipation of applications in section 4, we review the resolution of ADE singularities of ALE spaces in detail (which is self-contained and we hope is accessible to the reader). In section 4 we show how $N = 2$ Higgs mechanism in four dimensions is related to the resolution of certain singularities in type IIB string context and use this result to derive $N = 4$, $d = 3$ dual pairs. Also in this section we discuss the dual of toroidal compactifications of Exceptional tensionless strings down to three dimensions.

## 2 Basic Idea

We consider compactifications of type IIA and IIB strings on Calabi-Yau 3-folds. In such a compactification we generically obtain an effective $N = 2$, $d = 4$ theory with some number of $U(1)$'s, in which the vector multiplet moduli space (*Coulomb branch*) of the theory gets identified with the complex/Kähler moduli of Calabi-Yau and the hypermultiplet moduli space (*Higgs branch*) gets identified with the Jacobian variety over Kähler/complex moduli of Calabi-Yau in the type IIB/A respectively. In the latter case, we consider the Jacobian in order to take into account the RR field configurations on the Calabi-Yau.[1]

Depending on whether we put type IIA or IIB on a fixed 3-fold, in general we get inequivalent theories in 4 dimensions. However upon further compactification on $S^1$, they become equivalent by T-duality on $S^1$. The effective $N = 4$, $d = 3$ theories are therefore also equivalent, but their Coulomb and Higgs branches are exchanged. In fact, such an exchange symmetry in $N = 4$, $d = 3$ gauge theories were found in [29] and was called the mirror symmetry in 3 dimensions. That it should be a consequence of the T-duality of the type IIA and IIB theories was suggested in [39]. In principle, this should explain all the mirror symmetries of $N = 4$, $d = 3$ gauge theories which arise from type II theory on a Calabi-Yau 3-fold. In fact, in [30, 18], it was shown in detail how it works when the gauge group is a product of $U(1)$'s. In practice, however, it is difficult to apply this idea directly in non-abelian cases. This is because, as we will see below, we would need to find non-abelian generalization of the conifold transition [40], [41]. It turns out that the task is significantly simplified if we use mirror symmetry for Calabi-Yau threefolds. Let us describe our strategy to analyze the non-abelian case by first reviewing the abelian case.

### 2.1 Duality in the Abelian Case

As is well known, D-branes wrapped around cycles of Calabi-Yau give rise to solitons in this geometry. In particular if one considers type IIB with an $S^3$ inside a Calabi-Yau threefold $W$, by wrapping a $D3$ brane around $S^3$ we obtain a charged hypermultiplet [40] (charged under the $U(1)$ obtained by decomposition of the 4-form RR gauge potential as the volume form on

[1] In four dimensions the vector multiplets do not receive any quantum string corrections whereas the hypermultiplet moduli do. However if we go down to three dimensions on a further circle Coulomb branch also receives quantum corrections.

$S^3$ times a gauge field in space time). Moreover the mass of the hypermultiplet is proportional to the volume of $S^3$, which thus vanishes in the limit $S^3$ vanishes. This vanishing can be accomplished by changing the complex structure of Calabi-Yau. If we have more vanishing $S^3$'s than the number of $U(1)$'s then we can consider higgsing the $U(1)$'s. In particular it was shown in [41] (see also [49]) that this leads to a transition to a new Calabi-Yau in which we blow up some $S^2$'s, as anticipated in [50]. Let us denote the Calabi-Yau we started with by $W$ and the one we obtain after transition by $W^t$. In geometrical terms, we have tuned the complex moduli of $W$ to get a singular Calabi-Yau and then changed the Kähler structure of the singular space to obtain $W^t$ after transition to the Higgs branch. To be concrete let us assume that we Higgs a $U(1)^k$ system with $N > k$ hypermultiplets. Let $h^{p,q}$ denote the Hodge number of Calabi-Yau. Then we have

$$h^{2,1}(W) - k = h^{2,1}(W^t) \tag{1}$$

$$h^{1,1}(W) = h^{1,1}(W^t) - (N - k) \tag{2}$$

If we consider type IIA instead of type IIB, we have an interpretation of the same transition in terms of a (generically) inequivalent theory in 4 dimensions. In particular the inverse of the transition, namely $W^t \to W$ will have the interpretation of the Higgsing of $U(1)^{N-k}$ with $N$ flavors. Note in particular that what appears in the type IIB as the Higgs branch is now related to the Coulomb branch of a type IIA theory.

In 4 dimensions these two theories are inequivalent. However if we compactify the theories on an extra circle the story changes. This is because T-duality on the circle relates type IIA on $W \times S^1$ to type IIB on $W \times S^1$. Thus when we take the circles to be of the order of the string scale, after decoupling the excited modes of string, we obtain two effective 3-dimensional theories which should be equivalent. This duality of field theories in $3d$ is known as mirror symmetry [29], and the connection to the above transition in Calabi-Yau was noted in [30].

This duality symmetry, which we discussed in the abelian case above, has been extended to non-abelian gauge groups [30, 17, 18]. We would like to find the non-abelian realization of these transitions in Calabi-Yau compactifications in the same way we did for the abelian case above, and thus reduce the $N = 4$, $d = 3$ dualities to a perturbative symmetry of string theory.

## 2.2 Generalization to Non-Abelian Case

It is natural to expect that the derivation of the abelian duality symmetry in three dimensions involving the transition of Calabi-Yau will have non-abelian generalization. Finding this generalization will be useful as will reduce 3d duality symmetry of $N = 4$ theories to the knowledge available from string perturbation theory, plus physical interpretation of extremal transitions among Calabi-Yau in terms of Higgs/Coulomb branch transitions, which is more or less understood in terms of the D-brane solitons.

Mirror symmetry of Calabi-Yau is very important in understanding the non-abelian case. Let $M$ denote a Calabi-Yau threefold and $W$ be its mirror. By definition, this means that type IIA/B on $M$ gives the same theory as type IIB/A on $W$, where the role of complex deformations and Kähler deformations get exchanged. By now there is a lot of evidence for this symmetry [42] and some of it has been rigorized [43, 44]. Moreover there are hints that this symmetry is related to the more familiar $T$-duality ($R \to 1/R$) symmetry of toroidal compactification where one views the threefold as a $T^3$ fibered over $S^3$ [45, 46, 47] (see also [48].) We will consider type IIA on a local model of Calabi-Yau 3-fold, $M_K$ whose Kähler deformations give the Coulomb branch of an $N = 2$ theory in $d = 4$ (the subscript $K$ is there to remind us that we are considering varying the Kähler structure). We also consider the completely Higgsed branch which corresponds to an extremal transition of $M_K \to M_C^t$. The subscript $C$ on $M_C^t$ is to remind us that the complex structure variation corresponds to the Higgs branch of the theory. We thus consider

$$IIA(M_K, M_C^t) \tag{3}$$

as a local model for the Coulomb and Higgs branch of an $N = 2$ theory in $d = 4$ in the context of type IIA strings. Using mirror symmetry the same theory can be described equivalently as

$$IIA(M_K, M_C^t) = IIB(W_C, W_K^t) \tag{4}$$

Now we consider compactifying on the circle to get an $N = 4$ theory in $d = 3$. We thus have

$$IIA((M_K, M_C^t) \times S^1) = IIB((W_C, W_K^t) \times S^1) = IIA((W_C, W_K^t) \times \hat{S}^1) \tag{5}$$

where $\hat{S}^1$ denotes the T-dual circle. We thus conclude two type IIA models with (Coulomb,Higgs) branches given by the local model $(M_K, M_C^t)$ and $(W_K^t, W_C)$ which are inequivalent theories in

4 dimensions, will become equivalent in 3 dimensions, where the role of Kähler and complex deformations are exchanged.

This is a general correspondence between two theories in $3d$ and *it holds whether or not the local model of Calabi-Yau's correspond to any gauge systems*. In case that both the $M$ and $W$ lead to identifiable gauge systems we can then deduce dual gauge systems in $3d$. If one of them is a gauge system and the other is not we learn that a 3d $N = 4$ field theory may have a dual which is *not* a gauge system. Some cases of this type were conjectured in [29] and we will actually verify their conjecture. There are in principle also cases where neither side is a gauge system, and we would have a 3d field theory duality not involving gauge systems. We shall not consider this last case in this paper (but it may very well be the generic case).

For the purpose of identifying the 3d mirror for a gauge system we would need to know the local model for the Calabi-Yau 3-fold corresponding to a given group and matter. This can be done by geometric engineering of quantum field theory [3]. In particular if we are interested in pure $N = 2$ gauge system, in type IIA compactification we need to fiber an A-D-E singularity over $P^1$. The Kähler parameters corresponding to the blowing up of A-D-E singularity will correspond to Coulomb moduli of the corresponding gauge system. If we wish to get matter we will obtain it by "colliding singularities" which means that we consider intersecting $P^1$'s over which we have A-D-E singularities. Depending on what singularity is on top of intersecting $P^1$'s we will get matter in various representations [34, 35][2]. For example if we wish to get $U(n) \times U(m)$ with matter in bi-fundamental $(n, \overline{m})$ we consider a type IIA geometry with two intersecting $P^1$'s over one having an $A_{n-1}$ singularity and over the other an $A_{m-1}$ singularity and at the intersection point an $A_{n+m-1}$ singularity. The bi-fundamental $(n, \overline{m})$ can be interpreted as part of the decomposition of the adjoint matter of $U(n + m)$ and thus the corresponding bi-fundamental matter is localized near the intersection point [34, 35, 52]. In general if we have enough matter we can also Higgs the system and consider the complex deformations of the manifold which correspond to the Higgs branch. This in particular means that we go (classically) to the origin of Coulomb branch, i.e. blow down the A-D-E fibers and then deform the singularity of the manifold by changing the complex structure. For simple gauge systems such as $U(n)$ with fundamentals such a description is possible and is known [34], [35], but for more complicated systems it has not been worked out

[2]Not all intersecting singularities will give rise to matter, for example two $D$-type singularities intersecting gives rise to a superconformal $N = 2$ system which has no matter interpretation[51].

(our results below will amount to a description of this for a large number of cases). However what is known, in some simple cases [3] and in much greater generality in [4], is how to describe the exact Coulomb branch of gauge systems which are geometrically constructed in type IIA setup by applying local mirror symmetry and converting it to a type IIB compactification.

So our basic strategy is to start with a group $G$ (not necessarily simple) and representation $R$ and consider the local 3-fold $M_K$ which in type IIA gives rise to it. Then use the results [4] to construct the mirror manifold in type IIB, which we denote by $W_C$. Then we will explicitly construct what complete Higgsing means in this setup by finding $W_K^t$ and then consider what matter structure $W_K^t$ would correspond to if it were viewed in type IIA context. As a passing remark note that this also gives the $M_C^t$ manifold by applying mirror symmetry to $W_K^t$ (using the results in [4]), i.e. we will be able to write down the geometry corresponding to the non-abelian Higgs phenomenon.

## 2.3 Examples of Dual Pairs

Let us review some of the known examples of dual pairs of $N = 4$ gauge theories in three dimensions. Examples with $U(1)$ and $SU(2) = Sp(1)$ gauge groups were studied in [29], and they were generalized cases with higher rank gauge groups in [30, 31]. In these models, hypermultiplet moduli spaces do not receive any quantum corrections and can be read off directly from their classical Lagrangians by the hyperkähler quotient construction. On the other hand, their vector multiplet moduli spaces may be deformed by quantum effects.

