LECTURES ON PERTURBATIVE STRING THEORIES[1]

HIROSI OOGURI and ZHENG YIN
Department of Physics, University of California at Berkeley
366 Le Conte Hall, Berkeley, CA 94720-7300, U.S.A.
and
Theoretical Physics Group, Mail Stop 50A–5101
Ernest Orlando Lawrence Berkeley National Laboratory, Berkeley, CA 94720, U.S.A.

These lecture notes on String Theory constitute an introductory course designed to acquaint the students with some basic facts of perturbative string theories. They are intended as preparation for the more advanced courses on non-perturbative aspects of string theories in the school. The course consists of five lectures: 1. Bosonic String, 2. Toroidal Compactifications, 3. Superstrings, 4. Heterotic Strings, and 5. Orbifold Compactifications.

Contents

[1] Lectures given by H. Ooguri.

1 Lecture One: Bosonic String

It had been said that there are five different string theories – (1) open and closed superstrings (Type I), (2) non-chiral closed superstring (Type IIA), (3) chiral closed superstring (Type IIB), (4) heterotic string with $E_8 \times E_8$ gauge symmetry, and (5) heterotic string with $Spin(32)/\mathbb{Z}_2$ gauge symmetry. They are all formulated perturbatively as sums over two-dimensional surfaces. It had been known for a long time (and as we will learn in this course) that (2) can be related to (3), and (4) to (5), if we compacitify part of the target spacetime on S^1. These relations were discovered earlier since they hold in each order in the perturbative expansion of the theories. During the last two years, it has become increasingly clear that in fact all these five string theories are related to each other under various duality transformations. It seems likely that there is something more fundamental, which we may call *the theory*, and the five string theories describe various asymptotic regions of it.

One of the purposes of this year's TASI summer school is to guide students through this recent exciting development. It is hoped that students attending the school will someday reveal what *the theory* is about. First, however, the students have to understand its five known asymptotic regions. This is the purpose of this course. We will construct and analyze four perturbative string theories, (2), (3), (4) and (5). The type I theory containing open superstring will not be discussed here since it will be covered in Polchinski's lecture in this school. Due to the limited time, we cannot discuss computations of higher-loop string amplitudes at all. This important topic has been covered in previous TASI lectures, by Vafa [1] and by D'Hoker [2]. Due to the long history of the string theory, we were unable to make a complete bibliography of original papers. We apologize to numerous contributors to the subject for the omission. For works before 1987, we refer to the bibliography of [3].

1.1 Point Particle

Let us begin this lecture on string theory by recalling the relativistic action for a point particle moving in D-dimensional spacetime.

$$S = -m \int d\sigma \sqrt{-\dot{X}^2}, \tag{1}$$

where

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \sigma}, \quad \mu = 0, ..., D-1.$$

We are using the Minkowski metric $\eta_{\mu\nu}$ with signature $(-1, +1, \ldots, +1)$. It is an action defined over the worldline the particle traverses. Its canonical momenta are given by

$$P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = m \frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}. \tag{2}$$

However, they are not all independent but satisfy a constraint equation which is simply Einstein's relation between energy, momentum and mass.

$$P^2 = -m^2 \tag{3}$$

The constraint arises since (1) is invariant under worldline reparametrization: $\sigma \to \sigma' = f(\sigma)$. This is a gauge symmetry that we naturally expect since changing the parametrization scheme of the worldline should have *no* physical effect at all. It indicates the X's and P's are redundant as coordinates of the physical phase space. We can eliminate one of the X's by a choice of gauge. For example, we can set $X^0 = \sigma$ so the worldline time σ coincides with the physical time X^0. Constraint (3) then tells us how to eliminate one of the P's. Upon quantization the constraint (3) then becomes the requirement that physical states and observables should be gauge invariant.

1.2 Nambu-Goto Action

We now formulate string theory as an analogue of (1) on a two-dimensional *worldsheet*. The fields X^μ on the worldsheet Σ define an embedding of Σ in the D-dimensional spacetime. The pull-back of the Minkowski metric η to the worldsheet is called an induced metric:

$$\hat{g}_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}; \quad a, b = 0, 1; \quad \hat{g} \equiv \det \hat{g}_{ab}. \tag{4}$$

We then define the *Nambu-Goto* string action as proportional to the area of the worldsheet measured by the induced metric:

$$S = \frac{-1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\hat{g}}. \tag{5}$$

Because there are only bosonic degrees of freedom involved, we call it bosonic string. It can be shown that the dimensionful constant $T \equiv 1/(2\pi\alpha')$ gives the tension of the string.

The coordinate σ^0 is the "time" on the string worldsheet, σ^1 is the "space" coordinate along the closed string. In these lectures we only consider *orientable closed* strings, which means that the worldsheet can be assigned a definite orientation. By a reparametrization, we can let σ^1 range from 0 to 2π. Thus the Nambu-Goto action describes the motion of a string in spacetime, and the worldsheet is the trajectory it sweeps out (fig. 1). Unlike a particle, a string can have internal oscillations in addition to its center of mass motion. They include oscillation both transverse and longitudinal to the string worldsheet. As we will now demonstrate, however, only the transverse oscillations are physical.

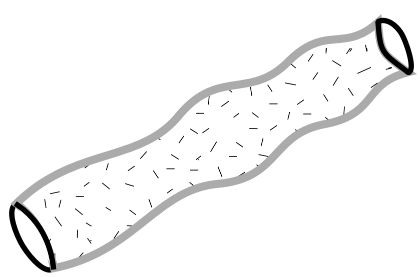


Figure 1. The Worldsheet of A String

The canonical momentum densities of the Nambu-Goto action (5) are given by

$$P_\mu(\sigma) = \frac{\delta L}{\delta \dot{X}^\mu(\sigma)} = \frac{1}{2\pi\alpha'}\sqrt{-\hat{g}}\ \hat{g}^{0\alpha}\ \eta_{\mu\nu}\partial_\alpha X^\nu. \tag{6}$$

Since (5) has reparametrization invariance on the worldsheet: $(\sigma^0, \sigma^1) \to (\sigma^{0\prime}, \sigma^{1\prime})$, one naturally expects an analog of the constraint (3). Indeed one finds

$$\begin{aligned} P \cdot \partial_1 X &= \frac{1}{2\pi\alpha'}\sqrt{-\hat{g}}\hat{g}^{0\alpha}\hat{g}_{\alpha 1} = \frac{1}{2\pi\alpha'}\sqrt{-\hat{g}}\delta^0_1 = 0, \\ P^2 &= -(\frac{1}{2\pi\alpha'})^2\hat{g}\hat{g}^{00} = -(\frac{1}{2\pi\alpha'})^2 g_{11} = -(\frac{1}{2\pi\alpha'})^2(\partial_1 X)^2, \end{aligned}$$

or

$$\left[P \pm \frac{1}{2\pi\alpha'}(\partial_1 X)\right]^2 = 0, \tag{7}$$

known as the *Virasoro constraints*.

By using the reparametrization invariance, we can bring the induced metric on any *coordinate patch* of the worldsheet to be a multiple of the standard Lorentzian metric:

$$\hat{g}_{ab} = \lambda\gamma_{ab},\ (\gamma_{ab}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Here λ may be a function of σ^a. This is called the *conformal gauge*. Note that this choice of gauge does not break spacetime Lorentz invariance. In this gauge, the momentum densities P_μ are given by

$$P_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu, \quad \dot{X} = \partial_0 X, \tag{9}$$

and the Virasoro constraints are

$$\begin{aligned} \partial_0 X \cdot \partial_1 X &= \hat{g}_{01} = 0, \\ (\partial_0 X)^2 + (\partial_1 X)^2 &= \hat{g}_{00} + \hat{g}_{11} = 0. \end{aligned}$$

In conformal gauge, there is still a residual gauge symmetry. It is called *conformal symmetry* because it only rescales the induced metric. To exhibit it, define the light-cone coordinates $\sigma^\pm \equiv \sigma^0 \pm \sigma^1$. It is not difficult to show that a coordinate transformation preserving the conformal gauge condition (8) must be of the form

$$\sigma^+ \to \sigma^{+\prime} = f(\sigma^+), \qquad \sigma^- \to {\sigma^-}' = h(\sigma^-). \tag{10}$$

In the light-cone coordinates,

$$-(d\sigma^0)^2 + (d\sigma^1)^2 = -d\sigma^+ d\sigma^-.$$

Since

$$d{\sigma'}^+ = f' d\sigma^+, \qquad d{\sigma'}^- = h' d\sigma^-,$$

(8) is indeed preserved as

$$d\sigma'^{+}d\sigma'^{-} = f'h'd\sigma^{+}d\sigma^{-}.$$

The worldsheet of a freely propagating string clearly looks like a tube. Choosing L_n and $\tilde{L}_n$, the Fourier components of f and h respectively, as the generators of conformal transformation on a cylinder, it is not difficult to find their commutators:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m}, \\ \left[\tilde{L}_n, \tilde{L}_m\right] &= (n-m)\tilde{L}_{n+m}, \\ \left[L_n, \tilde{L}_m\right] &= 0. \end{aligned} \tag{11}$$

We can completely fix this residual gauge symmetry, at the expense of spacetime Lorentz invariance. In the conformal gauge, the equation of motion for $X^\mu(\sigma)$ is

$$(\partial_0^2 - \partial_1^2)X^\mu = 0, \tag{12}$$

and its general solution is

$$X^\mu = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-).$$

Define $X^+ \equiv X^0 + X^1$, $X^- \equiv X^0 - X^1$. Taking advantage of this residual gauge symmetry, we can always choose local coordinates so that:

$$X^+ \equiv X^0 + X^1 = \alpha' p^+ \sigma^0, \tag{13}$$

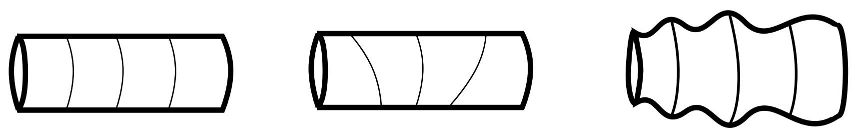
where p^+ is a constant[2] which, by (6), is related to the momentum density as $P^+ = p^+/2\pi$. Substituting this into (7), we obtain

$$\partial_\pm X^- = \frac{1}{\alpha' p^+} \sum_{i=2}^{D} \partial_\pm X^i \partial_\pm X^i. \tag{14}$$

This determines $X^- = X^0 - X^1$ up to a constant of integration x^-. Thus the gauge invariant information of a propagating string is given by x^-, p^+, and

[2]We keep p^+ rather than absorbing it into σ^0 so as to preserve the canonical Poisson bracket. In fact, since p^+ is a physical observable, we should not be able to gauge it away.

$X^i(\sigma^0, \sigma^1)$, $(i = 2, \ldots, D-1)$. As in the case of point particles, the worldsheet reparametrization invariance removes all degrees of freedom along the light-cone directions except for their zero modes. Intuitively, one can understand this as saying that oscillations tangential to the string can be absorbed by a worldsheet reparametrization (fig. 2).



a. No Oscillation b. Longitudinal Oscillation c. Transversal Oscillation

Figure 2. String Excitations

On length scales much larger than the string scale $\sqrt{\alpha'}$, which is the typical size of string, low lying excitations of string look like point particles and should form unitary representations of the Poincaré group. They are classified by the little group of their momenta in a certain Lorentz frame. As we will see, the first excitation level of the string makes a rank 2 tensor representation of $SO(D-2)$, including the trace, traceless symmetric and antisymmetric parts. The little groups for massless and massive particles in $(1, D-1)$ spacetime are $SO(D-2)$ and $SO(D-1)$ respectively. Since the rank 2 tensor of $SO(D-2)$ alone cannot be made into a representation of $SO(D-1)$, these excitations must correspond to massless particles. According to Weinberg's theorem [4], a massless rank 2 symmetric traceless tensor that is observed at low energy must describe graviton and implies general covariance. Hence a theory of closed strings must be, among other things, a theory of gravity.

1.3 First Quantization of String

For point particles, there are two roads from classical physics to quantum physics. The first quantization quantizes the worldline action and yields quantum mechanics (i.e. one-dimensional QFT) of the particles. The second quantization quantizes their spacetime action and yields a $(1, D-1)$-dimensional QFT. In string theory, the worldsheet is already two-dimensional, so we have a $(1,1)$-dimensional QFT theory already in the first quantization.

First, let us recall briefly the results for point particles. Quantization replaces P_μ by $-i\frac{\partial}{\partial X^\mu}$, so the Einstein constraint on physical states (3) becomes the Klein-Gordon equation on wavefunctions. In a similar vein, the quantization of supersymmetric (spinning) particle would give rise to the Dirac equation as the constraint equation, as we will discuss in lecture three.

Now for strings, let us choose the conformal gauge (8). In this gauge, the equation of motion (12) and the expression for canonical momentum (9) can be obtained from the action

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \\ &= \frac{1}{\pi\alpha'}\int d^2\sigma\,\partial_+ X^\mu \partial_- X_\mu . \end{aligned} \tag{15}$$

By varying the worldsheet metric away from γ_{ab}, we can find its (worldsheet) energy-momentum tensor $T^{ab} \sim \frac{\delta S}{\delta\gamma_{ab}}$. Since the action is conformally invariant, the trace of the classical energy-momentum tensor T vanishes. The remaining two components are

$$T_{++} = \frac{1}{\alpha'}(\partial_+ X)^2, \qquad T_{--} = \frac{1}{\alpha'}(\partial_- X)^2. \tag{16}$$

The reparametrization invariance of the Nambu-Goto action implies the first class constraints $T_{++} = 0$ and $T_{--} = 0$. In fact these are the Virasoro constraints (7) that we have seen before.

