On the rate loss and construction of source codes for broadcast channels

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Abstract— **In this paper, we first define and bound the rate loss of source codes for broadcast channels. Our broadcast channel model comprises one transmitter and two receivers; the transmitter is connected to each receiver by a private channel and to both receivers by a common channel. The transmitter sends a description of source (X, Y) through these channels, receiver 1 reconstructs X with distortion D_1, and receiver 2 reconstructs Y with distortion D_2. Suppose the rates of the common channel and private channels 1 and 2 are R_0, R_1, and R_2, respectively. The work of Gray and Wyner gives a complete characterization of all achievable rate triples (R_0, R_1, R_2) given any distortion pair (D_1, D_2). In this paper, we define the rate loss as the gap between the achievable region and the outer bound composed by the rate-distortion functions, i.e., $R_0+R_1+R_2 \geq R_{X,Y}(D_1, D_2)$, $R_0 + R_1 \geq R_X(D_1)$, and $R_0 + R_2 \geq R_Y(D_2)$. We upper bound the rate loss for general sources by functions of distortions and upper bound the rate loss for Gaussian sources by constants, which implies that though the outer bound is generally not achievable, it may be quite close to the achievable region. This also bounds the gap between the achievable region and the inner bound proposed by Gray and Wyner and bounds the performance penalty associated with using separate decoders rather than joint decoders. We then construct such source codes using entropy-constrained dithered quantizers. The resulting implementation has low complexity and performance close to the theoretical optimum. In particular, the gap between its performance and the theoretical optimum can be bounded from above by constants for Gaussian sources.**

I. Introduction

A broadcast system is a network in which a single sender is transmitting messages to a set of receivers simultaneously. The messages transmitted by the sender may include "common information" for several receivers and "private information" for each individual receiver. Broadcast systems play an important role in daily communications, e.g., a satellite sending TV programs or XM radio to a variety of users with different subscriptions, or a wireless base station updating traffic information to many hand-held devices from different manufacturers. In networks such as next generation wireless communication systems, where resources such as power and bandwidth are critically limited compared to the amount of information (e.g.,wireless multimedia, wireless internet access, or video) to be sent, it is important to compress data maximally before transmission. Zhao and Effros propose a practical broadcast system source code (BSSC) design and demonstrate the performance gain achieved by applying BSSCs rather than the traditional source code [1], [2], [3].

In [4], Gray and Wyner study source coding for the simple broadcast channel model shown in Figure 1. The model contains one transmitter and two receivers; the transmitter is connected to each receiver by a private channel and to both receivers by a common channel. The performance of a BSSC is given by $(R_0, R_1, R_2, D_1, D_2)$, where R_i and D_i are the expected rate for private channel i and distortion of the reproduction of receiver $i \in \{1, 2\}$, respectively, and R_0 is the expected rate for the common channel. The following theorem is the central result of [4].

Theorem 1: [4, Theorem 1] For any iid vector source $\{X_i, Y_i\}_{i=1}^{\infty}$ with density $f_{X,Y}(x, y)$ and distortion measure d, $(R_0, R_1, R_2, D_1, D_2)$ is BSSC-achievable if and only if there exists a conditional probability $Q_{W|X,Y}$ such that

$$\begin{cases} R_0 \geq I(X, Y; W), \\ R_1 \geq R_{X|W}(D_1), \\ R_2 \geq R_{Y|W}(D_2), \end{cases}$$

where $R_{X|W}(D)$ and $R_{Y|W}(D)$ are conditional rate-distortion functions.

Unfortunately, calculating the full achievable region is nontrivial even for simple sources due to the difficulty in the characterization of the random variable W.

The *rate loss*, which is the difference between the achievable rate and the corresponding rate-distortion function, has become a powerful performance analysis tool for network source codes (e.g., [5], [6], [7], [8], [9]). In this paper, we extend the rate loss definition to BSSCs. In particular, we define the rate loss of a BSSC achieving performance $(R_0, R_1, R_2, D_1, D_2)$ in the limit of large coding dimension as the vector (L_0, L_1, L_2), where $L_0 = R_0+R_1+R_2-R_{X,Y}(D_1, D_2)$, $L_1 = R_0+R_1-R_X(D_1)$, $L_2 = R_0+R_2-R_Y(D_2)$. Here $R_{X,Y}(D_1, D_2)$ is the joint rate-distortion function for source (X, Y), and $R_X(D_1)$ and $R_Y(D_2)$ are the rate-distortion functions for X and Y, respectively.