*Example 1*

A-model: $U(k)$ gauge group with 1 adjoint and $n$-fundamental matters
B-model: $\prod_{i=1}^{n} U(k)_i$ gauge group with a fundamental matter in $U(k)_1$. The is also a bi-fundamental for each $U(k)_i \times U(k)_{i+1}$ where $i = 1, ..., n$ and $U(k)_{n+1} = U(k)_1$,

The maximum Higgs branch of the A-model is the moduli space of $SU(n)$ instantons of degree $k$, and the maximum Higgs branch of the B-model is the Hilbert scheme (resolved symmetric product) of $k$-points on the $A_n$-type ALE space. There are also various mixed branches of these models. In [30], it is shown how these branches transform into each other under the duality, by taking into account quantum corrections to the vector multiplet moduli spaces. The duality transformation, which exchanges the mass and the FI parameters of the two models, was also found.

*Example 1'*

It is possible to eliminate the adjoint matter in the A-model by adding its mass term. According to the mirror map, this corresponds to turning on the FI parameter for the diagonal $U(1)$ of $U(k)^n$. We then obtain

A-model: $U(k)$ gauge group with $n$-flavors.

B-model: $\prod_{i=1}^{n-1} U(l_i)$ gauge group with

$$(l_1, l_2, ..., l_n) = (1, 2, ...(k-1), k, k, ..., k, (k-1), ..., 2, 1) \tag{6}$$

($l_i = k$ for $i = k, ..., (n-k)$). There is a fundamental for $U(l_k)$ and $U(l_{n-k})$, and a bi-fundamental for each $U(l_i) \times U(l_{i+1})$.

These models can be generalized to include arbitrary linear chain of $U(k_i)$ groups with bifundamental matter as well as possible extra fundamental matter and their dual turns out also to be of the same type (and it is easily derivable from case 4 below). Hanany and Witten [17] pointed out that these models can be constructed by using webs of NS 5-branes and D3 and D5-branes in type IIB string theory, and suggested that the duality in this case is a consequence of the $SL(2, Z)$ $S$-duality of the type IIB theory.

In the Abelian ($k = 1$) case, this example reduces to the $A_n$ type dual pairs of [29]. In this case, it was pointed out in [30] that the mirror symmetry is a consequence of the $T$-duality of the type IIA and IIB theories. In the following, we will see how this observation is generalized to the non-Abelian ($k > 1$) case.

*Example 2*

A-model: $Sp(k)$ gauge group with one antisymmetric representation and and $n$-fundamental matters.

B-model: $\prod_{i=1}^{n-3} U(2k)_i \times \prod_{i=1}^{4} U(k)_i$. There is a fundamental for $U(k)_1$. There is also a bi-fundamental for each $U(2k)_i \times U(2k)_{i+1}$ ($i = 1, ..., n-4$) and also for $U(2k)_1 \times U(k)_1$, $U(2k)_1 \times U(k)_2$, $U(2k)_{n-3} \times U(k)_3$ and $U(2k)_{n-3} \times U(k)_4$.

In this case, the maximum Higgs branch of the A-model is the moduli space of $SO(n)$ instantons of degree $k$, and the one for the B-model is the Hilbert scheme of $k$ points on the $D_n$-type ALE space.

*Example 2'*

As in the case of example 1', we can turn on the mass parameter for the matter in the antisymmetric representation in the A-model and the corresponding FI parameter for the B-model. The resulting mirror pair is:

A-model: $Sp(k)$ gauge group with $n$-fundamentals.
B-model: $\left[\prod_{i=1}^{n-2} U(l_i)\right] \times U(k)_1 \times U(k)_2$ gauge group with

$$(l_1, ..., l_{n-2}) = (1, 2, ..., (2k-1), 2k, ...., 2k) \tag{7}$$

($l_i = 2k$ for $i = 2k, ..., n-2$). There is a fundamental in $U(l_{2k})$. there is also a bi-fundamental for each $U(l_i) \times U(l_{i+1})$, ($i = 1, ..., n-3$), and also for $U(l_{n-2}) \times U(k)_1$ and $U(l_{n-2}) \times U(k)_2$.

The Abelian case ($k = 1$) corresponds to the $D_n$-type dual pair in [29]. We will verify this duality for general $k$ in this paper.

*Example 3*

In [29], it was conjectured that if we consider a model whose gauge group is a product of $U(l_i)$'s arranged on nodes of the affine $E_n$ ($n = 6, 7, 8$) Dynkin diagram with $l_i$ being equal to the Dynkin index of each node, its Coulomb branch is the moduli space of $E_n$ instantons of degree 1, and that it is dual to the compactification of tensionless $E_n$ string theories to three dimensions. We will verify this conjecture in this paper. In [4] the Coulomb branch for product of $U(kl_i)$ gauge groups arranged on nodes of the affine $E_n$ Dynkin diagram has been found, where $k$ is an arbitrary integer and $l_i$ is the Dynkin index of the corresponding node. It is shown there, using this result, that this system is dual to $k$ small $E_n$ instantons compactified to $d = 3$, as conjectured in [29], thus extending what we have found here for $k = 1$ to higher $k$'s.

*Example 4*

The example 1 can be further generalized [30] as

A-model: $\prod_{i=1}^{n} U(k)_i$ with $v_i$ fundamental matters in $U(k)_i$ and a bi-fundamental for $U(k)_i \times U(k)_{i+1}$ ($i = 1, ..., n$).
B-model: $\prod_{i=1}^{m} U(k)_i$ with $w_i$ fundamental matters in $U(k)_i$ and a bi-fundamental for $U(k)_i \times U(k)_{i+1}$ ($i = 1, ..., w$).

They make a mirror pair if a Young diagram with rows of lengths $v_1, ..., v_n$ is related to a diagram with rows of lengths $w_1, ..., w_m$ by transposition. This, in particular, means $n = \sum_i w_i$ and $m = \sum_i v_i$. It was pointed out in [18] that one can construct these models as webs of NS 5-branes, D3 and D5-branes, as in [17], and that the mirror symmetry follows from the $S$-duality of the type IIB theory. We expect that the methods of this paper (and [4]) can be generalized to also include this case as well as the example 1, thus covering all the cases conjectured.

# 3 Resolution of ADE Singularity

As discussed in the previous section we need to develop what Higgsing means geometrically and in particular in the context of type IIB compactifications on Calabi-Yau threefolds. Already in the abelian case, it is clear that one needs to go to a point on the complex moduli of type IIB side where there is a singularity (where some 3-cycles shrink) and blowup instead some 2-cycles at these points. It is thus not surprising that the non-abelian generalization would in particular involve understanding blowups and as it turns out of the singularities of A-D-E type for ALE spaces. In this section, we thus give a systematic description of the resolution of the A-D-E singularities that will be used in the next section. The equations

$$A_{n-1} \quad : \quad xy = z^n \tag{8}$$

$$D_n \quad : \quad x^2 + y^2 z = z^{n-1} \tag{9}$$

$$E_6 \quad : \quad x^2 + y^3 + z^4 = 0 \tag{10}$$

$$E_7 \quad : \quad x^2 + y^3 + yz^3 = 0 \tag{11}$$

$$E_8 \quad : \quad x^2 + y^3 + z^5 = 0 \tag{12}$$

describe complex surfaces embedded in the affine space $\mathbf{C}^3$ with coordinates $x, y, z$. Each of them has a singularity at $x = y = z = 0$ (we assume that $n \geq 2$ for $A_{n-1}$ and $n \geq 3$ for $D_n$) and its resolution means a smooth surface which is mapped to it in such a way that the map is an isomorphism except at the inverse image of the singular point $x = y = z = 0$. The resolution we are going to describe is the so called minimal resolution and it turns out that the inverse image of the point $x = y = z = 0$ consists of rational curves (i.e. $\mathbf{P}^1$'s) whose intersection matrix is the same as the Cartan matrix of the Lie algebra indicated by the name of its singularity type.

The resolution is carried out by sequential blow-ups of the ambient space $\mathbf{C}^3$ at the singular points of the surface. For $A_{n-1}$ and $D_n$ cases, this can be done more easily by sequential blow-ups of planes transversal to lines passing through the singular points.

### 3.1 Resolution of $A_{n-1}$ singularity

We can resolve the $A_{n-1}$ singularity $xy = z^n$ by a sequence of blow-ups of complex planes. We first resolve the simplest $A_1$ singularity. Let us blow up the $x$-$y$-$z$ space at $x = z = 0$. Namely, we replace the $x$-$y$-$z$ space by a union of two spaces — coordinatized by $(x, y, \widetilde{z})$ and $(\widetilde{x}, y, z)$ — which are mapped to the $x$-$y$-$z$ space by $(x, y, z) = (x, y, x\widetilde{z}) = (z\widetilde{x}, y, z)$. The $x$-$y$-$\widetilde{z}$ and the $\widetilde{x}$-$y$-$z$ spaces are glued by $\widetilde{z}\widetilde{x} = 1$ and $z = x\widetilde{z}$. The equation $xy = z^2$ of the $A_1$ singularity looks as $x(y - x\widetilde{z}^2) = 0$ in the $x$-$y$-$\widetilde{z}$ space and $z(\widetilde{x}y - z) = 0$ in the $\widetilde{x}$-$y$-$z$ space. If we ignore the piece described by $x = 0$ and $z = 0$ which is mapped to the $y$-axis $x = z = 0$, we obtain a union of two smooth surfaces — $U_1 = \{y = x\widetilde{z}^2\}$ in the $x$-$y$-$\widetilde{z}$ space and $U_2 = \{\widetilde{x}y = z\}$ in the $\widetilde{x}$-$y$-$z$ space. The surfaces $U_1$ and $U_2$ are coordinatized by $(x, \widetilde{z})$ and $(\widetilde{x}, y)$ respectively and are glued together by $\widetilde{z}\widetilde{x} = 1$ and $x\widetilde{z} = \widetilde{x}y$. Thus, we obtain a *smooth* surface. This is the resolution of the $A_1$ singularity. This surface is mapped subjectively onto the original singular $A_1$ surface $xy = z^2$: $(x, y, z) = (x, x\widetilde{z}^2, x\widetilde{z})$ on $U_1$ and $(x, y, z) = (\widetilde{x}^2y, y, \widetilde{x}y)$ on $U_2$. The inverse image of the singular point $x = y = z = 0$ is described by $x = 0$ in $U_1$ and by $y = 0$ in $U_2$. It is coordinatized by $\widetilde{z}$ and $\widetilde{x}$ which are related by $\widetilde{z}\widetilde{x} = 1$, and thus is a projective line $\mathbf{P}^1$.