Canonical quantization gives[3]

$$\left[\dot{X}^\mu(\sigma^0,\sigma^1), X^\nu(\sigma^0,\sigma^{1\prime})\right] = 2\pi\alpha' \imath \eta^{\mu\nu}\delta(\sigma^{1\prime} - \sigma^1) \tag{17}$$

The X's must be periodic in σ^1 with period 2π. After Fourier decomposition, we separate and recover the center of mass operators and the mode operators corresponding to excitations:

$$X^\mu = x^\mu + \alpha' p^\mu \sigma^0 + \sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\imath}{n}\{\alpha^\mu_n e^{-\imath n\sigma^+} + \tilde{\alpha}^\mu_n e^{-\imath n\sigma^-}\}$$

[3]In these lectures, $\imath$ denotes $\sqrt{-1}$.

$$
\begin{aligned}
&[x^\mu, p^\nu] = \imath \eta^{\mu\nu} \\
&[\alpha^\mu_n, \alpha^\nu_m] = n\eta^{\mu\nu}\delta_{n+m,0}, \quad [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta^{\mu\nu}\delta_{n+m,0} \\
&(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}, \qquad (\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n}.
\end{aligned} \tag{18}
$$

So the Hilbert space is the tensor product of $2 \times D$ infinite towers of harmonic oscillators, each labeled by positive integers (coming from α_n and $\tilde{\alpha}_n$) and that of the D-dimensional quantum mechanics (coming from the zero modes X^μ and P^μ):

$$
\bigotimes_{0\le\mu<D}^{n>0} \left\{ \left(\alpha^\mu_{-n}\right)^i |0\rangle \, | i = 0 \ldots \infty \right\} \bigotimes_{0\le\mu<D}^{n>0} \left\{ \left(\tilde{\alpha}^\mu_{-n}\right)^i |0\rangle \, | i = 0 \ldots \infty \right\} \bigotimes \Phi(X^\mu).
$$

The operator α^μ_{-n} (α^μ_n), with $n > 0$, creates (destroys) a quantum of left moving oscillation with angular frequency n along the X^μ direction in the spacetime. $\tilde{\alpha}^\mu_{-n}$ ($\tilde{\alpha}^\mu_n$) does the same for the right movers. This decomposition of degrees of freedom into essentially decoupled left and right movers is what makes many two-dimensional field theories so much more manageable compared to theories in higher dimensions.

One should note that because of the indefinite signature of the spacetime metric η in (18), the states created by the oscillators along the time direction may have negative norms. Such states are called *ghosts*, not to be confused with the Faddeev-Popov ghosts, which also enter the scene later. They cannot be present in the physical spectrum. As we will see presently, the quantum mechanical implementation of the constraints (7) eliminates them.

It also is convenient to Fourier transform the energy-momentum tensor T:

$$
\begin{gathered}
T \equiv T_{++} = \frac{1}{\alpha'}(\partial_+ X)^2 \equiv \sum_n L_n e^{-\imath n \sigma^+}; \\
\tilde{T} \equiv T_{--} = \frac{1}{\alpha'}(\partial_- X)^2 \equiv \sum_n \tilde{L}_n e^{\imath n \sigma^+}; \\
L_n = \sum_m \frac{1}{2}\alpha_{n-m}\alpha_m, \quad \tilde{L}_n = \sum_m \frac{1}{2}\tilde{\alpha}_{n-m}\tilde{\alpha}_m, \\
\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu.
\end{gathered}
$$

These L_n and $\tilde{L}_n$ are well defined except for $n = 0$, for which there is a normal ordering ambiguity. If we define

$$\begin{aligned} L_0 &= \sum_{n>0} \alpha_{-n}\alpha_n + \frac{1}{2}(\alpha_0)^2, \\ \tilde{L}_0 &= \sum_{n>0} \tilde{\alpha}_{-n}\tilde{\alpha}_n + \frac{1}{2}(\tilde{\alpha}_0)^2, \end{aligned} \tag{19}$$

the constraint for the $n = 0$ part would be $L_0 - a = 0$, $\tilde{L}_0 - \tilde{a} = 0$ where a and $\tilde{a}$ are constants reflecting the normal ordering ambiguity. The combination $(L_0 + \tilde{L}_0)$ is the Hamiltonian of the system generating a translation in σ^0 direction and $(L_0 - \tilde{L}_0)$ is the worldsheet momentum. Since

$$[L_0, \alpha_{-n}] = n\alpha_{-n},$$

the n-th oscillator has energy n, equal to its angular frequency. The same holds for the right movers.

We can try imposing

$$L_n - a\delta_{n,0} = 0, \quad \tilde{L}_n - \tilde{a}\delta_{n,0} = 0. \tag{20}$$

for all n, as constraints on physics states. However we run into problems immediately. It can be checked that the L's form a representation of the *Virasoro algebra*, which is the conformal algebra (11) with a nontrivial *central extension*:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\ \left[\tilde{L}_n, \tilde{L}_m\right] &= (n-m)\tilde{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\ \left[L_n, \tilde{L}_m\right] &= 0. \end{aligned} \tag{21}$$

In our case, the central charge c is equal to D, the spacetime dimension. Imposing $L_n|phys\rangle = 0$ for all $n \in \mathbb{Z}$ would be inconsistent with the commutation relation if $D \neq 0$. We may instead adopt the Gupta-Bleuler prescription and require that physical states be annihilated by half of L_n's

$$(L_n - a\delta_{n,o})|phys\rangle = 0, \quad n \geq 0. \tag{22}$$

We also define an equivalence relation among them:

$$|phys\rangle \sim |phys\rangle + L_{-n}\,|*\rangle\,, \quad n > 0. \tag{23}$$

We call a physical state *spurious* if it is a linear combination of $L_{-n}\,|*\rangle$ for some state $|*\rangle$. The true physical degrees of freedom are thus the equivalence classes of (23). Condition (22) implies that the matrix elements of L_n between physical states vanish for all n. This is consistent with (23), which says that any L_n has no physical effect on a physical state. It is the same story for the right movers.

As alluded earlier, these $2\times\infty$ set of constraints and equivalence conditions effectively remove 2 directions of oscillators of every mode from the physical spectrum if the theory is consistent. In the next section we demonstrate this explicitly for the first few excited states. Since L_n and $\tilde{L}_n$ have exactly the same property, in the following discussion we will concentrate on L_n, bearing in mind that the same results obtain for $\tilde{L}_n$. In particular, we will determine a by consistency requirement. Since we can repeat the same story for $\tilde{a}$, they must be the same, implying

$$(L_0 - \tilde{L}_0)\,|phys\rangle = 0. \tag{24}$$

This is known as the *level matching condition*.

1.4 Critical Dimensions

Another way to quantize the string is to start with the light-cone action and perform canonical quantization. In this gauge, the constraint equations (3) are explicitly solved and only the $(D-2)$ oscillatory excitations transverse to the string worldsheet remain. Whether we choose the light-cone gauge or the conformal gauge is a matter of convention and their results should agree unless there is an anomaly obstructing conformal invariance from becoming a full fledged quantum symmetry. As conformal invariance is the remnant of gauge symmetry on the worldsheet, an anomaly for it would spell disaster.

Let us look at the spectrum in the conformal gauge, taking into account the physical state condition (22) and the string gauge covariance (23). As a measure of oscillator excitation, define

$$N \equiv L_0 - \frac{1}{2}(\alpha_0)^2 = \sum_{n>0} n\alpha_{-n}\alpha_n \tag{25}$$

and similarly $\tilde{N}$ for the right moving sector. By (19) and the Einstein relation $m^2 = -k^2$, they also determine the mass of the states:

$$m^2 = \frac{4}{\alpha'}(N - a). \tag{26}$$

Therefore the constraint $L_0 = a$ is the *mass shell condition*. The level matching condition (24) implies that $N = \tilde{N}$. As mentioned above, it is sufficient to concentrate on the left movers.

Ground State — $N = 0$. There is no oscillator excitation and the states are simply $|k\rangle$ where

$$p^\mu |k\rangle = k^\mu |k\rangle, \quad \alpha_n |k\rangle = 0 \quad (n > 0).$$

The only nontrivial condition from (22) is the mass shell condition

$$(L_0 - a)|k\rangle = (\frac{\alpha'}{4}k^2 - a)|k\rangle = 0,$$

which implies

$$m^2 = -\frac{4}{\alpha'}a.$$

If $a > 0$, then the ground state would correspond to a tachyon. As it turns out, this is indeed the case for both bosonic string and superstring theory. In the latter, we will be able to consistently truncate the spectrum of the superstring so that the ground state tachyon is no longer present, but this seems impossible for the bosonic string theory. As we know from field theory, the presence of a tachyon indicates that we are perturbing around a local maximum of potential energy — we are at a wrong vacuum. However, to this date it is not known whether one can find an alternative vacuum for the the bosonic string theory so that it is free of tachyons.

First excited level — $N = 1$**.** The states are $e_\mu(k)\alpha_{-1}^\mu |k\rangle$. It is simple to deduce from (22) the following constraints:

$$(L_0 - a)|phys\rangle = 0 \quad \rightarrow \quad k^2 = (a-1)\frac{4}{\alpha'},$$
$$L_1 |phys\rangle = 0 \quad \rightarrow \quad k \cdot e = 0.$$

The equivalence relation (23) states that

$$e_\mu \sim e_\mu + \lambda k_\mu, \tag{27}$$

which has precisely the form of a gauge transformation in QED. However, this does not yields a spurious physical state unless $k^2 = 0$. This is familiar from QED: if states on this level are massless (i.e. $a = 1$), then physically there are only $(D-2)$ independent polarizations; otherwise there are $(D-1)$ polarizations. Since the light-cone gauge quantization gives $(D-2)$ polarizations, the anomaly-free requirement picks $a = 1$. Incidentally, this result can also be obtained if we determine the normal ordering prescription of L_0 in light-cone gauge by the ζ-function regularization.

Also by analogy to QED, $k \cdot e = 0$ can be interpreted as the Lorentz gauge condition. Combined with $k^2 = 0$, the massless Klein-Gordon equation, we obtain the Maxwell equation $\partial^\mu F_{\mu\nu} = 0$. These statements are precise in open string theory. In closed string, when we combine them with the their counterparts for the right movers, we obtain the equations of motion and gauge transformations appropriate for graviton, antisymmetric tensor, and dilaton fields.

2nd excited level — $N = 2$**.** The states take the form $[e_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\mu + e_\mu\alpha_{-2}^\mu]\,|k\rangle$. The mass shell condition reads

$$(L_0 - 1) = 0 \rightarrow k^2 = -\frac{4}{\alpha'},$$

where we have used the result $a = 1$. This means states at this level are massive. The other two nontrivial physical state conditions from L_2 and L_1 impose $(D+1)$ conditions on $e_{\mu\nu}$ and e_μ. They leave us with

$$\frac{1}{2}D(D+1) + D - (D+1) = \frac{1}{2}(D^2 + D) - 1$$

degrees of freedom in the polarization. On the other hand, light-cone quantization gives

$$\frac{1}{2}(D-1)(D-2)+D-2=\frac{1}{2}(D^2-D)-1$$

degrees of freedom. The deficit of D must be accounted for in the equivalence relation (23). The spurious states at level two are spanned by [4]

$$\chi_\mu L_{-1}\alpha_{-1}^\mu \left|k\right\rangle$$

and

$$(L_{-2}+\gamma L_{-1}^2)\left|k\right\rangle,$$

for some constants χ_μ and γ. Requiring the first type of spurious states to be physical leads to the condition

$$\chi\cdot k=0.$$

Therefore the spurious state of the first type accounts for $(D-1)$ degrees of freedom. Since we need to have D spurious states, the second type of state must also satisfy the physical state condition. It is left to students to verify that the L_1 constraint requires γ to be $\frac{3}{2}$ and the L_2 constraint fixes D to be 26. Thus we have arrived at the famous conclusion that bosonic string theory propagates in 26 dimensions.

We can continue this program to states of higher levels. The result is the same — only for $a=1$ and $D=26$ do we have an agreement between light-cone and conformal gauge. If one insists on considering $D\neq 26$ nonetheless, then it has been found that spacetime Lorentz invariance is completely lost in the light-cone gauge unless $D=26$ (ref. 204 in [3], Vol 1). On the side of the conformal gauge, although one can show there is no ghost in the tree level spectrum if $D\leq 26$ (refs. 65 and 202 of [3], Vol 1), they do show up as unphysical poles in one-loop string amplitudes unless $D=26$. Below we will mention very briefly the correct formulation of such *non-critical* string theory found by Polyakov (ref. 366 in [3], Vol 1).

[4]The particular choice of basis given here is purely a matter of convenience.

1.5 Massless Spectrum

Now let us examine the spectrum of massless states in 26-dimensional bosonic string theory. According to (26), the masses of the string states are integral multiple of $2/\sqrt{\alpha'}$. From last section, we see the massless particles arise from the first excited level of the string. Combining left and right moving sectors of the Fock space in accordance with the level matching condition (24), they have the form

$$e_{ij}\,\alpha^i_{-1}\tilde{\alpha}^j_{-1}\,|k\rangle\,, \quad (k^2=0; i,j=1,\ldots 24) \tag{28}$$

in the light-cone gauge. We may decompose e_{ij} into irreducible representations of $SO(24)$, each of which would correspond to a certain type of particle:

$$\begin{aligned} e_{ij} &= [\frac{1}{2}(e_{ij}+e_{ji})-\frac{1}{12}\delta_{ij}\mathrm{Tr}\,e]+[\frac{1}{2}(e_{ij}-e_{ji})]+[\frac{1}{24}\delta_{ij}\mathrm{Tr}\,e] \\ &\equiv [h_{ij}]+[B_{ij}]+[\delta_{ij}\Phi]. \end{aligned}$$

The traceless symmetric, antisymmetric, and trace parts of e_{ij} are denoted as h_{ij}, B_{ij} and Φ respectively. B_{ij} is known as the *antisymmetric tensor.* Φ is called *dilaton.* Being a massless scalar, Φ may develop a vacuum expectation value (VEV). We will later see that $\langle\Phi\rangle=\Phi_0$ shifts the string coupling constant κ to κe^{Φ_0}. h_{ij} is identified with the graviton because it observes general covariance. To see this we should choose the conformal gauge, which is 26-dimensional Lorentz covariant. Now we use Greek indices μ, ν, $\ldots$, ranging from 0 to 25, to label the tangent space. We mentioned earlier that the equivalence relations from L_{-1}, $\tilde{L}_{-1}$ have the spacetime interpretation of gauge transformations. It is not difficult to show that these gauge transformations act on $h_{\mu\nu}$ and $B_{\mu\nu}$ as

$$\begin{aligned} h_{\mu\nu} &\rightarrow h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu, \\ B_{\mu\nu} &\rightarrow B_{\mu\nu}+\partial_\mu\xi_\nu-\partial_\nu\xi_\mu. \end{aligned}$$

The first is simply diffeomorphisms acting on the spacetime metric in the Minkowski background. The second can be written in the language of differential forms as $B\rightarrow B+dA$. This suggests that the physical observable associated with the 2-form B should be its *3-form field strength* $H\equiv dB$.

1.6 Polyakov Action

There is another interpretation of the requirement $D = 26$, due to Polyakov. Consider the action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu, \tag{29}$$

where both, the metric g_{ab} as well as X^μ, are treated as dynamical variables. The worldsheet metric g_{ab} has no local propagating degrees of freedom. Classically, the equation for g requires it to be proportional to the induced metric (4). Substitute this back to (29) and we obtain the Nambu-Goto action (5), establishing their classical equivalence.