This paper describes source-independent upper bounds for the BSSC rate loss. Rate loss bounds are useful for several reasons. First, they describe the performance degradation associated with using the given code rather than the best traditional code with the same distortion(s). For example, L_0 describes

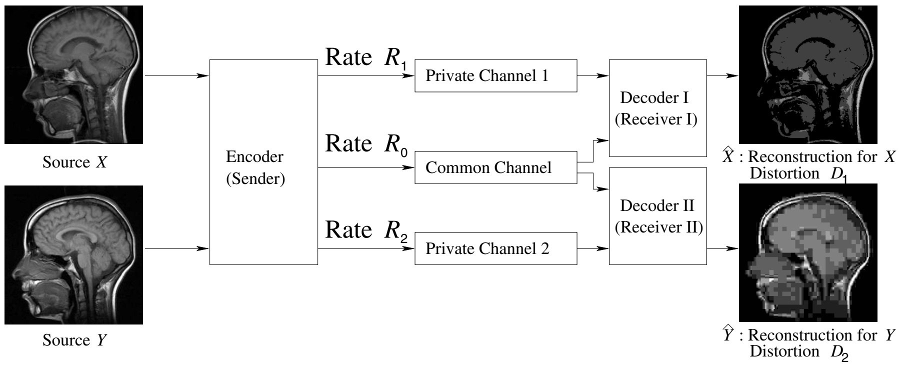

Fig. 1. A Broadcast System Source Code (BSSC). Two sources X and Y are encoded by the sender and transmitted through three channels. Receiver 1 receives messages from private channel 1 with rate R_1 bits per symbol (bps) and the common channel with rate R_0 bps and builds a reconstruction $\hat{X}$ for X with distortion D_1. Receiver 2 decodes messages from private channel 2 with R_2 bps and the common channel with rate R_0 bps and yields a reproduction $\hat{Y}$ for Y with distortion D_2.

the cost in total rate of using private channels in addition to the common channel rather than sending all the messages through the common channel. Second, they give new achievability results that provide elegant and often tight inner bounds on the region of achievable rate-distortion vectors. These inner bounds are definitely achievable but not necessarily the whole achievable region. They are far simpler to analyze than existing alternatives, for which solution requires a complex multidimensional optimization for every source and every distortion pair (D_1, D_2). Further, they often provide insight on the low-complexity near-optimal code design based on entropy constrained dithered quantization (ECDQ) (e.g., [10] and [11]).

ECDQ [12] is dithered uniform or lattice quantization followed by universal lossless entropy coding. ECDQ has low complexity and achieves single-resolution source coding performance that is provably close to the theoretical optimum [12], [13]. In this paper, we generalize the ECDQ algorithm to allow the design of the BSSC. We demonstrate the potential of the BSSC using ECDQ (BSSC-ECDQ) by deriving theoretical performance bounds. Paralleling the rate loss definition, we define the rate redundancy of a BSSC-ECDQ as the difference between its rates (measured as entropy) and the corresponding rate-distortion functions, i.e., the rate redundancy of a BSSC-ECDQ achieving rate distortion performance $(R_0, R_1, R_2, D_1, D_2)$ is the vector (C_0, C_1, C_2), where $C_0 = R_0 + R_1 + R_2 - R^n_{\mathbf{X},\mathbf{Y}}(D_1, D_2)$, $C_1 = R_0 + R_1 - R^n_{\mathbf{X}}(D_1)$, $C_2 = R_0 + R_2 - R^n_{\mathbf{Y}}(D_2)$. Here $R^n_{\mathbf{X},\mathbf{Y}}(D_1, D_2), R^n_{\mathbf{X}}(D_1), R^n_{\mathbf{Y}}(D_2)$ are the n-th order rate-distortion functions for stationary source $(\mathbf{X}, \mathbf{Y})$, and n is the block length of the universal lossless entropy encoder [13]. We then upper bound the rate redundancy of the BSSC-ECDQ. This result also leads to an achievable region for BSSCs on stationary sources.