If we started with higher $A_{n-1}$ singularity, the equation $xy = z^n$ looks as $y = x^{n-1}\widetilde{z}^n$ in the $x$-$y$-$\widetilde{z}$ space and $\widetilde{x}y = z^{n-1}$ in the $\widetilde{x}$-$y$-$z$ space (ignoring the trivial piece $x = 0$ and $z = 0$). It is smooth in the $x$-$y$-$\widetilde{z}$ plane but the part is the $\widetilde{x}$-$y$-$z$ has the $A_{n-2}$ singularity at $\widetilde{x} = y = z = 0$. Thus, the surface is not yet resolved but it has become *less singular*: $n$ has decreased by one. We can further decrease $n - 1$ by one by blowing up the $\widetilde{x}$-$z$ plane at $\widetilde{x} = z = 0$. Iterating this process, we can finally resolve the singular $A_{n-1}$ surface. It is straightforward to see that the resolved space is covered by $n$ planes $U_1$, $U_2$, $U_3$, ..., $U_n$ with coordinates $(x_1, z_1) = (x, \widetilde{z})$,

$(x_2 = \widetilde{x}, z_2)$, $(x_3, z_3)$, ..., $(x_n, z_n = y)$ which are mapped to the singular $A_{n-1}$ surface by

$$U_i \ni (x_i, z_i) \longmapsto \begin{cases} x = x_i^i z_i^{i-1} \\ y = x_i^{n-i} z_i^{n+1-i} \\ z = x_i z_i \end{cases} \tag{13}$$

The planes $U_i$ are glued together by $z_i x_{i+1} = 1$ and $x_i z_i = x_{i+1} z_{i+1}$. The map onto the singular $A_{n-1}$ surface is isomorphic except at the inverse image of the singular point $x = y = z = 0$. The inverse image consists of $n-1$ $\mathbf{P}^1$s $C_1$, $C_2$, ..., $C_{n-1}$ where $C_i$ is the locus of $x_i = 0$ in $U_i$ and $z_{i+1} = 0$ in $U_{i+1}$, and is coordinatized by $z_i$ and $x_{i+1}$ that are related by $z_i x_{i+1} = 1$. $C_i$ and $C_j$ do not intersect unless $j = i \pm 1$, and $C_{i-1}$ and $C_i$ intersect transversely at $x_i = z_i = 0$. It is also possible to show that the self-intersection of $C_i$ is $-2$. Thus, we see that the intersection matrix of the components $C_1, \ldots, C_{n-1}$ is the same as the $A_{n-1}$ Cartan matrix.

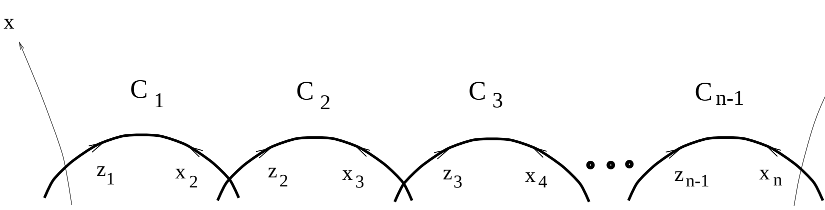



Figure 1: resolution of $A_{n-1}$ singularity

## 3.2 Resolution of $D_n$ singularity

Resolution of $D_n$ singularity ($n \geq 3$) is similar. Let us first blow up $x = z = 0$ and look at the equation $x^2 + y^2 z = z^{n-1}$ in the $x$-$y$-$\widetilde{z}$ space and in the $\widetilde{x}$-$y$-$z$ space. Ignoring the trivial piece given by $x = 0$ and $z = 0$ in the first and second patches respectively, we see $x^2 + y^2 \widetilde{z} = x^{n-2} \widetilde{z}^{n-1}$ in the $x$-$y$-$\widetilde{z}$ space and

$$z\widetilde{x}^2 + y^2 = z^{n-2} \tag{14}$$

in the $\widetilde{x}$-$y$-$z$. Let us assume $n > 3$ for the moment. Then, the surface is smooth in the $x$-$y$-$\widetilde{z}$ space, but the part in the $\widetilde{x}$-$y$-$z$ space (14) has a $D_{n-1}$ singularity at the origin $\widetilde{x} = y = z = 0$.

We can make it less singular by blowing up the $y$-$z$ plane at $y=z=0$. Iterating this process, we finally obtain a $D_3$ singularity. Now let us consider the $n=3$ case. After blowing up $x=z=0$, we see $x+y^2\tilde{z}=x\tilde{z}^2$ in the $x$-$y$-$\tilde{z}$ space and $z\tilde{x}^2+y^2=z$ in the $\tilde{x}$-$y$-$z$. Then, we see that there are two $A_1$ singularities at $x=y=\tilde{z}\mp 1=0$ (or equivalently $\tilde{x}\mp 1=y=z=0$). Blowing up again at $x=y=0$, we can resolve these $A_1$ singularities at the same time. In this way, we can resolve the singular $D_n$ surface.

For later use, we give an explicit description of the resolved surface. After the sequence of blow-ups, we obtain a 3-fold covered by $n$ open subsets $\mathcal{U}_1$, $\mathcal{U}_2$, …, $\mathcal{U}_n$ with coordinates $(s_1,t_1,z_1)=(x,y,\tilde{z})$, $(s_2=y,t_2=\tilde{x},z_2)$, …, $(s_n,t_n,z_n)$. These open sets are glued together by certain transition relations.[3] The projection to the $x$-$y$-$z$ space is given by

$$\left\{\begin{array}{lclcl} x &=& s_{2j-1}^{j} z_{2j-1}^{j-1} &=& s_{2j}^{j} t_{2j} z_{2j}^{j} \\ y &=& s_{2j-1}^{j-1} t_{2j-1} z_{2j-1}^{j-1} &=& s_{2j}^{j} z_{2j}^{j-1} \\ z &=& s_{2j-1} z_{2j-1} &=& s_{2j} z_{2j} \end{array}\right. \tag{15}$$

on $\mathcal{U}_1,\ldots,\mathcal{U}_{n-3},\mathcal{U}_{n-1}$. The expressions on $\mathcal{U}_{n-2}$ and $\mathcal{U}_n$ are somewhat irregular. For later use it is enough to write the expressions of $y$ and $z$:

$$y=\left\{\begin{array}{ll} z^{\frac{n}{2}-2} s_{n-2} t_{n-2} = s_n z^{\frac{n}{2}-2} & n\,:\,\text{even} \\ t_{n-2} z^{[\frac{n}{2}]-1} = s_n t_n z^{[\frac{n}{2}]-1} & n\,:\,\text{odd} \end{array}\right. \tag{16}$$

$$z = s_{n-2} t_{n-2} z_{n-2} = s_n z_n. \tag{17}$$

The resolved $D_n$ surface is given by

$$s_i + t_i^2 z_i = s_i^{n-1-i} z_i^{n-i} \qquad \text{in } \mathcal{U}_i \qquad (i\neq n-2,n) \tag{18}$$

$$s_{n-2} + t_{n-2} z_{n-2} = s_{n-2} z_{n-2}^2 \qquad \text{in } \mathcal{U}_{n-2} \tag{19}$$

$$\text{and} \qquad 1 + s_n t_n^2 z_n = z_n^2 \qquad \text{in } \mathcal{U}_n \tag{20}$$

This is mapped onto the singular $D_n$ surface by (15), and the map is an isomorphism except at the inverse image of the singular point $x=y=z=0$. The inverse image consists of $n$ rational curves $C_1,\ldots,C_n$ where $C_i$ $(i=1,\ldots,n-2)$ is the $z_i$-axis in $\mathcal{U}_i$ (i.e. $s_i=t_i=0$), and also the $t_{i+1}$-axis in $\mathcal{U}_{i+1}$ (i.e. $s_{i+1}=z_{i+1}=0$). $C_{n-1}$ and $C_n$ are the loci $t_{n-2}=z_{n-2}\mp 1=0$ parallel to

[3] For reference, we record the relations: $(s_j,t_j,z_j)=(s_{j+1}t_{j+1}z_{j+1},s_{j+1},t_{j+1}^{-1})$ for $j=1,\ldots,n-4$, $(s_{n-3},t_{n-3},z_{n-3})=(s_{n-2}t_{n-2}^2z_{n-2},s_{n-2}t_{n-2},t_{n-2}^{-1})$, and $(s_{n-2},t_{n-2},z_{n-2})=(z_{n-1}t_{n-1},s_{n-1},t_{n-1}^{-1})=(t_n^{-1},s_nt_n,z_n)$.

the $s_{n-2}$-axis in $\mathcal{U}_{n-2}$. $C_{i-1}$ and $C_i$ $(i = 2, \ldots, n-2)$ intersects transversely at $s_i = t_i = z_i = 0$, while $C_{n-2}$ intersects also with $C_{n-1}$ and $C_n$ at $s_{n-2} = t_{n-2} = z_{n-2} \mp 1 = 0$. There is no other intersection of distinct $C_i$'s. The self-intersection of $C_i$ in the resolved surface can be shown to be $-2$.

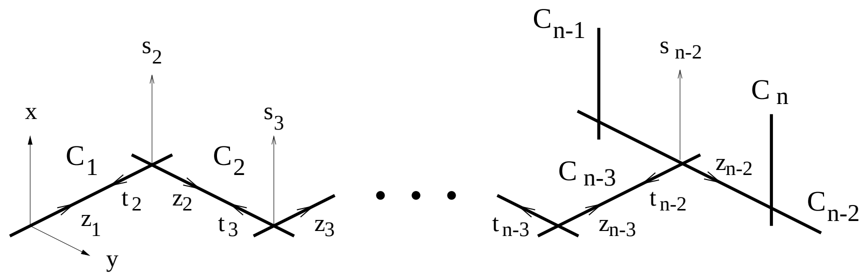



Figure 2: resolution of $D_n$ singularity

## 3.3 Resolution of $E_{6,7,8}$ singularities

Resolution of $E_{6,7,8}$ singularity is carried out by a sequence of blow-ups of $\mathbf{C}^3$. The blow up of $x$-$y$-$z$ space at the origin is a union of three spaces — coordinatized by $(x, y_1, z_1)$, $(x_2, y, z_2)$, and $(x_3, y_3, z)$ — which are glued together so that the map to the $x$-$y$-$z$ space can be defined by $(x, y, z) = (x, xy_1, xz_1) = (yx_2, y, yz_2) = (zx_3, zy_3, z)$. In particular, there are relations $y_1x_2 = 1$, $z_2y_3 = 1$, and $x_3z_1 = 1$. The inverse image of the origin $x = y = z = 0$ is a $\mathbf{P}^2$. This blow-up is shown in Figure 3.

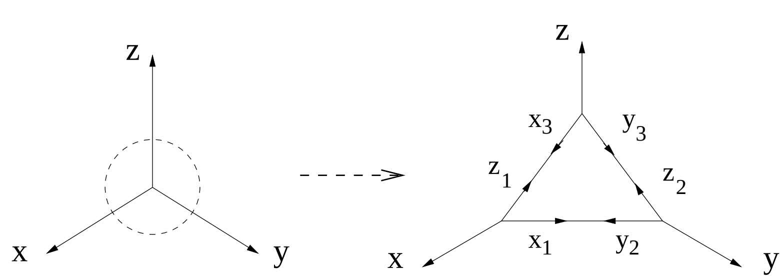



Figure 3: blow up of the $x$-$y$-$z$ space

$E_6$

Let us blow up the $x$-$y$-$z$ space at the origin. The $E_6$ equation $x^2 + y^3 + z^4 = 0$ looks as $1 + xy_1^3 + x^2 z_1^4 = 0$, $x_2^2 + y + y^2 z_2^4 = 0$, and $x_3^2 + zy_3^3 + z^2 = 0$ in the three patches where we ignore the $\mathbf{P}^2$ described by $x = 0$, $y = 0$, and $z = 0$ respectively. This surface is smooth in the first two patches, but has a singularity at the origin $x_3 = y_3 = z = 0$ of the third patch. In fact this is a $A_5$ type singularity as can be seen by completing the square of $z$. The inverse image of the singular point $x = y = z = 0$ in this surface is a line $\mathbf{P}^1$ defined by $x_2 = y = 0$ in the second patch and $x_3 = z = 0$ in the third patch.