In fact the worldsheet metric consists almost purely of gauge degrees of freedom. First the worldsheet metric has three independent degrees of freedom, two of which can be gauged away using worldsheet diffeomorphism, bringing the metric into the standard form

$$(g_{ab}) = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \tag{30}$$

in what is known as conformal coordinates. Furthermore, the Polyakov action (29) has the Weyl rescaling symmetry which allows us to scale λ to, say, 1. In this gauge, the equation of motion and canonical momenta can be obtained from the same conformal gauge action as for the Nambu-Goto action (15), so the same quantization procedure can be carried over.

There are two complications to this story. First, in general (30) can only be enforced in each coordinate patch. Between patches there can be global degrees of freedom left. Roughly speaking they describe the shape of the worldsheet and are known as *complex moduli*, for reasons to be discussed in Greene's lecture at this school. A simple example appears in the next section. Second, quantum mechanically the Weyl rescaling symmetry may became anomalous, and the algebra of conformal transformation (11) is not realized on the Hilbert space. It is deformed to be the Virasoro algebra with the central extension. The central charge c measures the violation of conformal invariance. As we saw, the central charge for X^μ is D, equal to the dimension of the spacetime. The Faddeev-Popov ghosts, which provide the correct normalization for the path integral respecting the reparametrization invariance, carry central charge -26. Since the conformal anomaly is additive, only when $D = 26$ does the anomaly from the X's cancel against that from the ghosts and give us a consistent theory. When $D \neq 26$, one can no longer gauge away λ and has to treat it as dynamical degrees of freedom, known as *Liouville field*. The resulting theories, known as *non-critical string*, are interesting in their own right but will not be discussed in this lecture. For reviews on this topic, see [5] and [6].

1.7 String Propagation and Interactions

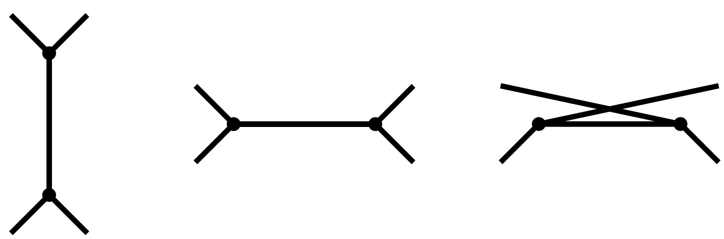
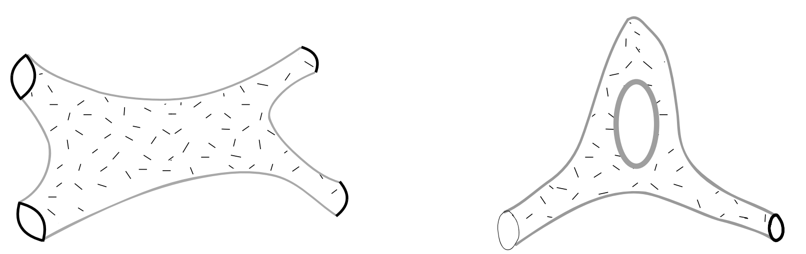


Figure 3. Some Feynman Diagrams for Point Particles



a. Tree-level 4-string scattering b. One-loop 2-string scattering

Figure 4. String Interaction

Point particles propagate in a straight line with amplitude given by their Feynman propagators. They interact at a well-defined point in the spacetime, where straight lines intersect at vertices. Each vertex also has some coupling constant associated with it. We calculate a scattering amplitude of them by drawing the corresponding Feynman diagrams, and multiplying together all the propagators and the coupling constants at each vertex (fig. 3). In string theory, the picture is similar (fig. 4). Propagation of string is represented by a tube. A slice of the worldsheet at any time determines a string state at that instant. However, because of worldsheet reparametrization invariance, no scheme of time slicing is preferred over others. This and the smooth joining and splitting of string tubes mean that there is no freedom in assigning coupling constants to any particular point. Indeed it will soon become clear that there is only one measure of string coupling, which is however a field carried by and distributed over the strings themselves.

To study string worldsheets of various topologies, it is convenient to choose the worldsheet metric to be Euclidean rather than Lorentzian. This can be done by performing a Wick rotation on the worldsheet:

$$\sigma^0 = -\imath\sigma^2$$

$$z \equiv \imath\sigma^+ = \sigma^2 + \imath\sigma^1, \quad \bar{z} = \imath\sigma^- = \sigma^2 - \imath\sigma^1,$$

$$X^\mu = x^\mu - \imath\alpha' p^\mu \mathrm{Re}\, z + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\imath}{n}\{\alpha_n^\mu e^{-nz} + \tilde{\alpha}_n^\mu e^{-n\bar{z}}\}$$

We will use this Euclidean notation from now on.

Figure 4 shows the worldsheet for a string-string scattering. Its amplitude is calculated by evaluating the Polyakov path integral over it. After gauging away arbitrary reparameterizations, the integration over the worldsheet metric g of Polyakov action is reduced to a sum of over all possible shapes and sizes of worldsheets of a given topology. Since the size of the worldsheet can be gauged away for critical string theory, this reduces to a finite dimensional integral over its moduli space, the space that parameterizes the shape of worldsheet with this topology. Worldsheet actions themselves do not tell us which topology of worldsheet we should choose, but analogy with Feynman diagrams suggests that handles in the worldsheet represent internal loops and we should sum over all number of handles. In fact the unitarity of the S-matrix dictates how to sum over topologies of the worldsheet.

As a simple and useful illustration, consider the one-loop vacuum to vacuum string amplitude (fig. 5). This has the physical interpretation of calculating the vacuum energy. There is no external string and the worldsheet is topologically a torus. By Weyl scaling we can always make it a flat torus, defined as the quotient of the complex plane by a lattice generated by 1 and τ — we identify points related by $n + m\tau$, $n, m \in \mathbb{Z}$ (fig. 6). τ is the complex moduli for the topological class of torus and cannot be gauged away by Weyl rescaling. The integration over g now reduces to an integration over the moduli parameter τ[5]. Nondegeneracy of the torus requires $\mathrm{Im}\,\tau \neq 0$, and by choices of basis of lattice vector we can require τ to live on the upper complex half-plane. Let us look at this from the worldsheet viewpoint. Choose the imaginary axis as worldsheet "time" and real axis as the spatial extent of the string. Then $\mathrm{Im}\,\tau$ is the worldsheet time. Worldsheet states evolve along it as usual with Hamiltonian $L_0 + \tilde{L}_0$. $\mathrm{Re}\,\tau$ is a spatial twist, generated by the worldsheet momentum $L_0 - \tilde{L}_0$. As there is no end to the string in this one-loop amplitude, the path integral sums over all states in the Hilbert space — it is a trace. In fact it is the partition function

$$\begin{aligned} Z(q) &\equiv \mathrm{Tr}\; q^{L_0 - \frac{D-2}{24}} \bar{q}^{L_0 - \frac{D-2}{24}} \\ &= (2\mathrm{Im}\,\tau)^{-D/2} (q\bar{q})^{-\frac{D-2}{24}} \left| \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{D-2}} \right|^2, \\ &\quad q = e^{2\pi\imath\tau}, \quad \mathrm{Im}\,\tau \geq 0. \end{aligned} \tag{31}$$

The number $\frac{1}{24}$ in the exponent is due to the conformal anomaly. A proper explanation would take us too far afield, but it can be found in, for example, §7.1 of [7]. The combination $(D-2)$ is easy to understand in light-cone gauge, but can also be obtained in the conformal gauge if one also includes the contribution from the Faddeev-Popov ghosts. Here it is sufficient to note here that with $D = 26$ we recover the correct normal ordering constant $a = \tilde{a} = 1$. We also note that the mass for states in a level can be read from the corresponding exponent for q and $\bar{q}$ outside the $(2\mathrm{Im}\,\tau)^{-D/2}$ factor. For example, the exponent for the tachyon is negative, and that for massless states are zero.

[5]However there are further discrete identification due to large diffeomorphisms, to be discussed in lecture two.

The $(2\mathrm{Im}\,\tau)^{-D/2}$ factor is the result of momentum integration, and here we have D rather than $(D-2)$ since the string zero modes are not affected by the light-cone gauge condition. The coefficient in front of a monomial in q counts the multiplicity of states with the corresponding mass (= the degree of the monomial). For example, from (31) one sees that there is just one tachyon. To complete the calculation of the amplitude, one also needs to integrate over the moduli parameter τ, which parameterizes the length and twist of the torus as discussed earlier. Observe that the integration over $\mathrm{Re}\,\tau$ enforces the level matching condition, as those terms with unequal exponents for q and $\bar{q}$ will vanish.

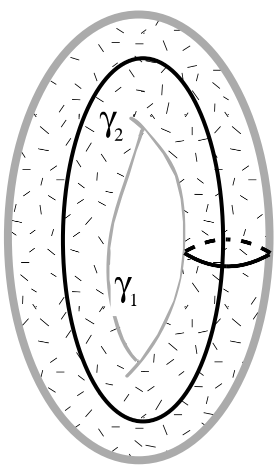


Figure 5. One loop vacuum to vacuum amplitude

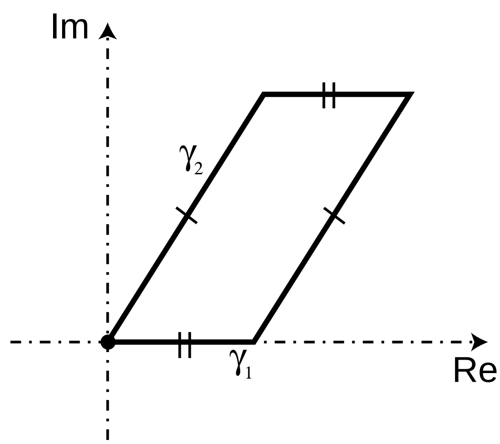


Figure 6. Torus and lattice

1.8 Conformal Field Theory

The conformal gauge action (15) is an example of a 2d *conformal field theory* (CFT). Although the details of CFT are outside the scope of this lecture (for extensive discussion on the subject, see for example [7] and [8]), we will now introduce some facts and concepts that will be useful. Consider a path integral calculation of a CFT over some Riemann surface, with some tubes extending to infinity. The field configurations at the ends of the tube correspond to states in the CFT Hilbert space. In string theory they represent external, asymptotic string states in a scattering process. We can perform arbitrary conformal transformations when evaluating the path integral of a CFT. Let us choose one that brings the tube C in (fig. 7) from infinity to within a finite distance from the scattering region. Because this would involve an infinite rescaling in the neighborhood of the end circle of tube C, the end circle, which has finite radius, will shrink to a point. Its effect should therefore be represented by the insertion of a local field operator at that point. It is called a *vertex operator*. Therefore there is a one-to-one correspondence between states and operators in CFT. In string theory, for example, a vertex operator taking momentum k has the form, : (oscillator part) $\times e^{ik\cdot X}$:, where :: denotes the normal ordering. The oscillator part of the operator is determined by its counterpart for the corresponding state. For example, the operator that creates an insertion of a massless operator of momentum k is

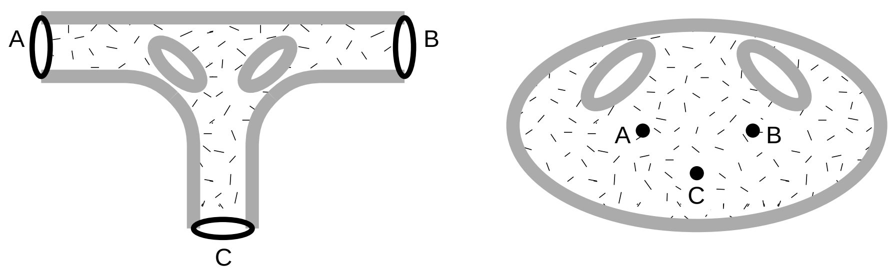
$$\xi_{\mu\nu} \colon \partial X^{\mu} \bar{\partial} X^{\nu} e^{ik\cdot X} \colon .$$

For the tachyon, the oscillator part is just the identity, so the vertex operator is simply : $e^{ik\cdot X}$:. Of course, not all vertex operators correspond to insertion of physical states. They have to obey the operatorial version of the physical state condition (22). The condition for the tachyon is simply $k^2 = 4/\alpha'$.

States that satisfy (22) with a and $\tilde{a}$ not necessarily equal to 1 are called *Virasoro primary* states of *conformal weight* $(a,\, \tilde{a})$[6]. The corresponding operators are called *Virasoro primary fields*. For a Virasoro primary operator Φ, its defining properties can be summarized in the *singular parts* of its operator product expansion (OPE) with the energy-momentum tensor:

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{a\phi(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi(w,\bar{w})}{(z-w)}, \\ \tilde{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{\tilde{a}\phi(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\phi(w,\bar{w})}{(\bar{z}-\bar{w})}. \end{aligned} \tag{32}$$

[6] So physical states are Virasoro primary states of conformal weight $(1, 1)$



7a. Before conformal transformation: asympototic states coming from infinity

7b. After conformal transformation: vertex operators inserted

Figure 7. 2-loop 3-string worldsheet, before and after conformal transformation

The Virasoro algebra (21) itself can be written as

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}, \tag{33}$$

and similarly for $\tilde{T}$ with no singularity between T and $\tilde{T}$. Thus T is almost a Virasoro primary field of weight $(2,0)$ except for its conformal anomaly. It is a fundamental property of a conformal field theory that its Hilbert space and operator content is a direct sum of often infinitely many irreducible representations of the left $\times$ right Virasoro algebra, each of which is generated by the action of the algebra on a highest weight state. The Virasoro primary fields of a CFT and their operator product expansion (OPE) completely characterize it. For later use, let us state the OPE between basic fields in the bosonic action (15):

$$\begin{aligned} \partial X^\mu(z) X^\nu(w) &\sim \frac{-\frac{\alpha'}{2}\eta^{\mu\nu}}{(z-w)}; \\ \partial X^\mu(z) \colon e^{\imath k\cdot X(w)}\colon &\sim \frac{-\imath\frac{\alpha'}{2}k^\mu}{(z-w)}; \\ \colon e^{\imath k_1\cdot X(z)}\colon\colon e^{\imath k_2\cdot X(w)}\colon &\sim |z-w|^{\alpha' k_1\cdot k_2}\colon e^{\imath k_1\cdot X(z)+\imath k\cdot X(w)}\colon. \end{aligned} \tag{34}$$

Let us sum over momentum and make a Fourier transformation, then the vertex operators for the massless particles are

$$h_{\mu\nu}(X(z,\bar{z}))\partial_z X^\mu \bar{\partial}_{\bar{z}} X^\nu$$

for the graviton field and

$$B_{\mu\nu}(X(z,\bar{z}))\partial_z X^\mu \bar{\partial}_{\bar{z}} X^\nu$$

for the antisymmetric tensor field. Now consider inserting coherent states of these fields — exponential of their integral over the worldsheet — in every correlation function we compute for the Polyakov action (29). Physically, this should be interpreted as vacuum expectation values for these spacetime fields. From the worldsheet viewpoint, they simply modify the action (29) into

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \{(\sqrt{g} g^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X))\partial_a X^\mu \partial_b X^\nu\}, \tag{35}$$

where $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We can also introduce a super-renormalizable term to (35) which corresponds to a VEV for the tachyon. Noting that the worldsheet scalar fields such as X^μ have zero scaling dimension, it is easy to see that the result is the most general renormalizable action we can write with X^μ and their derivatives in two dimensions. However, if we add a background for any one of the massive states, the corresponding operator would be non-renormalizable and would in general generate terms corresponding to the VEV's for all other massive states.