II. Rate Loss Bounds

For notational simplicity, assume (without loss of generality) that $E(X) = 0$ and $E(Y) = 0$. Further assume that the source (X, Y) is iid, that $d(x, \hat{x}) = (x-\hat{x})^2$ (the mean squared error (mse) distortion measure), that the differential entropies $h(X)$ and $h(Y)$ are finite, and that $0 < D_1 < \sigma_X^2 < \infty$ and $0 < D_2 < \sigma_Y^2 < \infty$.

For each fixed distortion pair (D_1, D_2), bounding the rate loss is equivalent to comparing the achievable rate region

$$\begin{aligned}\mathcal{R}(D_1, D_2) = \{&(R_0, R_1, R_2) | \\ &(R_0, R_1, R_2, D_1, D_2) \text{ is BSSC-achievable}\}.\end{aligned}$$

to the natural outer bound (or converse) composed of rate-distortion functions

$$\begin{aligned}\mathcal{R}_{\text{out}} &= \{(R_0, R_1, R_2) | \\ &\quad R_0 + R_1 + R_2 \geq R_{X,Y}(D_1, D_2), \\ &\quad R_0 + R_1 \geq R_X(D_1), R_0 + R_2 \geq R_Y(D_2)\}\end{aligned}$$

as illustrated in Figure 2. By the point-to-point rate-distortion theory, any point outside this outer bound is not achievable. The outer bound is composed of three planes. For any rate triple (R_0, R_1, R_2), the rate loss L_0 represents the gap between the rate triple and the plane designated $ACED$ ($R_0 + R_1 + R_2 = R_{X,Y}(D_1, D_2)$); the rate loss L_1 represents the gap between the rate triple and the plane designated DEB ($R_0 + R_1 = R_X(D_1)$); and the rate loss L_2 represents the gap between the rate triple and the plane designated CEB ($R_0 + R_2 = R_Y(D_2)$). Even for points on this outer bound, some rate loss can be made arbitrarily large. For example, for point A, $L_0 = 0$, but L_1 and L_2 can be arbitrarily large. Therefore, for any achievable rate triple (R_0, R_1, R_2), we need only bound $L = \min\{L_0, L_1, L_2\}$ since if any of L_0, L_1, and L_2 is small, then this achievable rate triple is close to one of the three planes, and thus close to $\mathcal{R}_{\text{out}}$. Bounding $L = \min\{L_0, L_1, L_2\}$ for all points on the lower boundary of the achievable region bounds the gap between the achievable region and $\mathcal{R}_{\text{out}}$.

Because of limited space, we only outline the proofs in this paper. Generally speaking, our proofs involve finding a Gaussian approximation of the reconstruction variable W in Theorem 1. Here, by choosing $W = (X + N_3, Y + N_4)$, where $N_3 \sim \mathcal{N}(0, D_3)$, $N_4 \sim \mathcal{N}(0, D_4)$, $N_3 \perp\!\!\!\perp N_4$,