Next we blow up the $x_3$-$y_3$-$z$ space at the origin. It turns out that the surface has an $A_3$ type singularity at one point. Continuing such process, we can finally resolve the singularity. The process is depicted in Figure 4. The bold lines or curves stands for the inverse image of the singular point $x = y = z = 0$.

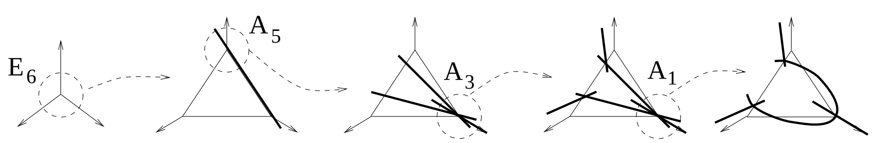



Figure 4: the process of resolution of $E_6$ singularity

The resolved surface is defined as a hypersurface in a 3-fold covered with five open subsets which we denote by $\mathcal{U}_1, \ldots, \mathcal{U}_5$ and coordinatize by $(x_1, y_1, z_1), \ldots, (x_5, y_5, z_5)$ respectively (here we have renamed the coordinates). These patches are glued together so that the projection to the $x$-$y$-$z$ space is defined in the following way:

$$\left\{ \begin{array}{llllll} x & = x_1 y_1 & = x_2 y_2^6 z_2 & = x_3 y_3^4 z_3^6 & = x_4 y_4^2 z_4^4 & = x_5 z_5^2 \\ y & = y_1 & = y_2^4 z_2 & = y_3^3 z_3^4 & = y_4^2 z_4^3 & = y_5 z_5^2 \\ z & = y_1 z_1 & = y_2^3 z_2 & = y_3^2 z_3^3 & = y_4 z_4^2 & = z_5 \end{array} \right. \tag{21}$$

The surface is defined by

$$x_1^2 + y_1 + y_1^2 z_1^4 = 0 \qquad \text{in} \quad \mathcal{U}_1, \tag{22}$$

$$x_2^2 + z_2 + z_2^2 = 0 \qquad \text{in} \quad \mathcal{U}_2, \tag{23}$$

$$x_3^2 + y_3 + 1 = 0 \qquad \text{in} \quad \mathcal{U}_3, \tag{24}$$

$$x_4^2 + z_4 y_4^2 + 1 = 0 \qquad \text{in} \quad \mathcal{U}_4, \tag{25}$$

$$x_5^2 + z_5^2 y_5^3 + 1 = 0 \qquad \text{in} \quad \mathcal{U}_5. \tag{26}$$

The inverse image of the singular point $x = y = z = 0$ is a union of six rational curves $C_1, C_2, C_{3+}, C_{3-}, C_{4+}, C_{4-}$ which are defined in the following. $C_1$ is the locus $x_1 = y_1 = 0$ in $\mathcal{U}_1$ and $x_2 = z_2 = 0$ in $\mathcal{U}_2$. $C_2$ is the locus $y_2 = x_2^2 + z_2 + z_2^2 = 0$ in $\mathcal{U}_2$ and $z_3 = x_3^2 + y_3 + 1 = 0$ in $\mathcal{U}_3$. $C_{3\pm}$ is the locus $y_3 = x_3 \mp i = 0$ in $\mathcal{U}_3$ and $z_4 = x_4 \mp i = 0$ in $\mathcal{U}_4$. $C_{4\pm}$ is the locus $y_4 = x_4 \mp i = 0$ in $\mathcal{U}_4$ and $z_5 = x_5 \mp i = 0$ in $\mathcal{U}_5$. These are depicted in the Figure 5. One can show that any of these rational curves has self-intersection $-2$ in the surface. Thus, the intersection matrix is the same as the $E_6$ Cartan matrix.

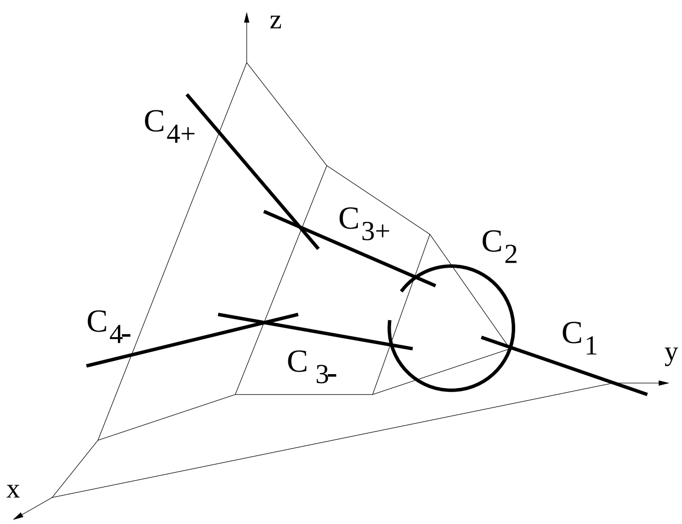



Figure 5: resolution of $E_6$ singularity

$E_7$

For $E_7$ we first blow up the $x$-$y$ plane at $x = y = 0$. Namely, we replace the $x$-$y$-$z$ space by a union of $x$-$\widetilde{y}$-$z$ and $\widetilde{x}$-$y$-$z$ space that are mapped to the $x$-$y$-$z$ space by $(x, y, z) = (x, x\widetilde{y}, z) = (y\widetilde{x}, y, z)$. The equation $x^2 + y^3 + yz^3 = 0$ looks as $x + x^2\widetilde{y}^3 + \widetilde{y}z^3 = 0$, and $y\widetilde{x}^2 + y^2 + z^3 = 0$ in the two patches. The surface is smooth in the $x$-$\widetilde{y}$-$z$ space but has a singularity at the origin $\widetilde{x} = y = z = 0$ of the second patch. Completing the square of $y$, we have

$$\left(y + \frac{\widetilde{x}^2}{2}\right)^2 - \frac{\widetilde{x}^4}{4} + z^3 = 0. \tag{27}$$

By putting, $x_6 = y + \widetilde{x}^2/2$, $y_6 = z$, $z_6 = \widetilde{x}/\sqrt{2i}$, we see that the singularity is of the $E_6$ type $x_6^2 + y_6^3 + z_6^4 = 0$. Now, we only have to resolve this $E_6$ singularity as done above.

The resolved $E_7$ surface is a hypersurface in a 3-fold covered by six open subsets $\mathcal{U}_1, \ldots, \mathcal{U}_5, \mathcal{U}_7$ with coordinates $(x_1, y_1, z_1), \ldots, (x_5, y_5, z_5), (x_7, y_7, z_7)$. These are glued so that the projection to the $x$-$y$-$z$ space is defined by

$$\begin{cases} x &= x_7 &= \sqrt{2i} z_6 (x_6 - i z_6^2) \\ y &= x_7 y_7 &= x_6 - i z_6^2 \\ z &= z_7 &= y_6 \end{cases} \tag{28}$$

where $x_6, y_6, z_6$ are expressed in $\mathcal{U}_1, \ldots, \mathcal{U}_5$ as in (21) under the replacement $x \to x_6, y \to y_6, z \to z_6$. We note that the coordinates of $\mathcal{U}_7$ and $\mathcal{U}_5$ are related by $x_7 = \sqrt{2i} z_5^3 (x_5 - i), y_7 = 1/(\sqrt{2i} z_5)$ and $z_7 = y_5 z_5^2$.

The surface is defined by

$$x_7 + x_7^2 y_7^3 + y_7 z_7^3 = 0 \qquad \text{in} \quad \mathcal{U}_7, \tag{29}$$

and by (22)-(26) in $\mathcal{U}_1, \ldots, \mathcal{U}_5$. The inverse image of the singular point $x = y = z = 0$ is the union of seven rational curves $C_1, C_2, C_{3\pm}, C_{4\pm}, C_7$ where the first six are as given in the description of $E_6$ surface and the last one $C_7$ is defined by $x_7 = z_7 = 0$ in $\mathcal{U}_7$ and $x_5 - i = y_5 = 0$ in $\mathcal{U}_5$. $C_7$ intersects only with $C_{4+}$ at one point $(x_5 - i = y_5 = z_5 = 0)$ and the intersection matrix of these curves is the same as the $E_7$ Cartan matrix.

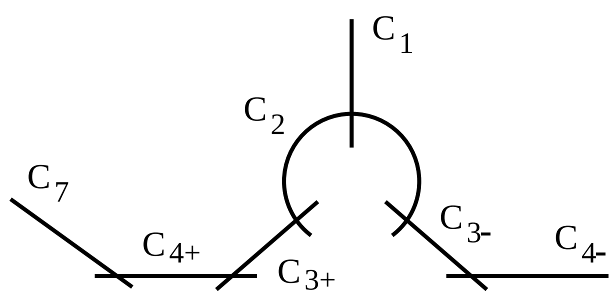



Figure 6: resolution of $E_7$ singularity

$E_8$

For $E_8$, we first blow up the $x$-$y$-$z$ space at the origin. The equation $x^2 + y^3 + z^5 = 0$ looks as $1 + x y_1^3 + x^3 z_1^5 = 0, x_2^2 + y + y^3 z_2^5 = 0$ and $x_3^2 + z y_3^3 + z^3 = 0$ in the three patches. The surface is smooth in the first two patches but has the $E_7$ singularity at the origin $x_3 = y_3 = z = 0$ of the third patch. Then, we just have to resolve this $E_7$ singularity.

The resolved $E_8$ surface is a hypersurface in a 3-fold covered by seven patches $\mathcal{U}_1, \ldots, \mathcal{U}_5, \mathcal{U}_7, \mathcal{U}_8$ with coordinates $(x_1, y_1, z_1), \ldots, (x_8, y_8, z_8)$ (here we renamed the coordinates). The 3-fold has a projection to the $x$-$y$-$z$ space defined by

$$\begin{cases} x &= x_8 y_8 &= x_7^2 y_7 &= \sqrt{2i} z_6 (x_6 - i z_6^2)^2 \\ y &= y_8 &= x_7 y_7 z_7 &= y_6 (x_6 - i z_6^2) \\ z &= y_8 z_8 &= x_7 y_7 &= x_6 - i z_6^2 \end{cases} \tag{30}$$

where $x_6, y_6, z_6$ are expressed in $\mathcal{U}_1, \ldots, \mathcal{U}_5$ as in (21) under the replacement $x \to x_6, y \to y_6, z \to z_6$. We note that the coordinates of $\mathcal{U}_7$ and $\mathcal{U}_5$ are related as in the $E_7$ case, and the coordinates of $\mathcal{U}_8$ and $\mathcal{U}_7$ are related by $x_8 = x_7/z_7, y_8 = x_7 y_7 z_7$ and $z_8 = 1/z_7$.

The surface is defined by

$$x_8^2 + y_8 + y_8^3 z_8^5 = 0 \qquad \text{in} \quad \mathcal{U}_8, \tag{31}$$

while it is defined in $\mathcal{U}_1, \ldots, \mathcal{U}_5, \mathcal{U}_7$ as in the $E_7$ surface. The inverse image of the singular point $x = y = z = 0$ is the union the eight rational curves $C_1, C_2, C_{3\pm}, C_{4\pm}, C_7$ and $C_8$ where $C_1$-$C_7$ are as given above in the description of the $E_7$ surface, and $C_8$ is defined by $x_8 = y_8 = 0$ in $\mathcal{U}_8$ and $x_7 = y_7 = 0$ in $\mathcal{U}_7$. The curve $C_8$ intersects only with $C_7$ at one point $(x_7 = y_7 = z_7 = 0)$. The intersection matrix of these curves is the same as the $E_8$ Cartan matrix.