Students may notice that the dilaton Φ is missing in this discussion. If we allow ourselves to use the worldsheet metric g_{ab} in addition to the scalar field X^μ, there is another operator of dimension two on the worldsheet, $R\Phi(X)$, where R is the worldsheet curvature. It is a long story to explain why this is a proper coupling of the worldsheet to the dilaton field[7]. The complete worldsheet action under the background of the massless fields is then

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \{(\sqrt{g} g^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X))\partial_a X^\mu \partial_b X^\nu + \alpha' \sqrt{g} R \Phi(X).\} \tag{36}$$

[7] Very briefly, this coupling is obtained by regularizing the dilaton vertex operator on a curved worldsheet and rescaling the background metric.

We note that if we let $\Phi \to \Phi + \Phi_0$, where Φ_0 is a constant, then $S \to S + \Phi_0\chi$, where $\chi = 2 - 2h - b$ is the Euler number of the worldsheet surface, h is the number of handles and b that of boundaries on the worldsheet. Since h is the number of loops in the string "diagram," shifting the dilaton field by a constant Φ_0 is equivalent to multiplying the string loop expansion parameter κ^2 by $e^{2\Phi_0}$. Looking closely enough, all string diagrams can be seen as combinations of ϕ^3 type of vertices and κ their coupling constant.

Recall that earlier on when we considered the simple case in which all the VEVs of these massless spacetime fields vanish, i.e. when the sigma model string action (36) is free, the decoupling of the conformal factor λ in the metric g_{ab} requires conformal invariance at the quantum level. This then led to the requirement of $D = 26$. Now with some of the VEV's being nonzero, quantum conformal invariance requires the vanishing of β functions:

$$0 = \beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + O(\alpha'^2) \qquad (37)$$

$$0 = \beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha'(\nabla^\lambda \Phi) H_{\lambda\mu\nu} + O(\alpha'^2) \qquad (38)$$

$$0 = \beta^\Phi = -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha'(\nabla\Phi)^2 - \frac{1}{24}\alpha' H^2 + O(\alpha'^2) \qquad (39)$$

These can be regarded as the equation of motion coming from a spacetime action of G, B, and Φ:

$$S = \frac{1}{2\kappa^2} \int d^{26}X \sqrt{-G} e^{-2\Phi} \{R - \frac{1}{12} H^2 + 4(\nabla\Phi)^2 + O(\alpha')\}. \qquad (40)$$

Here we see explicitly that the shift $\Phi \to \Phi + \Phi_0$ can be compensated by $\kappa \to \kappa e^{\Phi_0}$ for constant Φ_0.

In this action, the normalization of the Einstein-Hilbert term is not standard, and the sign of the dilaton kinetic term is wrong. We can cure these problems by a field redefinition:

$$G_{\mu\nu} = e^{-\Phi/6} \tilde{G}_{\mu\nu}, \qquad (41)$$

and the action (40) can be rewritten as

$$S = \frac{1}{2\kappa^2} \int d^{26}X \sqrt{-\tilde{G}} \{\tilde{R} - \frac{1}{12} e^{-\frac{1}{3}\Phi} H^2 - \frac{1}{6}(\nabla\Phi)^2 + O(\alpha')\}. \qquad (42)$$

Different choices of the metric correspond to different units of length (different rulers). $\tilde{G}$ is known as the *Einstein metric* while G is called the *string metric*.

2 Lecture Two: Toroidal Compactifications

The restriction on spacetime dimension by requiring quantum mechanical consistency is a striking result. Some analog of it may one day tell us why we live in three spatial and one temporal dimensions. However, as a candidate theory of everything, string theory faces the immediate criticism that it gives us *too many* dimensions. Later, when we come to superstring, the critical dimension will be lowered to $(9+1)$, but that is still 6 dimensions in excess. Naturally one entertains the possibility that the true spacetime takes the form of a direct product $M^4 \times K$, where M^4 is the 4-dimensional Minkowski space we recognize everyday and K an extremely tiny compact manifold that our crude probes of nature have so far failed to reveal. As you all know, this idea has existed in field theory in the form of Kaluza-Klein program long before string theory was invented. However, as we will presently see, string compactifications introduce interesting "stringy" effects not seen in the usual Kaluza-Klein schemes.

For a string propagating in a $M \times K$ background spacetime with constant VEV Φ for the dilaton, we may absorb Φ into the string coupling constant. The conformal gauge action is then

$$S = \frac{1}{4\pi} \int d^2 z \, (G_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu, \tag{43}$$

where we have set α' to 2 by choosing a unit of length. Because of the direct product structure of $M \times K$, S can be split into an external part S_M involving coordinates on M and an internal part S_K on K, which can be studied separately. The analysis of S_M is trivial and all the interesting consequences of compactification come from S_K. In this lecture we concentrate on the simplest possible choices for K: the tori. They are simply products of S^1 and are flat. One can choose constant metrics for them and the nonlinear sigma models describing string propagating on them are still free as a two-dimensional QFT. We will consider the spacetime as being $M^{26-D} \times T^D$. Although the ultimate goal of string theory is to describe the $D = 4$ world we live in, it turns out to be very instructive and enlightening to consider diverse choices of D. We will encounter many ideas useful for the rest of these lectures as well as many others to come in this school.

2.1 Lattice and Torus

We can always parameterize a flat torus T^D so that its metric G_{ij} is constant and the coordinates x^i have period 2π, i.e.

$$T^D \equiv \frac{\mathbb{R}^D}{\sim} \ , \tag{44}$$

where

$$X^i \sim X^i + 2\pi m^i \ , \qquad m^i \in \mathbb{Z} \ .$$

We will use indices i,j,... in this coordinate system. It turns out to be convenient to introduce a constant vielbein e_i^a and new coordinates X^a to bring the metric into the standard Euclidean form:

$$\begin{aligned} G_{ij} &= e_i^a e_j^b \delta_{ab}, \qquad a = 1, \ldots, D. \\ X^a &\equiv e_i^a X^i \end{aligned}$$

We use indices a, b, ... in these coordinates. In the new coordinates X^a, the periodicity condition is changed to

$$X^a \sim X^a + 2\pi e_i^a m^i. \tag{45}$$

In this way, instead of characterizing the size and shape of a torus by defining it with a fixed lattice ($\mathbb{Z}^D$) as in (44) with an arbitrary constant Riemannian metric G_{ij}, we can use the fixed metric δ_{ab} and an arbitrary nondegenerate D-dimensional lattice:

$$\begin{aligned} T^D &= \frac{\mathbb{R}^D}{2\pi\Lambda} \ , \\ \Lambda &\equiv \{e_i^a m^i ; m^i \in \mathbb{Z}\} \ . \end{aligned}$$

The momentum k^a conjugate to the coordinates X^a on the torus is quantized so that

$$k \cdot \Delta X \in 2\pi\mathbb{Z},$$

where $\Delta X \in 2\pi\Lambda$ is a lattice vector[8]. Therefore

$$k^i = G^{ij} n_j, \quad n_j \in \mathbb{Z}.$$

Namely the momentum k is in the dual lattice Λ^* of Λ,

$$k^a \in \Lambda^*,$$

$$\Lambda^* \equiv \{e^{*ai} n_i; \quad n_i \in \mathbb{Z}\}, \qquad e^{*ai} \equiv e^a_j G^{ij}.$$

Let us consider string compactification over $K = T^D$, with vanishing B for the time being,

$$S_K = \frac{1}{4\pi} \int d^2 z \, \delta_{ab} \partial X^a \bar{\partial} X^b. \tag{46}$$

The most general solution of the equation of motion for X is

$$\begin{aligned} X^a &= x^a + 2p^a \sigma^0 + \omega^a \sigma^1 + \sum_{n \neq 0} \frac{\imath}{n} \{\alpha^a_n e^{-\imath n \sigma^+} + \tilde{\alpha}^a_n e^{+\imath n \sigma^-}\} \\ &= x^a - \imath p^a (z + \bar{z}) - \imath \omega^a (z - \bar{z})/2 + \sum_{n \neq 0} \frac{\imath}{n} \{\alpha^a_n e^{-nz} + \tilde{\alpha}^a_n e^{-n\bar{z}}\}. \end{aligned} \tag{47}$$

This differs from the solution for $\mathbb{R}^d$ (18) in a new term linear in σ^1. As σ^1 goes from 0 to 2π, X^a is displaced by $2\pi\omega^a$. On a T^D, ω^a does not have to vanish, because the closed string can wind around a nontrivial loop on T^D, provided $\omega^a \in \Lambda$. For this reason ω^a is called the *winding number*. After canonical quantization, we find the same commutation relations between x^a, p^a, and the mode operators as in the last lecture. The ω^a's commute among themselves as well as with the other operators. So we can group states into *winding sectors* — eigensubspaces of ω^a. On the other hand, since p^a is the momentum conjugate to the center of mass position X^a, by our previous discussion it takes values in Λ^*.

Let us rewrite the mode expansion for X's in a more symmetrical form:

$$X^a = X^a_L + X^a_R;$$

[8]More precisely, eigenstates of k have wavefunction $e^{\imath k \cdot x}$ in the basis in which X is diagonal. The wavefunction of a scalar particle must be single-valued. Since X and $X + \Delta X$ represent the same point of the torus, $e^{\imath k \cdot X}$ must be equal to $e^{\imath k \cdot (X + \Delta X)}$.

$$X_L^a = x_L^a - \imath p_L^a z + \sum_{n\neq 0} \frac{\imath}{n} \alpha_n^i e^{-nz}, \qquad X_R^a = x_R^a - \imath p_R^a \bar{z} + \sum_{n\neq 0} \frac{\imath}{n} \tilde{\alpha}_n^i e^{-n\bar{z}};$$
$$x_L^a = \frac{1}{2} x^a - \theta^a; \qquad x_R^a = \frac{1}{2} x^a + \theta^a; \tag{48}$$
$$p_L^a = p^a + \frac{\omega^a}{2} = e^{*ai} n_i + \frac{1}{2} e_i^a m^i; \qquad p_R^a = p^a - \frac{\omega^a}{2} = e^{*ai} n_i - \frac{1}{2} e_i^a m^i.$$

Here we have introduced operators θ^a which are the canonical conjugates to the winding numbers ω^a. Their existence is ensured by the existence and uniqueness of a winding sector for each winding number. p_L and p_R are called *left* and *right momentum* respectively. They also appear in the energy-momentum tensor:

$$L_n = \sum_m \frac{1}{2} :\alpha_{n-m}\alpha_m:, \quad \alpha_0^a = p_L^a$$

$$\tilde{L}_n = \sum_m \frac{1}{2} :\tilde{\alpha}_{n-m}\tilde{\alpha}_m:, \quad \tilde{\alpha}_0^a = p_R^a$$

The OPE between the vertex operators $e^{ik\cdot X_L}$ is

$$:e^{\imath k_1 \cdot X_L(z)}::e^{\imath k_2 \cdot X_L(w)}: \sim (z-w)^{k_1\cdot k_2} :e^{\imath k_1\cdot X_L(z) + \imath k_2 \cdot X_L(w)}:, \tag{49}$$

with similar expression for the right movers.

The expressions for p_L and p_R suggest some "duality" between the winding number ω and the momentum p. Consider a pair of compactification lattices whose lattice vectors e_i^a and $e_i'^a$ are related as $e_i'^a = 2e^{*ai}$. These two compactifications give the same spectrum since their allowed values of the momenta are related as

$$p_L \leftrightarrow p_L'; \qquad p_R \leftrightarrow -p_R' \tag{50}$$

by interchanging the labels n_i and m_i.

2.2 Example: Compactification on S^1

Let us try out the above construction on the simplest case: compactification over a circle of radius R. Then the lattice structure is trivial:

$$e_1^1 = R; \qquad e_1^{*1} = \frac{1}{R}.$$

The allowed values for the momenta are simply

$$p_L = \frac{1}{R}n + \frac{R}{2}m, \qquad p_R = \frac{1}{R}n - \frac{R}{2}m. \tag{51}$$

The duality just mentioned also takes a simple form. Consider another theory compactified on radius $R' = \frac{2}{R}$. If we interchange n and m in (51), then we can identify the momentum operator for $R' = \frac{2}{R}$ with that of R with the isomorphism (50). Now extending this to an isomorphism of the fields in the two theories, the commutation relation between $x_{L,R}$ and $p_{L,R}$ forces us to require also

$$x_L \leftrightarrow x'_L; \qquad x_R \leftrightarrow -x'_R. \tag{52}$$

In order to have the spacetime interpretation of this duality as inverting the radius of (or equivalently the metric G_{ij} on) the circle, we need to transform the oscillators as well:

$$\alpha_n \leftrightarrow \alpha'_n; \qquad \tilde{\alpha}_n \leftrightarrow -\tilde{\alpha}'_n. \tag{53}$$

The isomorphism (50, 52, 53) can be summarized in a more compact form[9]:

$$X_L \leftrightarrow X'_L; \qquad X_R \leftrightarrow -X'_R. \tag{54}$$

This isomorphism of operators clearly translates into an isomorphism of the Hilbert space. To see this more explicitly, we can compute the partition function (31):

$$\begin{aligned} Z &= \operatorname{Tr} q^{L_0-1/24}\bar{q}^{L_0-1/24}, \\ &= \frac{1}{\left|q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\right|^2}\sum_{n,m} q^{\frac{1}{2}(n/R+mR/2)^2}\bar{q}^{\frac{1}{2}(n/R-mR/2)^2}, \\ & \quad q = e^{2\pi\imath\tau}, \quad \operatorname{Im}\tau \geq 0. \end{aligned} \tag{55}$$

It is invariant under $R \to \frac{2}{R}$[10]. To show that the two theories are actually equivalent, we have also to show that this map is an operator algebra isomorphism. This is easy, since both theories are free and their operator product

[9] As a side remark, we note that this is a two-dimensional version of the "electro-magnetic" duality discussed in the later courses of this school.