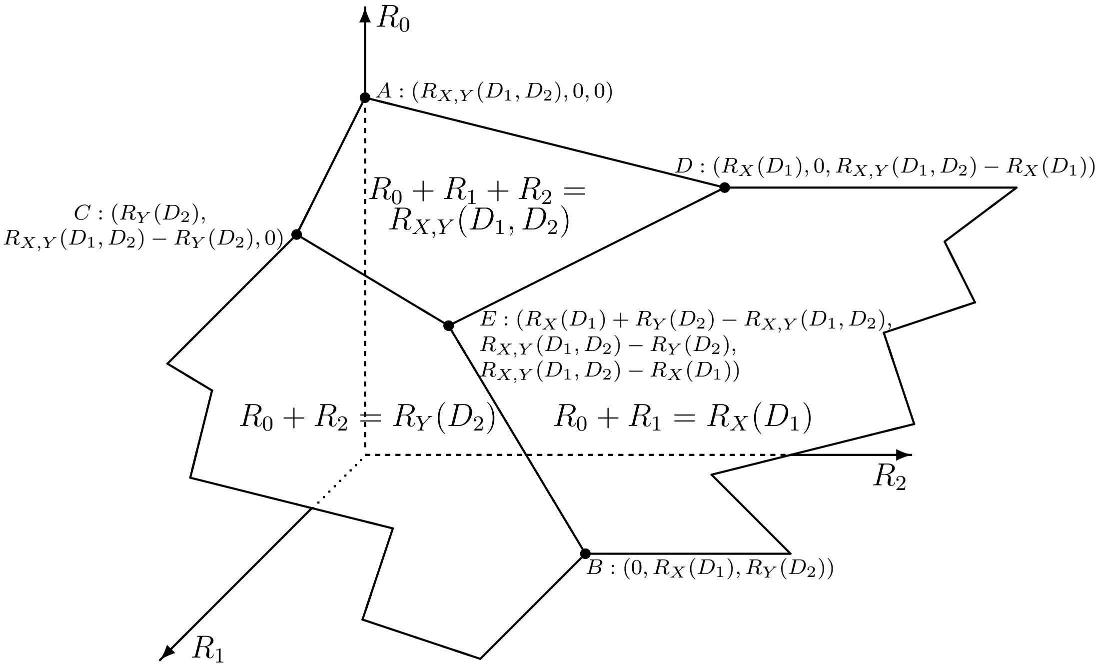

Fig. 2. The outer bound $\mathcal{R}_{\text{out}}$.

$(N_3, N_4) \perp\!\!\!\perp (X, Y)$, $D_3 \geq D_1$, and $D_4 \geq D_2$,[1] we can see that the rate triple

$$
\begin{aligned}
&(R_0(D_3, D_4), R_1(D_3, D_4), R_2(D_3, D_4)) = \\
&\quad (I(X, Y; X + N_3, Y + N_4), \\
&\quad I(X; X + N_1 | X + N_3, Y + N_4), \\
&\quad I(Y; Y + N_2 | X + N_3, Y + N_4)) \qquad (1)
\end{aligned}
$$

is achievable from Theorem 1, where $N_1 \sim (0, D_1)$, $N_2 \sim (0, D_2)$, $N_1 \perp\!\!\!\perp N_2$, and $(N_1, N_2) \perp\!\!\!\perp (X, Y, N_3, N_4)$. Then we bound the rate loss by bounding the gap between the outer bound $\mathcal{R}_{\text{out}}$ and this inner bound

$$
\begin{aligned}
\mathcal{R}_1 = &\{(R_0, R_1, R_2) | R_0 = R_0(D_3, D_4), \\
&R_1 = R_1(D_3, D_4), R_2 = R_2(D_3, D_4), D_3 \geq D_1, \\
&\text{and } D_4 \geq D_2\}.
\end{aligned}
$$

First we can show that points A and B in Figure 2 are always achievable, and there exist some achievable rate triples that are close to points C and D with $L_0 \leq 1$. The key to our proof is to find a good distortion pair (D_3, D_4) such that $D_3 \geq D_1$, $D_4 \geq D_2$, and the resulting rate triple $(R_0(D_3, D_4), R_1(D_3, D_4), R_2(D_3, D_4))$ defined in (1) is close to point E. Then we use the convexity of the achievable region to show that all other rate triples in (1) are almost as close to the outer bound.

[1]We use notation $A \sim \mathcal{N}(0, \sigma^2)$ to specify that A is a Gaussian random variable with mean 0 and variance σ^2 and notation $A \perp\!\!\!\perp B$ to specify that random variables A and B are independent. In addition, we use notation $(R_0(D_3, D_4), R_1(D_3, D_4), R_2(D_3, D_4))$ to represent the special achievable rate triple (R_0, R_1, R_2) when $W = (X + N_3, Y + N_4)$.

We first bound the rate loss for Gaussian sources. The following theorem states that for any Gaussian source, the entire lower boundary of the achievable region is always between $\mathcal{R}_{out}$ and the surface produced by shifting $\mathcal{R}_{out}$ upward by 2.21 bps.

Theorem 2: Given any iid Gaussian source (X, Y) and any distortion pair (D_1, D_2), for any rate triple (R_0, R_1, R_2) on the lower boundary of $\mathcal{R}(D_1, D_2)$, $L = \min\{L_0, L_1, L_2\} < 2.21$ bps.

For general sources, we derive the following distortion-dependent rate loss bounds.