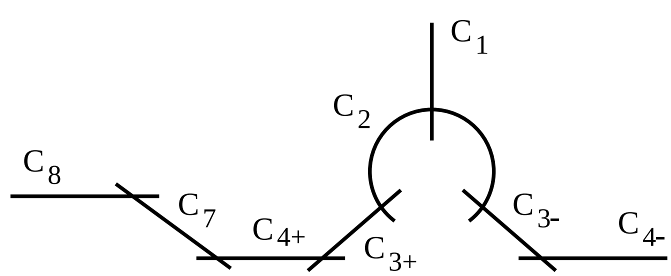



Figure 7: resolution of $E_8$ singularity

# 4 Geometry of Higgs Mechanism and $N = 4$ Dualities in 3 Dimensions

In this section we find dual pairs of three-dimensional $N = 4$ supersymmetric field theories obtained by Type II string compactifications on Calabi-Yau 3-fold times a circle, following the approach outlined in section 2. We find the duals of

1'. $\prod_{i=1}^{r} U(k_i)$ gauge theory with $n_i$-fundamentals for $U(k_i)$ and bi-fundamentals $(k_i, \overline{k}_{i+1})$

2'. $Sp(k)$ gauge theory with $n$-fundamentals.

We also find duals of

3. Theories arising from toroidal compactification down to three dimensions of one small $E_{6,7,8}$ instanton. The four dimensional versions of these theories has been considered in [53, 54, 55]. In particular we prove the conjecture of [29] for the dual of these theories.

The basic logic is as explained in Section 2. We start with Type IIA string theory compactified on $M \times S^1$ which gives the Coulomb branch of the (gauge) theory of interest where the Kähler moduli of the Calabi-Yau 3-fold $M$ corresponds to the vector moduli of the gauge theory. We perform the local mirror transform, obtaining a Type IIB string theory compactified on $W \times S^1$ where the vector moduli is now represented by the complex structure moduli of the mirror Calabi-Yau 3-fold $W$. Next, we consider transition to the Higgs branch corresponding to $W^t$ through the point where the 3-fold $W$ becomes singular. T-dualizing on the extra circle, we can equivalently view it as a Type IIA theory on $W^t \times S^1$ . Then, we can read the gauge symmetry and matter content by just looking at the geometry of the singular 3-fold, identifying the dual theory.

In this paper we skip the first process of local mirror transformation and refer the reader to the new paper [4]. Namely, we start with the Type IIB on $W \times S^1$ where we use the result of [4] to identify the geometry $W$ corresponding to the original gauge system (1',2',3 above). For the cases treated in this paper, there is a simplification [38] which is also useful. In the cases we study here $W$ can be defined by an equation of the form

$$F = uv \tag{32}$$

where $F$ is a holomorphic function (or a section of a line bundle) of some complex surface $\mathcal{S}$, and $u$ and $v$ are complex coordinates of another flat plane $\mathbf{C}^2$; the equation defines a hypersurface (3-fold) in the 4-fold $\mathcal{S} \times \mathbf{C}^2$. This 3-fold can be considered as an elliptic (or $\mathbf{C}^*$) fibration over $\mathcal{S}$ where the fibre acquires $A_{\ell-1}$ type singularity at the zero locus of $F$. Now, we use the correspondence of Type IIB on $A_{\ell-1}$ type singularity with Type IIA with $\ell$ NS fivebranes [28]. Then we can identify the Type IIB on $W \times S^1$ as the theory on the NS fivebrane with worldvolume $\{F = 0\} \times S^1 \times \mathbf{R}^3$ where we note that $\{F = 0\}$ is a Riemann surface embedded in the surface $\mathcal{S}$. This is the compactification on $S^1$ of the $d = 4$ $N = 2$ supersymmetric gauge theory with the Seiberg-Witten curve $\{F = 0\}$ [56, 38]. The results of [38] relating the non-compact $N = 2$ curve to the worldvolume theory of the type IIA (or equivalently M-theory) 5-brane has been recently interpreted in [57] as arising from the embedding of type IIA in M-theory. This has also been extended in [57] to the curves for the class of theories of the type 1' above where the M-theory fivebrane (or equivalently type IIA fivebrane) is embedded in some complex surface associated with the flavor symmetry. In other words, we could start with the theory on such Type IIA fivebrane[4] and obtain the Type IIB geometry $W$ through the correspondence of [28]. However, we stress that our main aim in this paper is to reduce non-trivial field theory dualities to classical symmetries of string theory. In particular the Type IIB geometry $W$ can be obtained only by knowledge of *classical* symmetries of string theory [4] (T-dualities), without making use of non-perturbative aspects of strings, for example how the branes of Type IIA arise from M-theory perspective.

The starting Calabi-Yau 3-fold on which we put Type IIB string theory to obtain the original gauge theories 1',2',3 are given by $F = uv$ where a special case of 1' we consider separately as 1'a:

1'a. $U(k)$ with $n$-fundamentals[5]

$$F = x + z^k + y \qquad \text{in the } A_{n-1} \text{ surface } xy = z^n \tag{33}$$

1'. $\prod_{i=1}^{r} U(k_i)$ with $n_i \times k_i$ and $(k_i, \overline{k}_{i+1})$

$$\begin{aligned} F \;=\; & x^{r+1} + z^{k_1} x^r + \cdots + z^{k_i + (i-1)n_1 + (i-2)n_2 + \cdots + n_{i-1}} x^{r-i+1} + \\ & \cdots + z^{k_r + (r-1)n_1 + \cdots + n_{r-1}} x + z^{rn_1 + \cdots + n_r} \end{aligned} \tag{34}$$

[4]Note that the strategy we are following in the Calabi-Yau language can be rephrased in this case by stating that in the type IIA context compactifying the NS 5-brane worldvolume theory on a circle and applying T-duality on the circle we obtain NS 5-brane of type IIB, from which we can also read off the mirror theory in 3 dimensions by using [28] or S-duality of type IIB viewing it as D5 branes.

[5]Note that if we take $k = 1$ this reduces to the abelian conifold transitions.

in the $A_{n_1+\cdots+n_r-1}$ surface $xy = z^{n_1+\cdots+n_r}$

2'. $Sp(k)$ with $n$-fundamentals

$$F = y - z^k \qquad \text{in the } D_n \text{ surface } x^2 + y^2 z = z^{n-1} \tag{35}$$

3. Critical $E_{6,7,8}$ tensionless string theories compactified to 4 dimensions

$$E_6 \;:\quad F = z \qquad \text{in the } E_6 \text{ surface } x^2 + y^3 + z^4 = 0 \tag{36}$$

$$E_7 \;:\quad F = z \qquad \text{in the } E_7 \text{ surface } x^2 + y^3 + yz^3 = 0 \tag{37}$$

$$E_8 \;:\quad F = z \qquad \text{in the } E_8 \text{ surface } x^2 + y^3 + z^5 = 0 \tag{38}$$

Some remarks are now in order.

• By $ADE$ surfaces, we mean the resolved surfaces described in the previous section.

• If we consider $F = 0$ as a Riemann surface factor of a IIA theory (or equivalently M-theory) fivebrane, the expressions (33) were derived in [38, 3] and the expressions (34) were obtained in [57]. This latter case has also been recently derived using just local mirror symmetry [4] extending the earlier work of [3]. In this case, we actually need to use $r$ different functions on $r$ different patches of the resolved surface, each proportional to the function $F$ in (34), which also arises naturally in [4]. (See Section 4.2 for detail)

• The description (35) for $Sp(k)$ gauge theory can be generalized to the case where the bare mass $m_i$ and adjoint vev $\phi_a$ are turned on:

$$F = y - \prod_{a=1}^{k} (z - \phi_a^2) \tag{39}$$

in the deformed $D_n$ surface (in the convention of [58])

$$x^2 + y^2 z = \frac{1}{z}\left(\prod_{i=1}^{n}(z + m_i^2) - \prod_{i=1}^{n} m_i^2\right) - 2\prod_{i=1}^{n} m_i\, y. \tag{40}$$

The curve $F = 0$ is exactly the same as the Seiberg-Witten curve for $Sp(k)$ gauge theory found in [59]. It should be a straightforward application of the local mirror transform to obtain the Calabi-Yau 3-fold $F = uv$ as the Type IIB geometry for this gauge theory. Also, we note that this curve $F = 0$ in the $D_n$ surface can be obtained as a factor of an M-theory fivebrane by generalizing the argument of [57] to the case where there is an orientifold six-plane parallel to the D sixbranes. Note that orientifolding converts the $A$-singularity associated to the D6 branes to $D$-singularity as is appearing in the above equation.

We consider the case where there is a complete Higgs phase. Specifically, in the class 1'a, $n \geq 2k$, in the calss 1' $n_i + k_{i-1} + k_{i+1} \geq 2k_i$ for any $i$, in the class 2'. $n \geq 2k + 2$. This condition is equivalent to non-asymptotic free condition of the corresponding four dimensional gauge theory. [6]

• The class 3 has been considered in [53, 54, 55] and in particular local mirror symmetry applied to this problem results in the description of the curve given above.

## 4.1 $U(k)$ gauge theory with $n$-fundamentals

We are interested in the locus of $F = 0$ in the surface, where the Calabi-Yau 3-fold $W$ described by $F = uv$ acquires $A$-type singularity. Suppose $F$ has zero of order $\ell$ along a rational curve ($\cong \mathbf{P}^1$) described by $z = 0$: $F \sim z^\ell$. Then, $W$ has $A_{\ell-1}$-type singularity $z^\ell = uv$ along the rational curve and leads to the $U(\ell)$ gauge symmetry in the Type IIA side (i.e. after T-duality on $S^1$). When the curve $z = 0$ is not a finite $\mathbf{P}^1$ but has an infinitely large volume with respect to the scale of interest, the gauge coupling (proportional to the inverse of the volume) is infinitesimally small compared to other couplings, and the $U(\ell)$ should be considered as a flavor symmetry.

Let us look at $F = x + z^k + y$ in the $i$-th patch $U_i$ of the resolved $A_{n-1}$ surface which is coordinatized by $(x_i, z_i)$. Since $(x, y, z)$ is expressed as (13), $F$ is given by

$$F = x_i^i z_i^{i-1} + x_i^k z_i^k + x_i^{n-i} z_i^{n+1-i}. \tag{42}$$

[6] The $N = 2$ results for Coulomb branch which we are using also make sense in the non-asymptotically free region. However there is another way to use the $N = 2$ results by embedding the non-asymptotically free theories in asymptotically free theories in four dimensions. For example, if we consider an $SU(k')$ gauge theory with flavor $n$ where $k'$ is chosen large enough $2k' > n$, there is a non-Baryonic branch of dimension $k(n-k)$ and at a generic point of the root of that branch the theory is identified as $U(k)$ gauge theory with $n$-flavors tensored with free $U(1)^{k'-k-1}$ Maxwell theory [60]. The curve at such a point is given by

$$x + z^{k'} + u_2 z^{k'-2} + \cdots + u_{k'-k} z^k + y = 0 \qquad \text{in the } A_{n-1} \text{ surface.} \tag{41}$$

Away from the $A_{n-1}$ singularity $x = y = z = 0$ the curve has genus $k' - k - 1$ and this is responsible for the free Maxwell theory part. Thus, the behavior of the curve near $x = y = z = 0$ is relevant for the $U(k)$ gauge theory with $n$-flavors. In such a region, the higher power $z^{k+j}$ in (41) is negligible compared to $z^k$. Thus, we may well start with (33). The same can be said about other cases.