[10] One can also evaluate the path integral on closed Riemann surfaces of arbitrary genus. There $R \to \frac{2}{R}$ is an invariance provided one shifts the constant dilaton field appropriately. See [9].

expansions can be computed exactly. Thus $R \to \frac{2}{R}$ is an exact symmetry of the action (43), on arbitrary Riemann surfaces. But is it really a symmetry of the spacetime theory that the worldsheet action describes? From the discussion of string perturbation in lecture one, we see that it is a symmetry of string theory order by order in string perturbation expansion. In fact, as we will see presently, it is a *gauge symmetry* of string theory.

2.3 Self-Dual Radius: $R = \sqrt{2}$

At the particular value of $R = \sqrt{2}$[11], the duality $R \to \frac{2}{R}$ maps R back to its original value and we expect something interesting to occur. Indeed, at this radius, the partition function (55) can be rewritten, after some elementary manipulation, as

$$Z = \left| \frac{1}{\eta} \sum_n q^{n^2} \right|^2 + \left| \frac{1}{\eta} \sum_n q^{(n+1/2)^2} \right|^2 , \qquad (56)$$

where Dedekind's η-function is simply the denominator in (55):

$$\eta \equiv q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

The first term in the sum is the modulus squared of $q^{-\frac{1}{24}}(1 + 3q + \cdots)$. The second term is that of $q^{-\frac{1}{24}}(2q^{\frac{1}{4}} + \cdots)$. They (the unsquared sums) are actually the character formulae of the two irreducible representation of an algebra, called $SU(2)$ affine Lie algebra at level $k = 1$. Where does the algebra comes from?

The contributions from (56) to the massless spectrum of the complete string theory are those terms first order in q and $\bar{q}$ inside the absolute values signs. Let us look at the left movers. The states are

$$\alpha_{-1} \left| p_L = 0 \right\rangle , \quad \left| p_L = \pm\sqrt{2} \right\rangle,$$

which respectively correspond to vertex operators

$$\partial X_L, \quad e^{\pm \imath \sqrt{2} X_L}. \qquad (57)$$

[11] If we put α' back, the self-dual radius would be $\sqrt{\alpha'}$ by dimensional analysis.

The first of the three exists for arbitrary radius R, but it is not difficult to show that the last two states only exist in the spectrum when $R=\sqrt{2}$. Using (34) and (49), one can evaluate the OPE's among them as

$$J^a(w)J^b(z) \sim \frac{k\delta^{ab}/2}{(w-z)^2} + \frac{i\epsilon^{ab}{}_c J^c(z)}{(w-z)}, \tag{58}$$

$$J^1 \equiv \frac{1}{2}(e^{\imath\sqrt{2}X_L} + e^{-\imath\sqrt{2}X_L}), \quad J^2 \equiv \frac{-\imath}{2}(e^{\imath\sqrt{2}X_L} - e^{-\imath\sqrt{2}X_L}), \quad j^3 \equiv \imath\frac{1}{\sqrt{2}}\partial X,$$

with $k=1$. Here ϵ^{ab}_c is the structure constant of $SU(2)$. This is precisely the definition of $SU(2)$ affine Lie algebra with level $k=1$. The same story is repeated for the right movers.

To construct a consistent bosonic string theory, we have also to take into account M^{25}, the external part of the spacetime. Let the coordinate for the S^1 be X^{25}. To make a vertex operator for the massless particle we have to choose, for both the left and right moving parts, contributions from either the coordinates on M^{25} or X^{25}. At generic radius, they are

$$\partial X^\mu \bar{\partial} X^\nu, \quad \mu,\nu = 0,\ldots,24,$$

which include the graviton, the antisymmetric tensor, and the dilaton in M^{25},

$$\partial X^\mu \bar{\partial} X^{25}, \qquad \partial X^{25} \bar{\partial} X^\mu,$$

which correspond to two $U(1)$ gauge fields[12] in M^{25}, and

$$\partial X^{25} \bar{\partial} X^{25},$$

a neutral scalar field in M^{25}. This is the usual massless part of Kaluza-Klein spectrum. But as we just found, at $R=\sqrt{2}$, there will be two additional gauge fields,

$$\partial X^\mu e^{\pm\imath\sqrt{2}X^{25}_R}, \qquad e^{\pm\imath\sqrt{2}X^{25}_L}\bar{\partial}X^\mu$$

and 8 more scalars

$$e^{\pm\imath\sqrt{2}X^{25}_L}e^{\pm\imath\sqrt{2}X^{25}_R}, \qquad \partial X^{25}_L e^{\pm\imath\sqrt{2}X^{25}_R}, \qquad e^{\pm\imath\sqrt{2}X^{25}_L}\partial X^{25}_R.$$

[12] Weinberg showed in [4] that a consistent theory of massless spin one field A_μ must have gauge invariance with A_μ as gauge field.

Thus we have in total $6 = 3 + 3$ gauge fields and $1 + 8 = 9 = 3 \times 3$ scalars at the massless level. The OPE (58) allows one to calculate tree level S-matrix elements among the gauge fields, and one finds that the gauge group is $SU(2)$. Hence we see that, at $R = \sqrt{2}$, we have an enhancement of gauge symmetry from $U(1)_L \times U(1)_R$[13] to $SU(2)_L \times SU(2)_R$ with a Higgs transforming in $(3, 3)$ under them. To make a connection between this observation and the $R \to \frac{2}{R}$ duality, let us make a digression into conformal field theory.

From the last lecture we see that conformal invariance and hence cancellation of conformal anomaly is crucial for a consistent string theory. Generic conformal field theories do not have a spacetime interpretation. Since only the spacetime in the uncompactified Minkowski space is observable, one may consider using arbitrary CFT to represent the effects of "compactification" even if they do not have any spacetime interpretation like that of (43). This is consistent as long as they have the right amount of central charge so that the total conformal anomaly still cancels. Therefore we should study the moduli spaces of CFT. Recall that for a given conformal field theory, we may perturb it by adding *marginal* operators to the action while maintaining conformal invariance. Therefore the space of marginal operators for a theory at a particular point on the moduli space of CFT is the tangent space at that point.

A marginal operator for two-dimensional conformal field theory is one with conformal dimension $(1, 1)$[14]. It is not too difficult to see that they are exactly those which create scalar massless particles in the Minkowski space. Therefore, for our case, there seem to be 9 independent directions to deform the $c = 1$ conformal field theory away from the self-dual radius, corresponding to giving VEV's to the $(3, 3)$ Higgs. When the VEV is turned on, the $SU(2)_L \times SU(2)_R$ gauge symmetry is spontaneously broken down to a $U(1)_L \times U(1)_R$. However, the same gauge symmetry tells us we can always choose a gauge so that only one component of them, say, the one coupled to $\partial X^{25} \bar{\partial} X^{25}$, has a nonzero VEV a. It has the simple spacetime interpretation of $a = R - \sqrt{2}$ when R is close to the self-dual value $\sqrt{2}$. Moreover, there is a residual Z_2 gauge symmetry, namely

[13] Here the subscript "L" ("R") just refers to their origin from the left (right) movers. It has nothing to do with spacetime chirality.

[14] In general, this is just a necessary condition. One must also require that after an infinitesimal deformation by themselves their conformal weights remain unchanged.

the Weyl group for either of the two $SU(2)$'s, which inverts the sign of the a. This is to first order the map $R = (\sqrt{2}+a) \to \frac{2}{R} = \sqrt{2} - a + O(a^2)$. Therefore at generic radius R, the T-duality $R \to \frac{2}{R}$ is the remnant of a spacetime gauge symmetry.

From this we also see that near the self-dual radius, the moduli space for the conformal field theory corresponding to compactification on a circle is one-dimensional and looks like figure 8. [15] If we try to go to smaller radius than $\sqrt{2}$, we will end up, via T-duality, with a larger radius. This is a hint that string theory possess a minimal length scale $\sqrt{\alpha'}$ and we cannot probe or define the physics at a smaller scale[16].

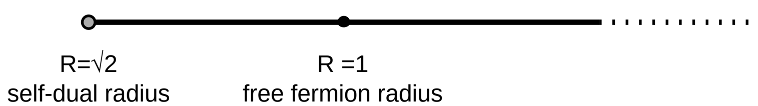


Figure 8. Moduli space of c=1 CFT
in the neighborhood of the self-dual radius.

2.4 $R = 1$

Let us also study the case $R = 1$ or equivalently $R = 2$. The motivation will be self-evident soon. By using the product formulae of the *theta functions*, we can rewrite the sums in the partition function into products:

$$\begin{aligned} Z &= \frac{1}{2}\left\{ \left| \frac{1}{\eta} \sum_n q^{1/2n^2} \right|^2 + \left| \frac{1}{\eta} \sum_n (q^{\frac{1}{2}n^2}(-1)^n) \right|^2 + \left| \frac{1}{\eta} \sum_n q^{\frac{1}{2}(n+\frac{1}{2})^2} \right|^2 \right\} \\ &= \frac{1}{2}\left| q^{-\frac{1}{24}} \right|^2 \left\{ \left| \prod_{r=1}^{\infty} (1+q^{r-\frac{1}{2}})^2 \right|^2 + \left| \prod_{r=1}^{\infty} (1-q^{r-\frac{1}{2}})^2 \right|^2 \right. \end{aligned}$$

[15] The complete moduli space is much more complicated, with a new branch and some discrete points. See §8.7 of [7] for an introduction.

[16] This statement requires significant qualification after D-branes come into the story, as discussed by S. Shenker in this school.

$$+\left|2q^{\frac{1}{8}}\prod_{r=1}^{\infty}(1+q^r)^2\right|^2\Bigg\}. \tag{59}$$

We will recover the same partition from worldsheet *fermions*.

The spectrum of momenta at $R=1$ is

$$p_L = n + \frac{1}{2}m; \quad p_R = n - \frac{1}{2}m, \quad m, n \in \mathbb{Z}. \tag{60}$$

To understand why this is related to the fermions, consider the following operators

$$\begin{aligned} \Psi_L(z) &= e^{\imath X_L(z)}, & \bar{\Psi}_L(z) &= e^{-\imath X_L(z)} \\ \Psi_R(\bar{z}) &= e^{\imath X_R(\bar{z})}, & \bar{\Psi}_R(\bar{z}) &= e^{-\imath X_R(\bar{z})}. \end{aligned} \tag{61}$$

One can calculate the OPE between them:

$$\Psi_L(z)\bar{\Psi}_L(w) \sim \frac{1}{(z-w)}, \qquad \Psi_R(\bar{z})\bar{\Psi}_R(\bar{w}) \sim \frac{1}{(\bar{z}-\bar{w})} \tag{62}$$

As we will see, these are precisely the OPE's for a massless free Dirac fermion Ψ on the worldsheet, with Ψ_L and Ψ_R being its two Weyl components. Thus we call them free fermion operators. To be precise, Ψ_L and $\bar{\Psi}_L$ do not have corresponding states in the spectrum since they carry momenta $p_L = \pm\frac{1}{2}$ and $p_R = 0$ and hence map states labeled by $(n, m) \in \mathbb{Z} \otimes \mathbb{Z}$ to states that are not. What we do have in the spectrum are operators bilinear in these fermions as $\Psi_L\Psi_R$, $\Psi_L\bar{\Psi}_R$.

As $\sigma^1 \to \sigma^1 + 2\pi$, there are two types of boundary conditions for the fermions. When $p_L \in \mathbb{Z} + \frac{1}{2}$ (and thus $p_R \in \mathbb{Z} + \frac{1}{2}$), they obey the periodic, *Ramond* (R), boundary condition. On the other hand, when $p_L \in \mathbb{Z}$ (and accordingly $p_R \in \mathbb{Z}$), the fermions obey the anti-periodic, *Neveu-Schwarz* (NS), boundary condition[17]. Clearly both types of boundary conditions show up in the lattice $(n, m) \in \mathbb{Z} \otimes \mathbb{Z}$. The free fermion operators do not mix between

[17] This statement requires some further elaboration. If $:\exp(\imath X_L):$ were given by $\exp\left(\sum_{n>0}(\imath/n)\alpha^i_{-n}e^{nz}\right)\,\exp(\imath x_L)\,\exp(-\imath p_L z)\,\exp\left(\sum_{n>0}(\imath/n)\alpha^i_{+n}e^{-nz}\right)$, by our definition NS (R) sector should be (anti-)periodic. However, in the the coordinate system we are using, where the worldsheet is a cylinder, there should be an additional factor of $e^{-z/2}$. This modifies the term linear in z on the exponent to $e^{(p_L-1/2)z}$ and gives the correct period-

R and NS boundary conditions. This suggests us to define two corresponding sectors of Hilbert space to host the free fermion operators. The periodic boundary condition happens when $(n,m) \in \mathbb{Z} \otimes (2\mathbb{Z}+1)$. We call them in the *R-R sector* since both Ψ_L and Ψ_R obey the R boundary condition. On the other hand, the anti-periodic boundary condition is realized in the *NS-NS sector* with $(n,m) \in \mathbb{Z} \otimes (2\mathbb{Z})$.

Hence the worldsheet boson describing compactification over a circle of radius $R = 1$ is *equivalent* to the worldsheet fermions after including both periodic and anti-periodic boundary conditions and then take a certain projection (explained below). This is called *bosonization* or *fermionization* depending on how you look at it.

Now let us study this from the fermion side. Consider the action for a massless Dirac spinor $\Psi(z)$ in $(1+1)$ dimension[18]:

$$\begin{aligned} S &= \frac{\imath}{2\pi} \int d^2\sigma\, \bar{\Psi}\gamma^a \partial_a \Psi, \\ &= \frac{\imath}{\pi} \int d^2\sigma\, \bar{\Psi}_L \bar{\partial} \Psi_L - \frac{\imath}{\pi} \int d^2\sigma\, \bar{\Psi}_R \partial \Psi_R, \\ & \bar{\Psi} \equiv \Psi^+ \gamma^0. \end{aligned} \tag{63}$$

Just like the bosonic theories we have discussed so far, the left and right movers decouple. In fact, it is a conformal field theory with the same central charge

icity. The simplest way to understand the origin of this factor is to note that the operator $:e^{\imath k X_L(z)}:$, when acting on the vacuum, should create a state with energy $k^2/2$ since this vertex operator's conformal weight is $(k^2/2, 0)$. In the cylindrical coordinate we are using, it should have a (Euclidean) time dependence of $e^{-k^2 t/2}$. Since it is holomorphically dependent on z, the correct factor is $e^{-k^2 z/2}$. In our case, $k = \pm 1$, and the factor is $e^{-z/2}$.

[18]In $(1,1)$-dimensional or in $(T, T+8k)$-dimensional spacetime, the Weyl condition can be compatible with the Majorana condition. For instance, on the worldsheet, which has signature $(1,1)$, one can define $\gamma^0 = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}$ which are purely imaginary. Then $\imath\gamma^a\partial_a$ is real and it is consistent with Dirac equation to require ψ to be real, i.e. Majorana. At the same time, $\gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal and real as well, so we can define $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, with ψ_L and ψ_R each being a *Majorana-Weyl* fermion. However, such a thing does not exist if the signature is (2,0), so to Euclideanize the worldsheet, we should combine pairs of Majorana-Weyl spinors ψ's into complex Weyl spinors Ψ's.