Theorem 3: Given any iid source (X, Y) and any distortion pair (D_1, D_2), for any rate triple (R_0, R_1, R_2) on the lower boundary of $\mathcal{R}(D_1, D_2)$,

$$
L = \min\{L_0, L_1, L_2\} \leq \frac{1}{2} \log 3 + \frac{1}{2} \log(1 + 2/h(d_1, d_2)),
$$

where $d_1 = D_1/\sigma_X^2$, $d_2 = D_2/\sigma_Y^2$, $d = \max\{d_1, d_2\}$, and $h(d_1, d_2)$ is defined as

$$
h(d_1, d_2) = \begin{cases} \sqrt{4d - 3d^2} - d, & \text{if } d < 1/3 \\ 2/3, & \text{otherwise.} \end{cases}
$$

This bound is primarily good for low resolution (or large distortions D_1 and D_2). For example, if $D_1 \geq \sigma_X^2/13$ and $D_2 \geq \sigma_Y^2/13$, then $L = \min\{L_0, L_1, L_2\} \leq 2$ bps for any point on the lower boundary of $\mathcal{R}(D_1, D_2)$ and any source.

III. The Relationship Between the Rate Loss Bound and the Inner Bound Derived in [4]

In bounding the rate loss, we are focusing on the inner bound $\mathcal{R}_1$. In this section, we consider the relationship between $\mathcal{R}_1$ and the inner bound $\mathcal{R}_2$ derived in [4], which is defined as

$$\begin{aligned}\mathcal{R}_2 &= \{(R_0, R_1, R_2) | R_0 = R_{X,Y}(D_3, D_4), \\ &\quad R_1 = R_{X|X_3,Y_4}(D_1), R_2 = R_{Y|X_3,Y_4}(D_2), \\ &\quad D_3 \geq D_1, D_4 \geq D_2, \text{ and } (X_3, Y_4) \text{ are the} \\ &\quad \text{random variables achieving } R_{X,Y}(D_3, D_4)\}.\end{aligned}$$

The following theorem demonstrates that these two inner bounds are close.

Theorem 4: For any point $(R_0, R_1, R_2) \in \mathcal{R}_1$, there exists a point $(\hat{R}_0, \hat{R}_1, \hat{R}_2) \in \mathcal{R}_2$ such that $0 \leq R_0 - \hat{R}_0 \leq 1$, $-1 \leq R_1 - \hat{R}_1 \leq 1.5$, and $-1 \leq R_2 - \hat{R}_2 \leq 1.5$. For any point $(\hat{R}_0, \hat{R}_1, \hat{R}_2) \in \mathcal{R}_2$, there exists a point $(R_0, R_1, R_2) \in \mathcal{R}_1$ such that $0 \leq R_0 - \hat{R}_0 \leq 1$, $-1 \leq R_1 - \hat{R}_1 \leq 1.5$, and $-1 \leq R_2 - \hat{R}_2 \leq 1.5$.

IV. BSSC design using ECDQ

In this section, we apply ECDQ to BSSC design. Let $(\mathbf{X}, \mathbf{Y})$ be the stationary vector source with variances (σ_X^2, σ_Y^2) and correlation coefficient ρ, i.e., for any integer i, $E(X_i^2) = \sigma_X^2$, $E(Y_i^2) = \sigma_Y^2$, and $E(X_i Y_i) = \rho \sigma_X \sigma_Y$. Further, let $Q_i(\cdot)$ be a K-dimensional lattice quantizer and dither $\mathbf{Z}_i$ be an n-dimensional vector composed of n/K K-dimensional random vectors independently and uniformly distributed on the basic cell of $Q_i(\cdot)$ for $i \in \{1, 2, 3, 4\}$. (Note that the basic cells of all the quantizers $Q_i(\cdot)$ $(i \in \{1, 2, 3, 4\})$ have the same shape with possibly different sizes.) In addition, $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, and $\mathbf{Z}_4$ are independent of each other and independent of $(\mathbf{X}, \mathbf{Y})$, and are known to both the transmitter and the receivers. Further, the universal entropy encoders use block length n, and K divides n. We consider two strategies in BSSC-ECDQ design, both use sequential coding schemes.