The locus $F = 0$ looks differently depending on $i$. We recall now that we are considering the case $n \geq 2k$. For $1 \leq i \leq k$, the first term is of lowest order both in $x_i$ and $z_i$. For $k+1 \leq i \leq n-k$ the lowest order is the second term, and for $n-k+1 \leq i \leq n$ it is the last term. Thus, $F$ factorizes as

$$F = x_i^i z_i^{i-1}(1 + x_i^{k-i} z_i^{k+1-i} + x_i^{n-2i} z_i^{n-2i+2}) \qquad i = 1, \dots, k, \tag{43}$$

$$F = x_i^k z_i^k (x_i^{i-k} z_i^{i-k-i} + 1 + x_i^{n-k-i} z_i^{n+1-k-i}) \qquad i = k+1, \dots, n-k, \tag{44}$$

$$F = x_i^{n-i} z_i^{n+1-i} (x_i^{2i-n} z_i^{2i-n-2} + x_i^{i-n+k} z_i^{i-n+k-1} + 1) \qquad i = n-k+1, \dots, n. \tag{45}$$

Recall that $x_i = 0$ in $U_i$ and $z_{i+1} = 0$ in $U_{i+1}$ defines a rational curve $C_i$. The curves $C_{i-1}$ and $C_i$ intersect transversely at one point $x_i = z_i = 0$. We note that the zero of the last factor in (43) - (45) defines a smooth curve $C$ which extends to infinity. It intersects only with $C_k$ and $C_{n-k}$. This can be seen by looking at the equation for $i = k, k+1$ and for $i = n-k, n-k+1$. For example, in $U_k$, $C_k$ is given by $x_k = 0$ while $C$ is given by $1 + z_k + x_k^{n-2k} z_k^{n-2k+2} = 0$, and they intersect at one point $x_k = 0$, $z_k = -1$ (if $n > 2k$; If $n = 2k$ where $C_k = C_{n-k}$, they intersect at two points $x_k = 0$, $z_k^2 + z_k + 1 = 0$). Likewise, it is easy to see that $C$ and $C_{n-k}$ intersect at one point transversely. From the above equations we see that $F$ has zeros at $C_i$ of order $i$ for $i = 1, \dots, k-1$, of order $k$ for $i = k, \dots, n-k$, and of order $n-i$ for $i = n-k+1, \dots, n-1$, and also a single zero at $C$. This is depicted in Figure 8.

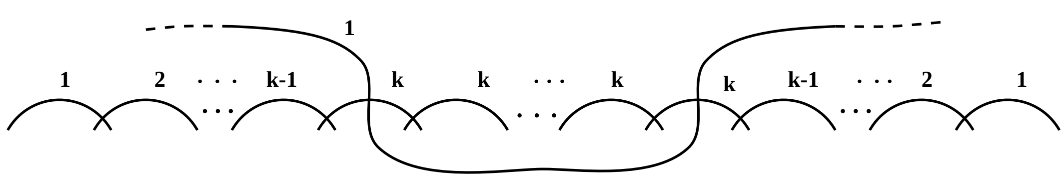



Figure 8: the zero and the order of $F$

Now, if we look at this geometry in the Type IIA side, we see that there is a gauge group $U(1) \cdot U(2) \cdots U(k-1) \cdot U(k)^{n-2k+1} \cdot U(k-1) \cdots U(2) \cdot U(1)$ coming from the $A$-type singularities along the rational curves $C_1, \dots, C_{n-1}$. Note that $C$, having infinite volume compared to others, does not lead to the gauge group. From the intersection of $C_i$ and $C_{i+1}$ we obtain the bi-fundamental of the neighboring gauge group, and from the intersection of $C$ with $C_k$ and $C_{n-k}$, we obtain fundamentals for the first and the last $U(k)$'s. In this way, we

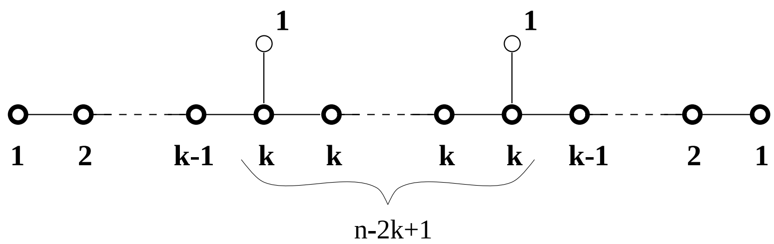



Figure 9: the mirror of $U(k)$ gauge theory with $n$ flavors

have identified the mirror gauge theory. Figure 9 depicts the quiver diagram describing the gauge and matter content of the mirror. To each node with index $\ell$ is associated a gauge (bold node) or flavor (normal node) group $U(\ell)$ and each edge connecting two nodes with indices $\ell_1$ and $\ell_2$ represents a hypermultiplet transforming as $(\ell_1, \overline{\ell_2})$ under $U(\ell_1) \times U(\ell_2)$.

## 4.2 Linear chain of $U(k_i)$ gauge groups

Next, we consider a theory with gauge group $\prod_{i=1}^{r} U(k_i)$ with $n_i$-fundamentals for $U(k_i)$ and bi-fundamentals $(k_1, \overline{k_2})$, $(k_2, \overline{k_3}), \ldots, (k_{r-1}, \overline{k_r})$. We assume the condition

$$n_i + k_{i-1} + k_{i+1} \geq 2k_i \tag{46}$$

for the existence of a complete Higgs phase. Before considering the mirror symmetry, we digress for a moment to provide the precise definition of the Calabi-Yau 3-fold $W$, or the "function" $F$. It turns out that we need different functions on different patches of the resolved $A_{n_1+\cdots+n_r-1}$ surface. This is *derived* in the new paper [4]. Here we see how this is *required* if we consider the gauge theory as coming from the worldvolume dynamics of a fivebrane in Type IIA or M-theory.

*Precise Definition of The Curve*

Let us consider an $N = 2$ gauge theory in four dimensions with gauge group $U(k_1) \times U(k_2)$ with $n_i$ massless fundamentals for $U(k_i)$ and a bi-fundamental $(k_1, \overline{k_2})$.[7] In the paper [57]

[7] We require asymptotic freedom $n_1 + k_2 < 2k_1, n_2 + k_1 < 2k_2$ for this part of the subsection.

using properties of M-theory fivebranes, it is shown that this theory can be described by a curve[8]

$$F := x^3 + g_1(z)x^2 + g_2(z)z^{n_1}x + z^{2n_1+n_2} = 0. \tag{47}$$

where $g_1(z)$ and $g_2(z)$ are polynomials in $z$ of degree $k_1$ and $k_2$ respectively. However, we must be careful about the precise definition of the curve if we consider it as embedded in the resolved $A_{n_1+n_2-1}$ surface. At a generic point in the Coulomb branch, we expect that the theory flows in the IR limit to a free Maxwell theory. This means that for generic $g_1(z)$ and $g_2(z)$ the curve should be smooth and irreducible. However, the curve (47) is not. $F$ is divisible by $x^2$ in the first $n_1$ patches $U_1, \ldots, U_{n_1}$ and by $z^{n_1}x$ in the last $n_2$ patches $U_{n_1+1}, \ldots, U_{n_1+n_2}$. In order to see this, we introduce variables $y' = y/z^{n_2}$ and $x' = x/z^{n_1}$. By using the formula (13) for the projection, we see that $y' = x_i^{n_1-i}z_i^{n_1+1-i}$ and $x' = x_i^{i-n_1}z_i^{i-n_1-1}$ on $U_i$, and thus that $y'$ is well-defined on the first $n_1$ patches while $x'$ is defined on the last $n_2$. By noting that $xy' = z^{n_1}$ and $x = x'z^{n_1}$, $x'y = z^{n_2}$, we see that $F$ is divisible by $x^2$ in the first $n_1$ patches while it is divisible by $z^{n_1}x$ in the last $n_2$ where

$$F/x^2 = x + g_1(z) + g_2(z)y' + z^{n_2}y'^2 \qquad \text{in } \; U_1, \ldots, U_{n_1} \tag{48}$$

$$F/(z^{n_1}x) = z^{n_1}x'^2 + g_1(z)x' + g_2(z) + y \qquad \text{in } \; U_{n_1+1}, \ldots, U_{n_1+n_2} \tag{49}$$

Thus, the precise definition of the curve is $F/x^2 = 0$ in the first $n_1$ patches and $F/(z^{n_1}x) = 0$ in the last $n_2$. This makes sense since the two functions are related by $(1/y')F/x^2 = F/(z^{n_1}x)$ where $y' \neq 0$ in the intersection region because $y'x' = 1$.

For the group $\prod_{i=1}^r U(k_i)$ with general $r$, the function $F$ is given by

$$\begin{aligned} F \;\; = \;\; & x^{r+1} + g_1(z)x^r + \cdots + g_i(z)z^{(i-1)n_1+(i-2)n_2+\cdots+n_{i-1}}x^{r-i+1} + \\ & \cdots + g_r(z)z^{(r-1)n_1+\cdots+n_{r-1}}x + z^{rn_1+\cdots+n_r} \end{aligned} \tag{50}$$

where $g_i(z)$ is a polynomial of degree $k_i$. In the resolved $A_{n_1+\cdots+n_r-1}$ surface, the curve is defined by $F/x^r = 0$ in the first $n_1$ patches, $F/(z^{n_1}x^{r-1}) = 0$ in the next $n_2$ patches, ..., $F/(z^{(i-1)n_1+\cdots+n_{i-1}}x^{r+1-i}) = 0$ in the next $n_i$ patches, ... etc.

*Dual Gauge Theory*

[8]Note that the $U(1)$'s have charged matter and thus infrared trivial, and thus do not affect the infrared dynamics of the non-abelian part in four dimensions.

We now identify the dual gauge theory. We present the detail for the case $r = 2$. The general case is treated in the same way.

As in the previous subsection, it is straightforward to determine the zero and the order of the function $F$ (34), or more precisely, of $F/x^2$ (48) in the first $n_1$ patches and of $F/(z^{n_1}x)$ (49) in the remaining $n_2$ patches, where $g_1(z) = z^{k_1}$ and $g_2(z) = z^{k_2}$. Without loss of generality, we may assume $k_1 \geq k_2$.

Using the expression (15), we see that the function looks as

$$F/x^2 = x_i^i z_i^{i-1} + x_i^{k_1} z_i^{k_1} + x_i^{n_1+k_2-i} z_i^{n_1+k_2+1-i} + x_i^{2n_1+n_2-2i} z_i^{2n_1+n_2+2-2i} \tag{51}$$

$$F/(z^{n_1}x) = x_i^{2i-n_1} z_i^{2i-n_1-2} + x_i^{i-n_1+k_1} z_i^{i-n_1+k_1-1} + x_i^{k_2} z_i^{k_2} + x_i^{n_1+n_2-i} z_i^{n_1+n_2+1-i} \tag{52}$$

in the first $n_1$ and the last $n_2$ patches respectively. As we will see below, for every $i$ there is one term $x_i^{\ell_i} z_i^{\ell_{i-1}}$ among the four of lowest order both in $x_i$ and $z_i$. Therefore, in each patch $U_i$, it factorizes as

$$x_i^{\ell_i} z_i^{\ell_{i-1}} f_i(x_i, z_i) \tag{53}$$

where $f_i(0,0) = 1$. The curves $f_i(x_i, z_i) = 0$, $i = 1, \ldots, n_1 + n_2$ glue up into one smooth curve $C$ that extends to infinity.[9] Thus, in each $U_i$ the function takes zero at $C_i$, $C_{i-1}$ and at $C$ of order $\ell_i, \ell_{i-1}$ and 1.