$c = 1$. Let us concentrate on the left movers. We will freely drop the subscript $_L$ without warning when there is no ambiguity. The mode expansions for the fermion fields are

$$\Psi(z) = \sum_r \Psi_r e^{-rz}; \qquad \bar{\Psi}(z) = \sum_r \Psi_r e^{-r\bar{z}} \tag{64}$$

Canonical quantization gives commutation relation between the modes:

$$\{\Psi_r, \bar{\Psi}_s\} = \delta_{r+s,0}, \tag{65}$$

from which one can derive the OPE (62) we previously calculated through bosonization. To make contact with the two sectors of Hilbert space discussed earlier, we note that the fermions are worldsheet spinors. As such, they can be either periodic or anti-periodic as $\sigma^1 \to \sigma^1 + 2\pi$, identified as R and NS sector respectively. The (anti-)periodicity also determines the modding r in (64). Therefore, $r \in \mathbb{Z}$ in R sector and $r \in \mathbb{Z} + \frac{1}{2}$ in NS sector.

Using bosonization, we can also find

$$: \Psi_L(z)\bar{\Psi}_L(z) : = \imath\partial X_L(z), \tag{66}$$

where : : denotes the non-singular part of the OPE $\Psi_L(w)\bar{\Psi}_L(z)$ in the limit $w \to z$. ∂X_L is the current associated with the $U(1)$ symmetry which shifts X_L by a constant. The charge for this current is its zero mode p_L. Since the fermion operators Ψ_L and $\bar{\Psi}_L$ carry $p_L = \pm 1$, the operator

$$F_L \equiv p_L$$

measures the fermion number. The bosonization rule (66) allows us to re-express the energy-momentum tensor for the bosonic theory in terms of the fermionic fields

$$T(z) = \sum_n L_n e^{-nz} = -\frac{1}{2} : \partial X \partial X : = -\frac{1}{2} : \bar{\Psi}\partial\Psi : -\frac{1}{2} : \Psi\partial\bar{\Psi} :,$$

in agreement with the energy-momentum tensor for the fermionic theory found by the usual means. In particular, its zero mode is

$$L_0 = \sum_{r>0} r \left(\Psi_{-r}\bar{\Psi}_r + \bar{\Psi}_{-r}\Psi_r\right) . \tag{67}$$

Now we are ready to compute the partition functions for the fermionic theory and have our final check of bosonization. We have for NS sector,

$$\mathrm{Tr}\, q^{L_0} = \prod_{r=1}^{\infty}(1+q^{r-\frac{1}{2}})^2,$$

$$\mathrm{Tr}\,(-1)^F q^{L_0} = \prod_{r=1}^{\infty}(1-q^{r-\frac{1}{2}})^2,$$

and for R sector,

$$\mathrm{Tr}\, q^{L_0} = (1+1)q^{\frac{1}{8}}\prod_{r=1}^{\infty}(1+q^r)^2,$$

$$\mathrm{Tr}\,(-1)^F q^{L_0} = (1-1)q^{\frac{1}{8}}\prod_{r=1}^{\infty}(1-q^r)^2 = 0.$$

Some explanation is warranted. The squares in all four expressions are due to that we have both Ψ_L and $\bar{\Psi}_L$ at each modding. In R sector, since the modding is even, we have the commutation relation

$$\{\Psi_0, \bar{\Psi}_0\} = 1.$$

This is represented by a single fermionic oscillator. Note we can also rewrite this as the Clifford algebra in two dimensions:

$$\{\psi_i, \bar{\psi}_j\} = 2\delta^{ij}, \qquad i,j = 1,2,$$

$$\Psi \equiv \frac{1}{2}(\psi^1 + \imath\psi^2).$$

Its straightforward generalization to higher even-dimensional Clifford algebra will be useful in the next lecture. The R ground states here therefore consist of two states: $|+\rangle$ and $|-\rangle$:

$$\Psi_0\,|-\rangle = 0, \quad \bar{\Psi}_0\,|+\rangle = 0,$$

$$\Psi_0\,|+\rangle = |-\rangle\,, \qquad \bar{\Psi}_0\,|-\rangle = |+\rangle\,.$$

States built from $|+\rangle$ and $|-\rangle$ with nonzero modes give identical contribution to $\mathrm{Tr}\, q^{L_0}$. However they have opposite $(-1)^F$ parity as

$$\left\{\Psi_0, (-1)^F\right\} = 0, \quad \left\{\bar{\Psi}_0, (-1)^F\right\} = 0. \tag{68}$$

Therefore their contributions to $\text{Tr}\,(-1)^F q^{L_0}$ are equal in magnitude but opposite in sign. The R ground states have charges $p_L = \pm\frac{1}{2}$, and their conformal weight is $\frac{1}{8}$ which accounts for the $q^{1/8}$ factor in the R sector partition function. In fact one can map the NS ground state to the two R ground states using operators

$$\sigma^{\pm} \equiv e^{\pm \imath \frac{1}{2}\phi_L}$$

with conformal weight $\frac{1}{8}$[19]. Using the OPE (49) one can show that they transform into each other under, and flip the boundary condition of, the free fermion operators. They are called spin field operators and form a representation of Clifford algebra in two dimensions under the actions of Ψ in OPE.

Combining the left and right movers, we can rewrite (59) as

$$Z = \text{Tr}_{\,\text{NS}-\text{NS}\oplus\text{R}-\text{R}}\, P\, q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24} \tag{69}$$

where we introduce a projection operator

$$P \equiv \big(1 + (-1)^{F_L+F_R}\big)/2. \tag{70}$$

This P projects out states with odd number of fermions from NS-NS and R-R sectors to obtain the Hilbert space for the original bosonic theory.

2.5 Modular Invariance and Narain's Condition

Let us continue the discussion of string on S^1 with $R = 1$. As mentioned before, the partition function $\text{Tr}q^{L_0-1/24}\bar{q}^{\tilde{L}_0-1/24}$ for a 2d theory can be interpreted as evaluating its path integral on a torus (fig. 6). The path integral formally involves integrating over all possible field fluctuations, which must satisfy appropriate boundary conditions when a nontrivial manifold is involved. The theory is that of free fermions (spinors). Spinors cannot be defined on all manifolds. But when they can, there is often more than one consistent but inequivalent way to do so, called *spin structures*. A proper explanation of these matters is outside the scope of these lectures but can be found in §12

[19] As mentioned earlier, the boundary condition of the left and right moving sector are correlated. The complete operator that does this is $\exp(\pm\imath\frac{1}{2}\phi_L \pm \imath\frac{1}{2}\phi_R)$.

of [3]. As a matter of fact, there are four consistent ways to define spinors on a torus, corresponding to choosing periodic or anti-periodic boundary conditions as a spinor goes around either of the two independent nontrivial cycles shown in figure 5. After identifying γ_2 as "time" and γ_1 as the spatial extent of the string, R (NS) sector corresponds to (anti-)periodic boundary condition around the latter. Because $(-1)^F$ anticommutes with Ψ, its insertion in the trace flips the boundary condition around γ_2. It can be shown that without its insertion, that the boundary condition is antiperiodic. These situations are summarized in figure 9. Therefore we can interpret the projection and the sum over R-R and NS-NS sectors in (69) as a sum over all possible spin structures, but what is the reason behind this summation?

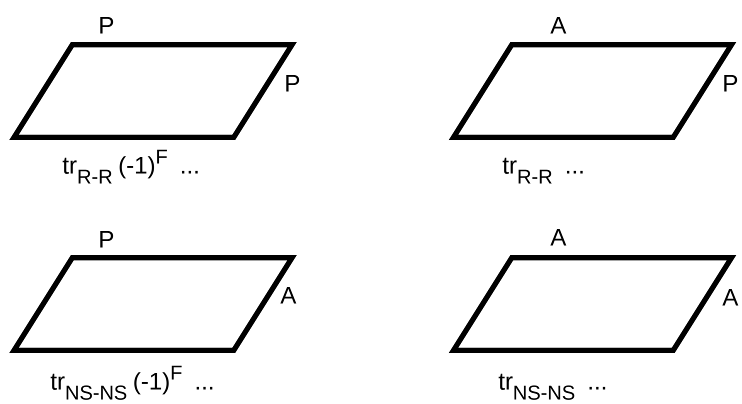


Figure 9. Spin Structure of T^2

To understand this we need to introduce the notion of modular invariance. As mentioned in the last lecture, we can characterize the shape of a torus by a complex parameter τ taking value in the upper complex half-plane. However, not all distinct values of τ correspond to distinct tori. In fact, define the operations $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$. The operation T corresponds to changing one of the lattice basis defining the torus and S to swapping the basis. They generate *large diffeomorphisms* of the torus, which cannot be smoothly connected to the identity map. It should be clear that they map the different spin structures into each other. Only the spin structure which is periodic around both cycles is invariant. This is not surprising since the corresponding partition function vanishes identically. For the bosonic theory, the counterpart of spin structures correspond to the different windings around the target space S^1 as the coordinate X goes around the two cycles of the worldsheet torus. When we sum over all possible value for the center of mass momentum and the winding numbers, we are summing over all these different contributions, rendering the partition function invariant under S and T — it is *modular invariant*. Therefore to have an equivalence between the bosonic and fermionic theories, we *must* sum over spin structures on the fermionic side.

As a side note, S and T generates the group $\mathrm{SL}(2,\mathbb{Z})$, the group of 2×2 matrices with integral elements and unit determinant:

$$T : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This group will appear time after time throughout this school[20]. Here we merely note that they have the interpretation of changing the basis (e^1, e^2) of the lattice defining torus:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : \begin{pmatrix} e^{1\prime} \\ e^{2\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \end{pmatrix}$$

Their action on the moduli τ is

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}.$$

This discrete identification divides the upper complex plane into infinite number of *fundamental domains*, each of which is a single cover of the true moduli space for the torus.

We will now demonstrate the modular invariance of the partition for the most general class of toroidal compactification of the bosonic string. Recall

[20]It eventually made its way to the official T-shirt for this school.

our earlier expression for the left and right moving momenta:

$$p_L^a = p^a + \frac{\omega^a}{2} = e^{*ai} n_i + \frac{1}{2} e_i^a m^i; \qquad p_R^a = p^a - \frac{\omega^a}{2} = e^{*ai} n_i - \frac{1}{2} e_i^a m^i.$$

Let us combine them into one $(D+D)$-component column vector:

$$\hat{p} = \begin{pmatrix} p_L^a \\ p_R^a \end{pmatrix}.$$

This construction treats Λ and Λ^* on equal footing as

$$\hat{p} = \hat{e}^{*i} n_i + \hat{e}_j m^j, \tag{71}$$

where

$$\hat{e}_j = \frac{1}{2} \begin{pmatrix} e_j^a \\ -e_j^a \end{pmatrix}; \qquad \hat{e}^{*i} = \begin{pmatrix} e^{*ai} \\ e^{*ai} \end{pmatrix}. \tag{72}$$

Hence $\hat{p}$ takes value in a $(D+D)$-dimensional lattice $\hat{\Lambda}$ spanned by $\{\hat{e}^{*i}\}$ and $\{\hat{e}_j\}$.

We also define a metric of signature $(D,\, D)$ on this $2d$-dimensional space:

$$\hat{\delta} = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & -\delta_{ab} \end{pmatrix}$$

This metric captures some of the most important properties of the lattice $\hat{\Lambda}$. Because of (72),

$$\hat{e}_i \cdot \hat{e}_j = 0; \qquad \hat{e}^{*i} \cdot \hat{e}^{*j} = 0; \qquad \hat{e}_i \cdot \hat{e}^{*j} = \delta_i^j, \tag{73}$$

$$(\hat{e}^{*i} n_i + \hat{e}^j m^j) \cdot (\hat{e}^{*i} n_i' + \hat{e}^j m^{j\prime}) = n_i m^{i\prime} + n_j' m^j \tag{74}$$

which implies (1) if $q \in \hat{\Lambda}$, then $q \cdot q \in 2\mathbb{Z}$ and (2) the dual lattice of $\hat{\Lambda}$ is $\hat{\Lambda}$ itself. Such a lattice is called even, because q^2 is *even*, *self-dual*, because $\hat{\Lambda}^* = \hat{\Lambda}$, and *Lorentzian*, because of the signature of the metric with respect to which the conditions are imposed.

Now consider as the internal part of string "compactification" a conformal field theory the same as that of (46) except that its momenta live on some general $(D+D)$-dimensional lattice $\hat{\Lambda}$. Its partition function is

$$Z_{\hat{\Lambda}} = \frac{1}{|\eta(q)|^{2D}} \sum_{(p,\tilde{p}) \in \hat{\Lambda}} q^{\frac{1}{2}p^2} \bar{q}^{\frac{1}{2}\tilde{p}^2}$$

What requirement should we impose on Λ? Recall that string theory has worldsheet diffeomorphism invariance. Modular transformations are diffeomorphisms of the torus that cannot be contracted smoothly to the identity. They are residual gauge symmetries after gauge fixing. Like the Weyl rescaling symmetry, it might be anomalous quantum mechanically. In the last lecture, we see that conformal invariance at the quantum level is responsible for removing unphysical states from the string spectrum and determining the critical dimension. Similarly, modular invariance would imply we need only to integrate over a fundamental domain as the moduli space of the torus. This turns out to be essential for preventing ultraviolet divergences in string theory (§8.2 of [3]). It is natural to ask what kind of $\hat{\Lambda}$ would ensure the modular invariance of $Z_{\hat{\Lambda}}$.

Since the modular group $\mathrm{SL}(2,\mathbb{Z})$ is generated by S and T, it is sufficient to require that Z is invariant under both of them. For the T-transformation,

$$Z_{\hat{\Lambda}}(\tau+1) = \frac{1}{|\eta|^{2D}} \sum_{(p,\tilde{p})\in\Lambda} q^{\frac{1}{2}p^2} \bar{q}^{\frac{1}{2}\tilde{p}^2} e^{2\imath\pi\frac{1}{2}(p^2-\tilde{p}^2)}.$$

Since $(p^2-\tilde{p}^2)\in 2\mathbb{Z}$ for an even Lorentizan lattice $\hat{\Lambda}$, the partition function is invariant under T. For the S-transformation, we make use of the *Poisson resummation* (see the appendix), and the identity

$$\eta(-\frac{1}{\tau}) = \sqrt{-\imath\tau}\,\eta(\tau).$$

One then finds

$$Z_{\hat{\Lambda}}(-\frac{1}{\tau}) = \mathrm{vol}(\hat{\Lambda}^*) Z_{\hat{\Lambda}^*}(\tau).$$

Here $\mathrm{vol}(\hat{\Lambda})$ denotes the volume of the unit cell of the lattice $\hat{\Lambda}$. Since

$$\mathrm{vol}(\hat{\Lambda})\mathrm{vol}(\hat{\Lambda}^*) = 1$$

for any lattice $\hat{\Lambda}$, $\mathrm{vol}(\hat{\Lambda}^*) = 1$ provided $\hat{\Lambda}$ is self-dual. In this case, the above equation gives

$$Z_{\hat{\Lambda}}(-\frac{1}{\tau}) = Z_{\hat{\Lambda}}(\tau).$$

Therefore if $\hat{\Lambda}$ is an even self-dual Lorentzian lattice, $Z_{\hat{\Lambda}}$ is modular invariant and is a candidate for consistent string compactification. This is known as *Narain's condition* (ref. 340 in [3], Vol 1).