A. Strategy 1

First, the transmitter describes the joint entropy coded descriptions of $Q_3(\mathbf{X} + \mathbf{Z}_3)$ and $Q_4(\mathbf{Y} + \mathbf{Z}_4)$ (With a little abuse of notations, we also use Q_3 and Q_4 to represent these quantized values) conditioned on $(\mathbf{Z}_3, \mathbf{Z}_4)$ to both receivers (as the common information). Let the variance of $\mathbf{Z}_i$ be D_i, $i \in \{1, 2, 3, 4\}$. Based on $Q_3(\mathbf{X} + \mathbf{Z}_3)$ and $Q_4(\mathbf{Y} + \mathbf{Z}_4)$, both the transmitter and the two receivers can build partial reconstructions $\mathbf{X}_3 = Q_3(\mathbf{X} + \mathbf{Z}_3) - \mathbf{Z}_3$ and $\mathbf{Y}_4 = Q_4(\mathbf{Y} + \mathbf{Z}_4) - \mathbf{Z}_4$, resulting in distortions D_3 and D_4 for $\mathbf{X}$ and $\mathbf{Y}$, respectively. Define $\hat{\mathbf{X}}_3 = \mathbf{X} - \mathbf{X}_3$ and $\hat{\mathbf{Y}}_4 = \mathbf{Y} - \mathbf{Y}_4$. The transmitter then transmits the entropy coded description of $Q_1(\hat{\mathbf{X}}_3 + \mathbf{Z}_1)$ conditioned on $\mathbf{Z}_1$ and common information $Q_3, Q_4, \mathbf{Z}_3, \mathbf{Z}_4$ to receiver 1. Receiver 1 reconstructs $\mathbf{X}_1 = \mathbf{X}_3 + Q_1(\hat{\mathbf{X}}_3 + \mathbf{Z}_1) - \mathbf{Z}_1$. Similarly, the transmitter transmits the entropy coded description of $Q_2(\hat{\mathbf{Y}}_4 + \mathbf{Z}_2)$ conditioned on $\mathbf{Z}_2$ and common information $Q_3, Q_4, \mathbf{Z}_3, \mathbf{Z}_4$ to receiver 2, and receiver 2 reconstructs $\mathbf{Y}_2 = \mathbf{Y}_4 + Q_2(\hat{\mathbf{Y}}_4 + \mathbf{Z}_2) - \mathbf{Z}_2$.

Notice that $D_i = E(Z_i^2) = E(\mathbf{X} - \mathbf{X}_i)^2$ $(i = 1, 3)$ and $D_i = E(Z_i^2) = E(\mathbf{Y} - \mathbf{Y}_i)^2$ $(i = 2, 4)$. Further, let $\mathbf{N}_i$ be distributed as $-\mathbf{Z}_i$, $i = 1, 2, 3, 4$, then following [14], the rates are

$$\begin{aligned}R_0 &= \frac{1}{n} I(\mathbf{X}, \mathbf{Y}; \mathbf{X} + \mathbf{N}_3, \mathbf{Y} + \mathbf{N}_4) \\ R_1 &= \frac{1}{n} I(\mathbf{X}; \mathbf{X} + \mathbf{N}_1 | \mathbf{X} + \mathbf{N}_3, \mathbf{Y} + \mathbf{N}_4) \\ R_2 &= \frac{1}{n} I(\mathbf{Y}; \mathbf{Y} + \mathbf{N}_2 | \mathbf{X} + \mathbf{N}_3, \mathbf{Y} + \mathbf{N}_4).\end{aligned}$$

Note that the rates here resemble the rates in the definition of $\mathcal{R}_1$, except that random vectors replace random variables and uniformly distributed random vectors $(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4)$ replace Gaussian random variables (N_1, N_2, N_3, N_4).

Similar to the rate loss analysis, we only need to bound $C = \min\{C_0, C_1, C_2\}$ for any rate triple. Analyzing the rate redundancy leads to the following theorem.

Theorem 5: For arbitrary stationary source $(\mathbf{X}, \mathbf{Y})$ and distortion pair (D_1, D_2), the rate redundancy of BSSC-ECDQ can be bounded by

$$C \leq 2 \log(2\pi e G_K) + \frac{1}{2} \log 3 + \frac{1}{2} \log(1 + 2/h(d_1, d_2)),$$

where $d_1 = D_1/\sigma_X^2$, $d_2 = D_2/\sigma_Y^2$, $d = \max\{d_1, d_2\}$, and $h(d_1, d_2)$ is defined as

$$h(d_1, d_2) = \begin{cases} \sqrt{4d - 3d^2} - d, & \text{if } d < 1/3 \\ 2/3, & \text{otherwise.} \end{cases}$$

For an arbitrarily iid Gaussian source, the rate redundancy $C = \min\{C_0, C_1, C_2\}$ associated with using scalar quantizers can always be bounded from above by 5.74 bps.