Next, we identify the lowest order term and determine $\ell_i$. For both of the expressions (51) and (52), the following holds: For $i \leq k_1$ the first term is lower than the second, for $i \leq n_1 + k_2 - k_1$ the second term is lower than the third, for $i \leq n_1 + n_2 - k_2$ the third term is lower than the last. The complete Higgs condition (46) yields $k_1 \leq n_1 + k_2 - k_1 \leq n_1 + n_2 - k_2$. Thus, the lowest order term is the first term for $1 \leq i \leq k_1$, the second term for $k_1 + 1 \leq i \leq n_1 + k_2 - k_1$, the third term for $n_1 + k_2 - k_1 + 1 \leq i \leq n_1 + n_2 - k_2$, and the last term for $n_1 + n_2 - k_2 + 1 \leq i \leq n_1 + n_2$. Note that $n_1 + k_2 - k_1 \leq n_1$ by the assumption $k_1 \geq k_2$. Thus,

[9] In the special case $n_1 + k_2 = 2k_1, n_2 + k_1 = 2k_2$, $f_i(x_i, z_i)$ factorizes into two, but the intersection takes place at a point away from $x = y = z = 0$ and is irrelevant for the dynamics of interest.

we have

$$
\ell_i = \begin{cases} i & i = 1, \ldots, k_1 \\ k_1 & i = k_1 + 1, \ldots, n_1 + k_2 - k_1 \\ n_1 + k_2 - i & i = n_1 + k_2 - k_1 + 1, \ldots, \min\{n_1, n_1 + n_2 - k_2\} \\ \begin{cases} k_2 & i = n_1 + 1, \ldots, n_1 + n_2 - k_2 \quad \text{if } n_1 \leq n_1 + n_2 - k_2 \\ 2n_1 + n_2 - 2i & i = n_1 + n_2 - k_2 + 1, \ldots, n_1 \quad \text{if } n_1 > n_1 + n_2 - k_2 \end{cases} \\ n_1 + n_2 - i & i = \max\{n_1, n_1 + n_2 - k_2\} + 1, \ldots, n_1 + n_2. \end{cases} \tag{54}
$$

The function $f_i(x_i, z_i)$ is of the following form

$$
f_i(x_i, z_i) = 1 + z_i + \mathcal{O}(x_i z_i), \qquad i = k_1, n_1 + k_2 - k_1, n_1 + n_2 - k_2 \tag{55}
$$

$$
f_i(x_i, z_i) = 1 + x_i + \mathcal{O}(x_i z_i), \qquad i = k_1 + 1, n_1 + k_2 - k_1 + 1, n_1 + n_2 - k_2 + 1 \tag{56}
$$

$$
f_i(x_i, z_i) = 1 + \mathcal{O}(x_i z_i), \qquad \text{otherwise.} \tag{57}
$$

(Here we assume that the three values of $i$ in (55) are well-separated. Other case can also be treated.) Thus, the curve $C$ intersects with $C_{k_1}$, $C_{n_1+k_2-k_1}$ and $C_{n_1+n_2-k_2}$ transversely.

In summary, the function (51)-(52) has zero at $C$ of order one and at $C_1, C_2, \ldots, C_{n_1+n_2-1}$ of order

$$
1, 2, \ldots, k_1 - 1, \underbrace{k_1, \ldots, k_1}_{n_1+k_2-2k_1+1}, k_1 - 1, \ldots, k_2 + 1, \underbrace{k_2, \ldots, k_2}_{n_2-k_2+1}, k_2 - 1, \ldots, 2, 1 \qquad \text{or}
$$

$$
1, 2, \ldots, \underbrace{k_1, \ldots, k_1}_{n_1+k_2-2k_1+1}, k_1 - 1, \ldots, 2k_2 - n_2 + 1, 2k_2 - n_2, 2k_2 - n_2 - 2, \ldots, n_2 + 2, n_2, n_2 - 1, \ldots, 2, 1
$$

if $n_2 \geq k_2$, or $n_2 < k_2$ respectively. Thus, we have identified the mirror gauge theory. The gauge group and the matter content are described by the quiver diagram in Figure 10. Here

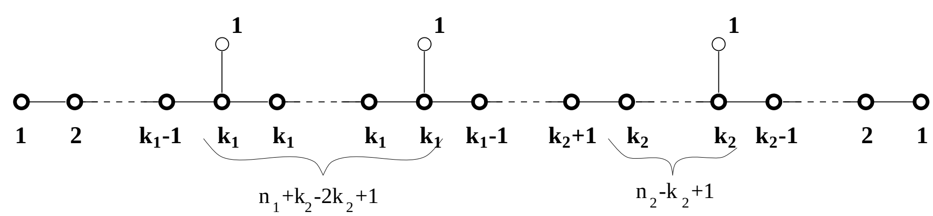



Figure 10: the mirror in the case $n_2 \geq k_2$

we present the mirror for the case $n_2 \geq k_2$. The mirror for the other case is obtained by an obvious replacement of the chain of ranks.

## 4.3 $Sp(k)$ gauge theory with $n$-fundamentals

In this subsection, we find the mirror of $Sp(k)$ gauge theory with $n$-fundamental hypermultiplets. We assume $n \geq 2k+2$.

It is straightforward to determine the zero and the order of $F = y - z^k$ in the resolved $D_n$ surface. Recall that the resolved $D_n$ surface is defined as a hypersurface (18)-(20) in a 3-fold covered by $n$ patches $\mathcal{U}_1, \ldots, \mathcal{U}_n$. Recall also that $y$ and $z$ are expressed in the patch $\mathcal{U}_i$ by (15) and (16)-(17). Let us look at the function $F$ in the $2j$-th patch $\mathcal{U}_{2j}$, $2j \leq n-3$. From the expression (15), we see that

$$F = s_{2j}^j z_{2j}^{j-1}(1 - s_{2j}^{k-j} z_{2j}^{k-j+1}) \qquad j = 1, \ldots, k-1, \tag{58}$$

$$F = s_{2k}^k z_{2k}^{k-1}(1 - z_{2k}) \qquad j = k, \tag{59}$$

$$F = s_{2j}^k z_{2j}^k (s_{2j}^{j-k} z_{2j}^{j-k-1} - 1) \qquad j = k+1, \ldots, [(n-3)/2] \tag{60}$$

The last factor has a single zero at a curve $C$ which extends to infinity. $F$ also has zeros at $s_{2j} = 0$ and $z_{2j} = 0$. We now recall the defining equation of the surface

$$s_{2j} + t_{2j}^2 z_{2j} = s_{2j}^{n-1-2j} z_{2j}^{n-2j}.$$

We see that there are two branches of zeros of $F$ for each $j$: $s_{2j} = z_{2j} = 0$ and $s_{2j} = t_{2j} = 0$ which corresponds to the rational curves $C_{2j-1}$ and $C_{2j}$ respectively. Near the first branch $s_{2j} = z_{2j} = 0$, $(t_{2j}, z_{2j})$ is a good coordinate, i.e. $s_{2j}$ can be uniquely expressed in terms of $t_{2j}$ and $z_{2j}$ by the defining equation. Since $t_{2j} \neq 0$ generically, $s_{2j} \sim z_{2j}$ near $C_{2j-1}$. Hence $F \sim z_{2j}^j z_{2j}^{j-1} = z_{2j}^{2j-1}$ for $j \leq k$ while $F \sim z_{2j}^{2k}$ for $j > k$. Thus, the zero of $F$ at $C_{2j-1}$ is of order $2j-1$ for $j \leq k$ and order $2k$ for $j > k$. Near the second branch $s_{2j} = t_{2j} = 0$, $(t_{2j}, z_{2j})$ is again a good coordinate, and $s_{2j} \sim t_{2j}^2$ for $z_{2j} \neq 0$. Thus, $F \sim t_{2j}^{2j}$ for $j \leq k$ and $F \sim t_{2j}^{2k}$ for $j > k$ near $C_{2j}$. Namely, $F$ has a zero at $C_{2j}$ of order $2j$ for $j \leq k$ and $2k$ for $j > k$. By looking at the equation for $j = k$, we see that the infinite curve $C$ and the rational curve $C_{2k}$ meet at the point $s_{2k} = t_{2k} = 0, z_{2k} = 1$. For $\mathcal{U}_{2j-1}$ the analysis is similar (although we need some care for the factorization of $F$). In summary, in the part of the surface in the patches $\mathcal{U}_1, \ldots, \mathcal{U}_{n-3}$, $F$ has zeros at $C_1, C_2, \ldots, C_{2k-1}, C_{2k}, C_{2k+1}, \ldots, C_{n-3}$ and $C$ of order $1, 2, \ldots, 2k-1, 2k, 2k, \ldots, 2k$ and 1 respectively.

Let us now look at the function $F$ in $\mathcal{U}_{n-2}$. By looking at the expressions (16)-(17) carefully, we see that $y$ is divisible by $z^k$, and $y/z^k - 1$ of $F = z^k(y/z^k - 1)$ has a single zero at $C$.

Now, we consider the zero of $z^k = (s_{n-2}t_{n-2}z_{n-2})^k$. By looking at the defining equation of the surface

$$s_{n-2}(z_{n-2}^2 - 1) = t_{n-2}z_{n-2}$$

we see that there are four branches of zero: $s_{n-2} = z_{n-2} = 0$, $s_{n-2} = t_{n-2} = 0$, $t_{n-2} = z_{n-2} - 1 = 0$ and $t_{n-2} = z_{n-2} + 1 = 0$, which corresponds to $C_{n-3}$, $C_{n-2}$, $C_{n-1}$ and $C_n$ respectively. Near $C_{n-3}$ where $s_{n-2} = z_{n-2} = 0$ and $t_{n-2} \neq 0$, the surface is coordinatized by $(t_{n-2}, z_{n-2})$ and $F \sim s_{n-2}^k z_{n-2}^k \sim z_{n-2}^{2k}$ has zero at $C_{n-3}$ of order $2k$, as we have seen. Near $C_{n-2}$ where $s_{n-2} = t_{n-2} = 0$ and $z_{n-2} \neq 0$, the surface is again coordinatized by $(t_{n-2}, z_{n-2})$ and $F \sim s_{n-2}^k t_{n-2}^k \sim t_{n-2}^{2k}$ has zero at $C_{n-2}$ od order $2k$. Near $C_{n-1}$ or $C_n$ where $t_{n-2} = z_{n-2} \mp 1 = 0$ and $s_{n-2} \neq 0$, the surface is coordinatized by $(s_{n-2}, z_{n-2})$, and $F \sim t_{n-2}^k \sim (z_{n-2} \mp 1)^k$ has zero at $C_{n-1}$ and $C_n$ of order $k$. The zero of $F$ in the part in $\mathcal{U}_{n-1}$ and $\mathcal{U}_n$ can be seen in the same way, but it turns out that there is no additional zero than what we have found.