Let us now figure out what is the moduli space of such a compactification. Any nontrivial $O(D, D)$ rotation would map one even self-dual Lorentzian lattice into a different one. The converse is a mathematical fact: any two even self-dual Lorentzian lattices are related by some $O(D, D)$ rotation. Therefore the space of such lattices is simply $O(D, D)$. However, not all of them correspond to different compactifications. The spectrum for the $(26 - D)$-dimensional theory is determined by p_L^2 and p_R^2. They are left invariant by $O(D) \times O(D)$, the maximal compact subgroup of $O(D, D)$, acting independently on the left and right momenta respectively. Therefore the space of vacua is *locally* $O(D, D)/(O(D) \times O(D))$, of dimension D^2.

In fact a *spacetime* interpretation can be given to such a construction. In (46) we have set to zero the background antisymmetric tensor field B. The dimension of the space of possible G is only $D(D + 1)/2$. However B's contribution to the total energy vanishes as long as its field strength H is zero. This allows us to give to B arbitrary constant VEV while staying in the vacua. Since B contains $D(D-1)$ independent components, this fully accounts for the dimension of the space of vacua. Indeed, by canonically quantizing the action

$$S_K = \frac{1}{4\pi}\int d^2z\,(G_{ab} + B_{ab})\partial X^a \bar{\partial} X^b,$$

one finds that (72) is modified:

$$\hat{e}_j = \frac{1}{2}\begin{pmatrix} e^{*ai}B_{ji} + e_j^a \\ e^{*ai}B_{ji} - e_j^a \end{pmatrix}; \qquad \hat{e}^{*i} = \begin{pmatrix} e^{*ai} \\ e^{*ai} \end{pmatrix}. \tag{75}$$

It is easy to verify that this satisfies Narain's condition.

Just as for the moduli of the worldsheet torus, there are further discrete identifications of points in the moduli space of a toroidal compactification. Let us now find what they are. The toroidal compactification does not affect the oscillators, and the operator algebra works out as usual. All that distinguishes one compactification from another is the lattice $\hat{\Lambda}$ in which the left and right momenta live. Thus we arrive at the important conclusion that any two toroidal compactifications are *equivalent* if their lattices are the same, i.e. they differ only by a change of lattice basis. The most general change of basis is an element of $\mathrm{SL}(2D, \mathbb{Z})$, acting on the labels of lattice basis. But when we parameterize the space of even self-dual Lorentzian lattices acting by $O(D, D)$ on some reference lattice of the form (72), the inner product matrix of the basis vectors always takes the $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ form in (73). Therefore the analog of the modular group for the vacua is contained in $O(D, D; \mathbb{Z})$, the stabilizer of (73) in $\mathrm{SL}(2D, \mathbb{Z})$. It is easy to identify some of its elements. For example, the analog of $T : \tau \to \tau + 1$ is adding to B_{ij} an integral antisymmetric matrix. The analog of $S : \tau \to -1/\tau$ is to change the basis of the compactification lattice Λ. And then there are the generalizations of the $R \to \frac{2}{R}$ symmetry. Since $T^D \sim (S^1)^D$, there are now D of them. Consideration in a similar vein to that for $R \to \frac{2}{R}$ duality shows that they are *gauge* symmetries. The detailed forms of these discrete transformations can be found in [9]. They do not commute with each other but actually generate the whole $O(D, D; \mathbb{Z})$, the *T-Duality* group for compactification over T^D. The moduli space for such compactifications can therefore be written as $O(D, D; \mathbb{Z}) \backslash O(D, D)/(O(D) \times O(D))$.

Appendix: Poisson Resummation

Consider a function f defined on $\mathbb{R}^n$ and its Fourier transform f^*:

$$f(x) = \int \frac{d^n k}{(2\pi)^n} \, e^{\imath k \cdot x} f^*(k).$$

Let Λ be some lattice in $\mathbb{R}^n$ and Λ^* be its dual, then one finds

$$\begin{aligned} \sum_{m \in \Lambda} f(m) &= \int \frac{d^n k}{(2\pi)^n} \, f^*(k) \sum_{m \in \Lambda} e^{\imath k \cdot m} \\ &= \mathrm{vol}(\Lambda^*) \sum_{n \in \Lambda^*} f^*(2\pi m). \end{aligned}$$

3 Lecture Three: Superstrings

The bosonic string theory we studied in the last two lectures has displayed some very interesting structures, yet it conspicuously lacks one important ingredient: fermions. In the real world, we of course know that fermions are the basic constituents of matter. So we should find some way to incorporate them into string theory if the latter is to become a theory of reality. From the last lecture, we see that under certain conditions a theory of bosons can be equivalent to a theory of fermions. That was in the context of worldsheet, while what we really want are spacetime fermions. However the two are not unrelated. As we have seen, the Ramond sector of a theory of worldsheet fermions furnishes a representation of the Clifford algebra with the worldsheet fermion operators, which carry spacetime Lorentz indices, playing the role of gamma matrices. In this lecture we will indeed see how to build a theory of *spacetime* fermions out of *worldsheet* fermions with worldsheet supersymmetry. At the end of the day we will find that the annoying tachyon has disappeared. Moreover we will find a symmetry between *spacetime* bosons and fermions.

3.1 From Superparticle to Superstring

Let us start at a more humble level and try to construct an action for a superparticle by adding new fields to the worldline action for the point particle (1). In fact there is more than one way to do it, but we will consider what is called the *spinning particle*[21].

First let us write an action with a worldline einbein

$$S = -\frac{1}{2}\int d\sigma \left\{ e\dot{X}^2 - \frac{m^2}{e} \right\}. \tag{76}$$

This is to (1) what the Polyakov action is to the Nambu-Goto action. The einbein e is a Lagrange multiplier rather than a dynamical variable. By solving equations of motion for e and substituting the solution back to (76), we regain (1) for $m \neq 0$. For $m = 0$ the latter fails but (76) is still valid as an action for a massless particle. Let us supersymmetrize the action (76) when $m = 0$[22]. We add worldsheet Majorana spinors ψ^μ as the superpartners of X^μ and a

[21] Another approach, which exhibits spacetime fermion and supersymmetry manifestly, can also be generalized to string theory — the Green-Schwarz action. We will mention it briefly below. For more details, see §5 of [3]

[22] If the *cosmological constant* $m \neq 0$, worldline supersymmetry, if present at all, must be spontaneously broken. To keep the action supersymmetric one must introduce an additional fermion as the Nambu-Goldstone particle, which decouples from the rest of the theory in the limit $m \to 0$. We will not consider this case, since for string theory, Weyl rescaling invariance forces m to vanish even for the bosonic string.

worldsheet gravitino ν as the superpartner of e.

$$S=-\frac{1}{2}\int d\sigma \left\{e\dot{X}^2-\imath e\psi\dot{\psi}-2\imath\nu\dot{x}\psi\right\}. \tag{77}$$

Clearly this action is invariant under worldline diffeomorphisms. As implied above, it also has a local supersymmetry:

$$\delta X^\mu=\imath\theta\psi^\mu; \qquad \delta\psi=\theta\dot{X}^\mu;$$

$$\delta e=-2\imath\theta\nu; \qquad \delta\nu=\dot{\theta}e-\frac{1}{2}\theta\dot{e}.$$

Just as in the Polyakov action, ν and e do not have dynamical degrees of freedom. Their equations of motion are algebraic and serve to impose constraints on the physical phase space. Variation of the action with respect to ν implies

$$\dot{X}\cdot\psi=0. \tag{78}$$

Canonical quantization for this action yields

$$-\imath\frac{\partial}{\partial X^\mu}=P_\mu=g_{\mu\rho}\dot{X}^\rho,$$
$$\{\psi^\mu,\psi^\rho\}=g^{\mu\rho}.$$

Therefore ψ^μ realizes the Clifford algebra for the spacetime, for which the Hilbert space forms a representation. The spinning superparticle is a spacetime spinor. The constraint equation (78) then states that physical states satisfy the Dirac equation as befit for a spinor:

$$\imath\gamma^\mu\partial_\mu\langle x|phys\rangle=0.$$

It is simple to generalize this to strings. The supersymmetrization of the bosonic Polyakov action (29) is:

$$S=\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-g}\;\;\Big\{g^{ab}(\partial_a X^\mu\partial_b X_\mu+\imath\bar{\psi}_\mu\lambda^a\partial_a\psi^\mu)$$
$$+\bar{\chi}_a\lambda^b\lambda^a(\partial_b X^\mu+\frac{1}{2}\bar{\psi}_\mu\psi^\mu\chi_b)\Big\}\;.$$

Here λ^a are the worldsheet Dirac matrices. New field contents include D worldsheet spinors ψ^μ that transform in spacetime as a tangent vector, and a worldsheet Rarita-Schwinger field χ_a with no spacetime index. The action has four local symmetries: the worldsheet diffeomorphism and Weyl rescaling symmetries already present for the bosonic string, and their superpartners: local supersymmetry and super-Weyl transformation. Classically they together allow one to gauge away the metric g and the Rarita-Schwinger field χ_a, and impose constraints on the physical phase space. In the *superconformal* gauge, g_{ab} can be set to $\lambda\gamma_{ab}$ and χ_a to 0. Again, there are potential anomalies. The new Faddeev-Popov ghosts introduced by gauge fixing the local fermionic symmetries raise the central charge for the ghost sector to -15. On the other hand, the contribution from the ψ's increases the matter sector central charge to $\frac{3}{2}D$. Therefore the critical dimension for them to cancel is now $D = 10$.

Like the conformal gauge, the superconformal gauge is preserved by some residual gauge symmetries, which are called *superconformal transformations*. The superconformal gauge action,

$$S = \frac{1}{\pi\alpha'} \int d^2\sigma \, \{\partial_+ X \cdot \partial_- X + \imath\psi_L \cdot \partial_- \psi_L + \imath\psi_R \cdot \partial_+ \psi_R\},$$

is the supersymmetric extension of (15). It is a superconformal field theory (SCFT), a conformal field theory with additional structures and algebra reflecting its superconformal symmetry. Gauge fixing them leads us again to light-cone gauge, where 2 directions of the oscillatory excitations are taken away from both the X's and the Ψ's. Manifest Lorentz covariance is lost but the constraints are explicitly solved. Back in the superconformal gauge, the same result should be obtained if we impose constraint conditions on the physical states in the same way as we did for the bosonic string. The constraint conditions correspond to the vanishing of the matrix elements of T ($\tilde{T}$), left (right) moving energy-momentum tensor, and G ($\tilde{G}$), left (right) moving super-current, between physical states.

As discussed in the last lecture, there are two sectors of Hilbert space for a worldsheet fermion, with different boundary conditions. Spacetime Lorentz covariance requires all the left (right) moving fermions to be in the same sector, but we let the choice for left and right movers be independent. Hence the superstring has 4 sectors: NS-NS, NS-R, R-NS and R-R, in contrast to what we did in the last lecture. As usual, left and right moving operators decouple, and we will concentrate on the left movers:

$$T = \sum_n L_n e^{-nz} = -\frac{1}{2}\partial X \cdot \partial X - \frac{1}{2}\psi \cdot \partial\psi,$$

$$G = \sum_n G_n e^{-nz} = \imath\psi \cdot \partial X.$$

Because ∂X's have integer modding, the modding of G is the same as that of ψ's: $r \in \mathbb{Z}$ in R sector; $r \in \mathbb{Z} + \frac{1}{2}$ in NS sector. The superconformal algebra is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8}(n^3 - n)\delta_{n+m,0}.$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}(r^2 - \frac{1}{4})\delta_{r+s,0}$$

$$[L_n, G_r] = (\frac{1}{2}n - r)G_{n+r}.$$

The corresponding OPE's can be found in §12 of [8].

We learned from the last lecture that the R sector realizes the Clifford algebra, therefore they transform as spacetime spinors. We also see that for every complex Ψ or, equivalently, every pair of real ψ, an R sector ground state is $\frac{1}{2} \times \left(\frac{1}{2}\right)^2 = \frac{1}{16}$ higher in L_0 eigenvalue, its conformal weight, than the NS ground state. Therefore it is natural to shift L_0 by $-\frac{D}{16}$ for the R sector so that the R ground state also obeys $L_0 = 0$. With this definition of L_0, we have

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8}n^3\delta_{n+m,0},$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}r^2\delta_{r+s,0},$$

for the R sector. In particular $G_0^2 = L_0$. This is the rigid supersymmetry algebra in 2 dimensions. From a spacetime point view, G_0 is the Dirac operator $\partial\!\!\!/$, L_0 the d'Alembertian operator $\Box$. So $G_0^2 = L_0$ translates into the identity $\partial\!\!\!/^2 = \Box$. The constraints $T \sim 0$, $G \sim 0$ in particular contain the Dirac and the Klein-Gordon equations.

The constraints on physical states are

$$(G_r - b\delta_{r,0})\,|phys\rangle = 0, \quad (L_n - a\delta_{n,0})\,|phys\rangle = 0, \qquad r, n \geq 0.$$

In the R sector, the relation between G_0 and L_0 implies $a = b^2$. Equivalence with the light-cone gauge spectrum then leads to $a = 0$ and $D = 10$, as the students should verify. Below we briefly demonstrate the procedure for the NS sector.

Ground State

$$(L_0 - a)\,|k\rangle = 0, \Rightarrow m^2 = -k^2 = -2a \tag{79}$$

First excited level

$$(L_0 - a)e_\mu \psi^\mu_{-1/2}\,|k\rangle = 0 \Rightarrow m^2 = -k^2 = 1 - 2a\frac{\alpha'}{4}.$$

$$G_{1/2} e_\mu \psi^\mu_{-1/2}\,|k\rangle = 0 \Rightarrow k \cdot e = 0.$$

$$G_{-1/2}\,|k\rangle \sim 0 \Rightarrow e^\mu(k) \sim e^\mu(k) + \xi k^\mu.$$

These two conditions remove 2 degrees of freedom, in agreement with the light-cone gauge, only if this level is massless. Hence $a = \frac{1}{2}$[23] and the tachyon remains, for the time being.