B. Strategy 2

We can further explore the correlation between $\mathbf{X}$ and $\mathbf{Y}$ to build more sophisticated reconstructions. Assume that the variance of stationary $\mathbf{Z}_i$ is σ_i^2 for $i \in \{1, 2, 3, 4\}$. Again, the transmitter first describes the joint entropy coded description of $Q_3(\mathbf{X} + \mathbf{Z}_3)$ and $Q_4(\mathbf{Y} + \mathbf{Z}_4)$ conditioned on $(\mathbf{Z}_3, \mathbf{Z}_4)$ to both receivers (as the common information). Let α_i for $i \in \{1, 2, 3, 4\}$ and β_i for $i \in \{3, 4\}$ be constants chosen later. Based on the received common information, the two receivers can build partial reconstructions $\mathbf{X}_3 = \alpha_3(Q_3(\mathbf{X} + \mathbf{Z}_3) - \mathbf{Z}_3) + \beta_3(Q_4(\mathbf{Y} + \mathbf{Z}_4) - \mathbf{Z}_4)$ with distortion D_3 and $\mathbf{Y}_4 = \beta_4(Q_4(\mathbf{X} + \mathbf{Z}_3) - \mathbf{Z}_3) + \alpha_4(Q_4(\mathbf{Y} + \mathbf{Z}_4) - \mathbf{Z}_4)$ with distortion D_4. Again, define $\hat{\mathbf{X}}_3 = \mathbf{X} - \mathbf{X}_3$ and $\hat{\mathbf{Y}}_4 = \mathbf{Y} - \mathbf{Y}_4$. The transmitter then transmits the entropy coded description of $Q_1(\hat{\mathbf{X}}_3 + \mathbf{Z}_1)$ conditioned on $\mathbf{Z}_1$ and common information $Q_3, Q_4, \mathbf{Z}_3, \mathbf{Z}_4$ to receiver 1, which reconstructs $\mathbf{X}_1 = \mathbf{X}_3 + \alpha_1(Q_1(\hat{\mathbf{X}}_3 + \mathbf{Z}_1) - \mathbf{Z}_1)$. Similarly, the transmitter transmits the entropy coded description of $Q_2(\hat{\mathbf{Y}}_4 + \mathbf{Z}_2)$ conditioned on $\mathbf{Z}_2$ and common information $Q_3, Q_4, \mathbf{Z}_3, \mathbf{Z}_4$ to receiver 2, and receiver 2 reconstructs $\mathbf{Y}_2 = \mathbf{Y}_4 + \alpha_2(Q_2(\hat{\mathbf{Y}}_4 + \mathbf{Z}_2) - \mathbf{Z}_2)$.

We choose α_i and σ_i^2 for $i \in \{1, 2, 3, 4\}$ and β_i for $i \in \{3, 4\}$ to optimize the rate-distortion performance. This approach is called "pre/post-filtering" in [15]. In the previous strategy, $\alpha_i = 1$ and $\sigma_i^2 = D_i$ for $i = 1, 2, 3, 4$ and $\beta_i = 0$ for

$i = 3, 4$.) The optimization for $\mathbf{X}_3$ and $\mathbf{Y}_4$ leads to

$$\begin{aligned}
\alpha_3 &= \frac{(1-\rho^2)\sigma_X^2\sigma_Y^2 + \sigma_X^2\sigma_4^2}{\Lambda}, \quad \beta_3 = \frac{\rho\sigma_X\sigma_Y\sigma_3^2}{\Lambda}, \\
\alpha_4 &= \frac{(1-\rho^2)\sigma_X^2\sigma_Y^2 + \sigma_Y^2\sigma_3^2}{\Lambda}, \quad \beta_4 = \frac{\rho\sigma_X\sigma_Y\sigma_4^2}{\Lambda}, \\
D_3 &= \frac{\sigma_X^2\sigma_3^2((1-\rho^2)\sigma_Y^2 + \sigma_4^2)}{\Lambda}, \\
D_4 &= \frac{\sigma_Y^2\sigma_4^2((1-\rho^2)\sigma_X^2 + \sigma_3^2)}{\Lambda},
\end{aligned}$$