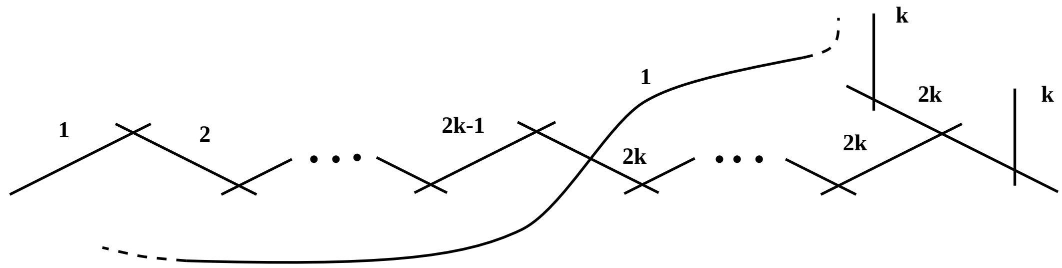



Figure 11: the zero and the order of $F$

In summary, $F$ has zeros at $C_1, C_2, \ldots, C_{2k-1}, C_{2k}, C_{2k+1}, \ldots, C_{n-3}, C_{n-2}, C_{n-1}, C_n$ and $C$ of order $1, 2, \ldots, 2k-1, 2k, 2k, \ldots, 2k, 2k, k, k$ and 1 respectively. The curves $C_1, \ldots, C_n$ are rational curves of finite volume whose intersection is dictated by the $D_n$ Dynkin diagram, while $C$ extends to infinity. The curves $C$ and $C_{2k}$ intersects transversely. Thus, we have identified the mirror gauge theory. It is given by the quiver diagram in Figure 12.

## 4.4 Compactifications of Exceptional Tensionless String Theories

In this subsection, we find mirrors of theories with global $E_{6,7,8}$ symmetry corresponding to compactification of theories with small $E_{6,7,8}$ instantons down to three dimensions and show

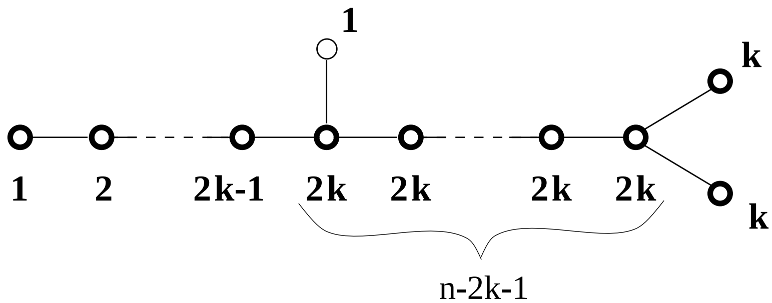


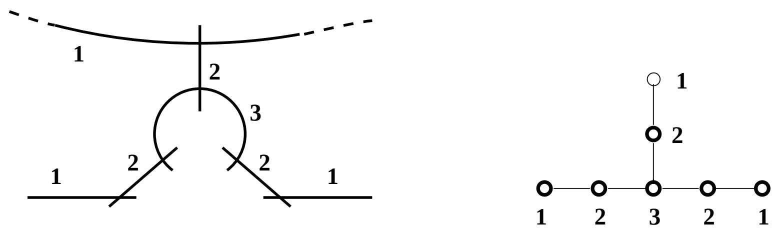

Figure 12: the mirror of the $Sp(k)$ gauge theory with $n$ fundamentals

that they are ordinary gauge systems, as anticipated in [29]. As noted before this is a rather interesting example in that it dualizes a gauge system to another quantum field theory which is expected not to have an ordinary lagrangian description.[10] As in the previous cases, we only have to determine the zero and the order of the function $F = z$ in the resolved $E_n$ surfaces described in Section 3.3. Thus, we follow the notation of the suitable part in that section.

*$E_6$ Theory*

It is a straightforward matter to see the zero and the order of $F = z$ if we look at the expression (21) for $z$. For example, $z = y_1 z_1$ in $\mathcal{U}_1$, and thus it has zero at $z_1 = 0$ and $y_1 = 0$. By the defining equation of the surface (22), the zero locus consists of the curve $z_1 = x_1^2 + y_1 = 0$ which we denote by $C$, and the locus $C_1$ of $x_1 = y_1 = 0$. Note that the curve $C$ extends to infinity while $C_1$ is a rational curve with finite volume. They intersect at one point $x_1 = y_1 = z_1 = 0$. Near $C$ where generically $y_1 \neq 0$, the surface is coordinatized by $(x_1, z_1)$ and $z \sim z_1$ has zero at $C$ of order 1. Near $C_1$ where $x_1 = y_1 = 0$ and $z_1 \neq 0$, the surface is coordinatized again by $(x_1, z_1)$ and $z \sim y_1 \sim x_1^2$ has zero at $C_1$ of order 2. The zero and the order of $z$ in other patches $\mathcal{U}_2, \ldots, \mathcal{U}_5$ can be determined in the same way. In summary, $F = z$ has zeros at $C_1, C_2, C_{3+}, C_{3-}, C_{4+}, C_{4-}$ and $C$ of order $2, 3, 2, 2, 1, 1$ and 1 respectively (see Figure 13 (a)). The curve $C$ intersects with $C_1$ at one point and extends to infinity. The way these curves intersect is dictated by the affine $E_6$ Dynkin diagram, where the affine node corresponds to the infinite curve $C$. Thus, the mirror theory is a gauge theory whose gauge and matter content is as given by the quiver diagram in Figure 13.

[10]That the critical E-theories do not have a lagrangian description (with finite parameters) is strictly speaking not proven. One can at least rule that out as far as ordinary gauge systems with matter are concerned.




(a) (b)

Figure 13: (a) depicts the zero and the order of $F$ for the $E_6$ theory. (b) is the quiver diagram showing the mirror gauge theory.

*$E_7$ Theory*

The zero and the order of $F = z$ can be seen by looking at the expression for $z$ in (28). In $\mathcal{U}_7$, $z = z_7$ and it has zero at $z_7 = 0$. By looking at the defining equation of the surface (29), we see that there are two components: $x_7 = z_7 = 0$ which is the rational curve $C_7$, and the curve $1 + x_7 y_7^3 = z_7 = 0$ which we denote by $C$. The latter curve $C$ extends to infinity and does not intersect with $C_7$. Since $dz_1 \neq 0$ in the surface near both $C$ and $C_7$, $z = z_1$ has single zeros at $C$ and $C_7$. In the part of the surface in the patches $\mathcal{U}_1, \ldots, \mathcal{U}_5$, we must look at $y_6$ which is equal to "$y$" in the formulae (21) for $E_6$ case. For example in $\mathcal{U}_5$, $F = z$ is given by $y_6 = y_5 z_5^2$. Thus, it has zero at $z_5 = 0$ or $y_5 = 0$. By the defining equation (26), the zero locus is $C_{4\pm}$ given by $z_5 = x_5 \mp i = 0$ in the former case, while it is the curves defined by $y_5 = x^5 \mp i = 0$ in the latter case. One of the latter curves $y_5 = x^5 - i = 0$ is the rational curve $C_7$. The other one $y_5 = x^5 + i = 0$ is actually the infinite curve $C$, as can be seen by looking at the relations $x_7 = \sqrt{2i} z_5^3 (x_5 - i), y_7 = 1/(\sqrt{2i})$. The rational curves $C_7$ and $C_{4+}$ intersect at one point as we have seen in Section 3.3, while the curve $C$ intersects only with $C_{4-}$ at one point $x_5 + i = y_5 = z_5 = 0$. Near $C_{4\pm}$ where $z_5 = x_5 \mp i = 0$ and $y_5 \neq 0$ generically, the surface is coordinatized by $(y_5, z_5)$ and $y_6 = y_5 z_5^2 \sim z_5^2$ has zero at $C_{4\pm}$ of order 2. The zero and the order of $z = y_6$ in other patches can be determined in the same way. In summary, $F = z$ has zeros at $C_1, C_2, C_{3+}, C_{3-}, C_{4+}, C_{4-}, C_7$ and $C$ of order $2, 4, 3, 3, 2, 2, 1$ and 1 respectively (see Figure 14 (a)). The curve $C$ intersects with $C_{4-}$ at one point and extends to infinity. The intersection of these curves is dictated by the affine $E_7$ Dynkin diagram where the infinite curve $C$ corresponds to the affine node. Thus, the mirror theory is a gauge theory whose gauge and matter content is as given by the quiver diagram in Figure 14.

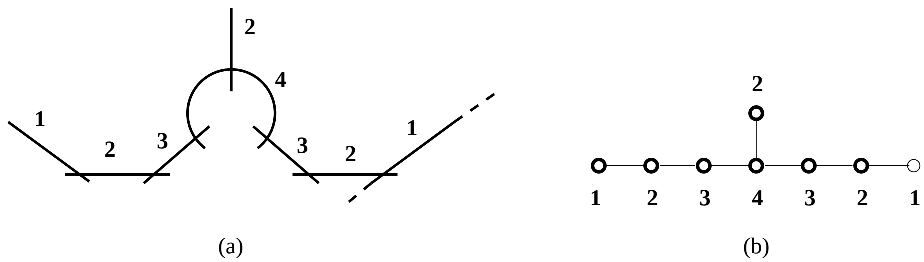



Figure 14: (a) depicts the zero and the order of $F$ for the $E_7$ theory. (b) is the quiver diagram showing the mirror gauge theory.

*$E_8$ Theory*

For $E_8$ theory, we must look at the expression of $z$ in (30). In the patch $\mathcal{U}_8$, $F = y_8 z_8$ has a single zero at the curve $C$ defined by $z_8 = x_8^2 + y_8 = 0$, and also a double zero at the rational curve $C_8$. The curve $C$ extends to infinity and intersects with $C_8$ at one point $x_8 = y_8 = z_8 = 0$. The zero and the order of $F = x_7 y_7 = x_6 - i z_6^2$ in other patches can be determined in the same way without much effort (for the expression of $x_6$ and $z_6$, use the formulae for $x$ and $z$ in (21)). In summary, $F = z$ has zeros at $C_1, C_2, C_{3+}, C_{3-}, C_{4+}, C_{4-}, C_7, C_8$ and $C$ of order $3, 6, 5, 4, 4, 2, 3, 2$ and $1$ respectively (see Figure 15 (a)). The intersection of these curves is given by the affine $E_8$ Dynkin diagram where the affine node corresponds to the infinite curve $C$. Thus, the mirror theory is a gauge theory whose gauge and matter content is as given by the quiver diagram in Figure 15.

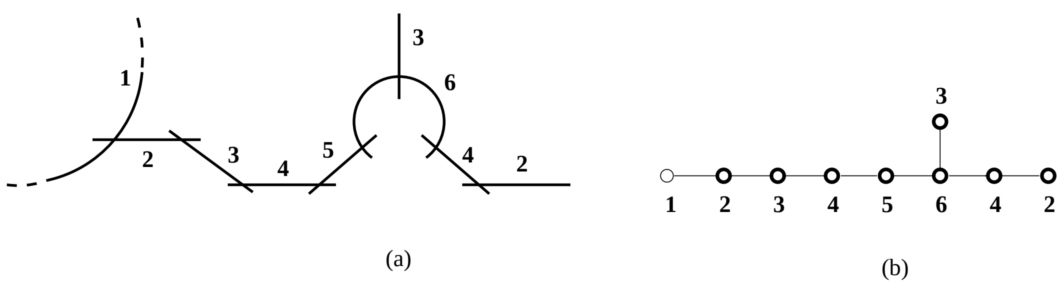



Figure 15: (a) depicts the zero and the order of $F$ for the $E_8$ theory. (b) is the quiver diagram showing the mirror gauge theory.

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