where $\Lambda = (\sigma_X^2 + \sigma_3^2)(\sigma_Y^2 + \sigma_4^2) - \rho^2\sigma_X^2\sigma_Y^2$. Let $\mathbf{N}_i$ be distributed as $-\mathbf{Z}_i$, $i = 1, 2, 3, 4$, then the rate of the common message is:

$$R_0 = \frac{1}{n} I(\mathbf{X}, \mathbf{Y}; \mathbf{X} + \mathbf{N}_3, \mathbf{Y} + \mathbf{N}_4).$$

We also know that

$$\begin{aligned}
D_1 &= (1-\alpha_1)^2 D_3 + \alpha_1^2\sigma_1^2, \\
D_2 &= (1-\alpha_2)^2 D_4 + \alpha_2^2\sigma_2^2.
\end{aligned}$$

Therefore, minimizing D_1 and D_2 results in

$$\begin{aligned}
\alpha_1 &= \frac{D_3}{\sigma_1^2 + D_3}, \quad D_1 = \frac{\sigma_1^2 D_3}{\sigma_1^2 + D_3}, \\
\alpha_2 &= \frac{D_4}{\sigma_2^2 + D_4}, \quad D_2 = \frac{\sigma_2^2 D_4}{\sigma_2^2 + D_4}.
\end{aligned}$$

The rates of the private informations for the two receivers are

$$\begin{aligned}
R_1 &= \frac{1}{n} I(\mathbf{X}; \mathbf{X} + \mathbf{N}_1 | \mathbf{X} + \mathbf{N}_3, \mathbf{Y} + \mathbf{N}_4), \\
R_2 &= \frac{1}{n} I(\mathbf{Y}; \mathbf{Y} + \mathbf{N}_2 | \mathbf{X} + \mathbf{N}_3, \mathbf{Y} + \mathbf{N}_4).
\end{aligned}$$

The code using strategy 2 yields better rate distortion performance than the code using strategy 1 in general. Therefore the previous rate redundancy results hold for this strategy.

V. SUMMARY

In this paper, we define and bound from above the rate loss for BSSCs. The rate loss bounds demonstrate that the outer bound $\mathcal{R}_{\text{out}}$, albeit unachievable in general, may be quite close to the achievable region which is generally difficult to characterize.

The rate loss bounds also serve as upper bounds on the performance penalty associated with using the BSSC rather than traditional source codes without private channels. For example, the constant rate loss bounds for Gaussian sources remove the concern that the performance penalty may be arbitrarily large for both receivers.

It is worth mentioning that the rate loss in this paper is indeed the gap between total rates such as $(R_0 + R_1 + R_2) - R_{X,Y}(D_1, D_2)$ or $(R_0 + R_1) - R_X(D_1)$. We can also consider the gap between individual rates. For example, from the proof of Theorem 2, for any rate triple (R_0, R_1, R_2) on the lower boundary of $\mathcal{R}_{\text{out}}$, the rate triple $(R_0 + 0.74, R_1 + 0.74, R_2 + 0.74)$ is always achievable for any joint Gaussian sources.

However, these bounds are not tight in general. For example, we can show that for any point (R_0, R_1, R_2) on the triangle ACD or line segments BC and BD in Figure 2, the rate triple $(R_0 + 0.5, R_1 + 0.5, R_2 + 0.5)$ is achievable for any source and any distortion pair.

Finally, we propose a new practical BSSC algorithm using ECDQ. The advantages of this design are that it has low computational complexity and low storage requirements, and we can characterize the exact rate distortion performance of the resulting codes. In addition, we have obtained upper bounds on the performance gap between the resulting codes and the theoretical optimal codes, which are similar to the rate loss bounds.

VI. ACKNOWLEDGMENT

The authors wish to thank Dr. Qian Zhao for many helpful exchanges. This material is based upon work partially supported by NSF Grants No. CCR-0220039 and the Caltech's Lee Center for Advanced Networking.

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