3rd excited level

For reasons similar to the case of the second excited level of bosonic string, we require

$$(G_{-3/2} + \gamma G_{-1/2} L_{-1}\,|k\rangle \qquad -k^2 = 2$$

to be physical in order to have the same number of states as in the light-cone gauge. This implies $\gamma = 2$ and $D = 10$. Thus we obtain again that the critical dimension for superstring is 10.

Define

$$N = -\frac{1}{2} + \sum_{n>0} (n\alpha_{-n} \cdot \alpha_n + (n - \frac{1}{2})\,\psi_{-n+1/2} \cdot \psi_{n-1/2}) \tag{80}$$

[23] This agrees with $a = 0$ for the R sector and the relative normalization of L_0 for the two sectors if and only if $D = 10 - 2$ — another consistency check.

for the NS sector and

$$N = \sum_{n>0} n\,(\alpha_{-n}\cdot\alpha_n + \psi_{-n}\cdot\psi_n) \tag{81}$$

for the R sector, and similarly define $\tilde{N}$ for the right movers. Then the level matching condition is again $N = \tilde{N}$. The mass shell condition can be written as

$$m^2 = 2N.$$

3.2 Spacetime Supersymmetry

If superstring theory has spacetime supersymmetry, then its one-loop vacuum amplitude should vanish due to cancellation between bosons and fermions. We know from the first lecture that such an one-loop amplitude correspond to the partition function of the worldsheet action. The partition function also tells us the spectrum of the theory, and unbroken supersymmetry would imply a perfect matching between bosonic and fermionic spectra. As we are working on a closed string theory, we need to glue the left and right movers together to obtain a physical state or vertex operator. As the R sector realizes the Clifford algebra, spacetime bosons should come from the NS-NS and R-R sectors and fermions from the NS-R and R-NS sectors. Hence the superstring partition function takes the form:

$$Z = (Z_{\rm NS}\bar{Z}_{\rm NS} + Z_R\bar{Z}_R) - (Z_{\rm NS}\bar{Z}_R + Z_R\bar{Z}_{\rm NS}) = |Z_{\rm NS} - Z_{\rm R}|^2,$$

where $Z_{\rm NS}$ and $Z_{\rm R}$ are the partition function for the (left moving) NS and R sectors respectively. For Z to vanish, $Z_{\rm NS} - Z_{\rm R}$ must be zero. However[24]

$$\begin{aligned} Z_{\rm NS} &= {\rm Tr}_{\,\rm NS}\, q^{L_0 - 12/24} \\ &= \left[\frac{q^{-1/24}}{\prod_{n=1}^{\infty}(1-q^n)}\right]^8 \left[q^{-1/24}\prod_{n=1}^{\infty}(1+q^{n-1/2})^2\right]^4 \\ &= q^{-1/2}\left[\frac{\prod_{n=1}^{\infty}(1+q^{n-1/2})}{\prod_{n=1}^{\infty}(1-q^n)}\right]^8 \end{aligned}$$

[24]We are calculating in the light-cone gauge, or equivalently we have taken into account the ghost contribution. We also neglect to write the trivial factor of $(2{\rm Im}\,\tau)^{-5}$ from integrating over momentum. It is an instructive exercise for the students to justify the various factors and powers of q based on the discussion from the last lecture.

is definitely not equal to

$$\begin{aligned} Z_{\mathrm{R}} &= \mathrm{Tr}_{\mathrm{R}}\, q^{L_0-12/24} \\ &= \left[\frac{q^{-1/24}}{\prod_{n=1}^{\infty}(1-q^n)}\right]^8 \left[2q^{-1/24}q^{1/8}\prod_{n=1}^{\infty}(1+q^n)^2\right]^4 \\ &= 2^4 q^{-1/2}\left[\frac{\prod_{n=1}^{\infty}(1+q^n)}{\prod_{n=1}^{\infty}(1-q^n)}\right]^8 \end{aligned}$$

Therefore as it is there is no spacetime supersymmetry. In fact, the ground state in the NS sector is a tachyon, whereas in the R it is massless. Anyhow, the tachyon's presence would indicate vacuum instability, in direct conflict with supersymmetry. It is therefore clear that to have spacetime supersymmetry we have to truncate the spectrum consistently so that the tachyon disappears. This reminds us of the projection operator P introduced in the last lecture to obtain a bosonic theory from a fermionic one. However, since we want to remove the tachyon, the projection operator should be defined as

$$P = \frac{1}{2}(1-(-1)^F)$$

for the NS sector, with the ground state having fermion number $F = 0$, where

$$F = \sum_{n\geq 0} \psi_{-n-1/2}\cdot\psi_{n+1/2}.$$

Its most important property is

$$\left\{(-1)^F, \psi^{\mu}\right\} = 0.$$

The lowest level that survives this projection consists of 8 massless fields with (spacetime) vector indices. It is easy to see that this projection keeps states with integral values of N as defined in (80).

In the R sector, we want to project out half of the ground states because there are 16 of them at the start. This can also be accomplished with $(-1)^F$, if we define its action on the ground states carefully. As a representation of the 10-dimensional Clifford algebra, the Ramond ground states make a Dirac spinor. It can be split into two irreducible representations of $Spin(10)$. They are distinguished by their chirality and are mapped into each other with any odd power of gamma matrices, i.e. the zero modes of ψ^μ. Let us define

$$(-1)^F = \pm\gamma^{11} \times (-1)^{F'}, \tag{82}$$

where γ^{11} is the 10-dimensional chirality operator defined as usual in terms of the products of the gamma matrices and

$$F' = \sum_{n\geq 1} \psi_{-n} \cdot \psi_n$$

The projection operator $P = \frac{1}{2}(1 - (-1)^F)$ will project out spinors of either chirality depending on the choice of sign in (82). Although the overall choice of sign is merely a convention, it will become clear in the next section that the relative sign between left and right movers matters greatly.

Now let us compute the partition function again. Inserting the projector in the trace, one finds that

$$\begin{aligned}
Z_{\rm NS}(P) &= \mathrm{Tr}_{\rm NS}\, P\, q^{L_0 - 8/24} \\
&= \frac{1}{2} q^{-1/2} \left[\frac{\prod_{n=1}^\infty (1+q^{n-1/2})^8 - \prod_{n=1}^\infty (1-q^{n-1/2})^8}{\prod_{n=1}^\infty (1-q^n)^8} \right], \\
Z_{\rm R}(P) &= \mathrm{Tr}_{\rm R}\, P\, q^{L_0 - 8/24} = \frac{1}{2} \mathrm{Tr}_{\rm R} q^{L_0 - 8/24} \\
&= \frac{2^4}{2} q^{-1/2} \left[\frac{\prod_{n=1}^\infty (1+q^n)}{\prod_{n=1}^\infty (1-q^n)} \right]^8 .
\end{aligned}$$

Again, the partition function with completely periodic spin structure vanishes. Amazingly, these two truncated partition functions are equal, thanks to Jacobi's *aequatio identica satis abstrusa*:

$$q^{-1/2} \left\{ \prod_{n=1}^\infty (1+q^{n-1/2})^8 - \prod_{n=1}^\infty (1-q^{n-1/2})^8 \right\} = 2^4 \prod_{n=1}^\infty (1+q^n)^8$$

This result is so remarkable that it is worthwhile to understand it in a different light. In the light-cone gauge, we may group the 8 transverse into 4 pairs and define

$$\Psi^i = (\psi^{2i-1} + \imath\psi^{2i})/\sqrt{2}$$

and their conjugate $\bar{\Psi}^i$, for $i = 1, \ldots, 4$. Their modes obey the commutation relations

$$\left\{\Psi^i_m, \Psi^j_n\right\} = 0, \quad \left\{\bar{\Psi}^i_m, \bar{\Psi}^j_n\right\} = 0$$
$$\left\{\Psi^i_m, \bar{\Psi}^j_n\right\} = \delta^{ij}\delta_{m+n,0}.$$

We may bosonize them:

$$\Psi^i = e^{\imath\phi^i_L}.$$

The (left) momentum for the left moving bosons take values in a charge lattice in $\mathbb{R}^4$, which is different for the NS and R sectors. The 8 massless states in NS sector have charge vectors of the form $(\pm1, 0, 0, 0), (0, \pm1, 0, 0), ..., (0, 0, 0, \pm1)$. They are in fact the weight vectors[25] for the vector representation of $so(8)$. The 16 ground states in the R sector have charges $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1)$. From the commutation relations for Ψ's and $\bar{\Psi}$'s in the R sector, we see that their zero modes — the gamma matrices — can be expressed in terms of fermion creation and annihilation operators. Using this procedure one can explicitly construct the representations of Clifford algebra of any dimension. The $\pm$ sign in each entry of the charge vectors for the massless states reflects the occupation number of a corresponding fermionic oscillator. We can define the fermion number operator F directly in terms of Ψ as

$$F = \bar{\Psi}^i_0\Psi^i_0 + F'.$$

Thus the chirality of the massless states is given by the parity of the number of minus signs in their charge vectors. Not surprisingly, the charge vectors for (anti-)chiral states turn out to be the weight vectors for the (conjugate-) spinor representation of $so(8)$. Now there is a *triality* symmetry of the Dynkin diagram of $so(8)$ (see fig. 10), which gives isomorphisms among the vector, spinor, and conjugate spinor representations of $so(8)$. In fact, these isomorphisms extend beyond these three representations and hence the massless states. $so(8)$ has four conjugacy classes of representations[26], three of which are represented by the vector and two spinor ones respectively. After GSO projection, the charge lattice for the NS sector consists of the weight vectors in the vector

[25] Weight vectors and conjugacy classes of representation will be defined in the next lecture.
[26] See the last footnote.

class, while that of the R sector consists of either of the two spinor classes. The equality $Z_{\rm NS}(P) = Z_{\rm R}(P)$ is then a consequence of triality.

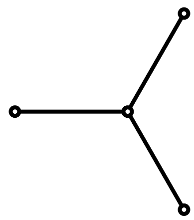


Figure 10. Triality of so(8)

The triality also allows us to make contact with another approach to superstring theory, which is discussed in chapter 5 of [3]. Instead of the spinning particle action (77) which is found to describe a spacetime fermion only after quantization, one can define an action with spacetime spinor built in and with manifest spacetime supersymmetry. It can be generalized to describe superstrings (the Green-Schwarz approach). To find a relation between the Green-Schwarz approach and the Neveu-Schwarz-Ramond approach that we are studying here, let us consider the spin field operators which map the NS ground state to R ground states:

$$S^\alpha = e^{\frac{i}{2}(\pm\phi^1 \pm\phi^2 \pm\phi^3 \pm\phi^4)}, \qquad \alpha = (\pm, \pm, \pm, \pm). \tag{83}$$

These spin fields S^α transform as a spinor of $so(8)$ with chirality determined by the number of minus signs. Furthermore they all have conformal weight $4 \times \frac{1}{8} = \frac{1}{2}$ so they are also worldsheet spinors. They are in fact the spinor variables used in the Green-Schwarz approach, in the light-cone gauge. The field redefinition (83) demonstrates the equivalence between Neveu-Schwarz-Ramond and Green-Schwarz superstrings.

3.3 Massless Spectrum

Now let us examine the massless particles in superstring theory for their spacetime meaning. We will use the language of the covariant superconformal gauge, therefore our counting will be off-shell. For NS-NS sector, we clearly get the same fields as for bosonic string: the dilaton Φ, the metric $G_{\mu\nu}$ and the antisymmetric tensor field $B_{\mu\nu}$. For the NS-R and R-NS sectors, the Ramond parts transform as spacetime spinors λ_L or λ_R. In fact they are Majorana-Weyl spinors. The NS parts are of course vectors, so we have two 10-dimensional Rarita-Schwinger fields. The only known way to incorporate such fields consistently is to couple them to the supergravity current. They are therefore the gravitinos. So a GSO projected superstring theory contains $N = 2$ supergravity. Depending on the choice of the relative sign in defining $(-1)^{F_L}$ and $(-1)^{F_R}$, we have two inequivalent possibilities, corresponding to the relative chirality of the surviving λ_L and λ_R. If we choose opposite chiralities, we obtain the type IIA superstring theory whose low energy effective theory is the type IIA supergravity. The type IIA theory is non-chiral and can be obtained by dimensional reduction from 11-dimensional supergravity. This is the first and simplest evidence for the relation between type IIA string theory and a theory in eleven dimensions, "*M theory.*" M theory is discussed by Duff and Schwarz at this school. If we choose the same chirality for both left and right movers, we obtain the type IIB superstring theory. The corresponding type IIB supergravity is chiral and potentially anomalous. Cancellation of gravitational anomaly in type IIB supergravity was shown by Alvarez-Gaumé and Witten (ref. 20 in [3], Vol 1).

More novelties come from the R-R sectors. Here the massless states transform as the products of two spinors. Contracting them with antisymmetrized products of gamma matrices, we see that they are related to antisymmetric tensors of rank 0 to 10. However, because the spinors making the products are chiral, not all the possibilities can appear. For the type IIA theory, λ_L and λ_R are of the opposite chiralities, and we obtain even rank tensors

$$F^{\{0\}} \equiv \bar{\lambda}_L \lambda_R, \quad F^{\{2\}}{}_{\mu\nu} \equiv \bar{\lambda}_L \gamma_{\mu\nu} \lambda_R, \cdots .$$

On the other hand, the type IIB theory contains odd rank tensors

$$F^{\{1\}}{}_{\mu} \equiv \bar{\lambda}_L \gamma_{\mu} \lambda_R, \quad F^{\{3\}}{}_{\mu\nu\rho} \equiv \bar{\lambda}_L \gamma_{\mu\nu\rho} \lambda_R, \cdots .$$

Here $\gamma_{\mu_1 \ldots \mu_n}$ is the antisymmetrized product of n gamma matrices. Moreover they are not all independent. There is an important γ-matrix relation:

$$\epsilon_{\mu_1\dots\mu_n}{}^{\rho_{n+1}\dots\rho_{10}}\gamma_{\rho_{n+1}\dots\rho_{10}} \sim \gamma^{11}\gamma_{\mu_1\dots\mu_n}.$$

Because of the GSO projection, Ψ_L and Ψ_R both have definite eigenvalue of γ^{11}. Therefore

$$F^{\{n\}} \sim *F^{\{10-n\}}. \tag{84}$$

In particular, $F^{\{5\}}$ is self-dual. The students should verify that the number of independent components of the antisymmetric tensor fields, taking into account these relations, is equal to that of the tensor product of two Majorana-Weyl spinors. What kind of fields are they? It is not difficult to show that the massless Dirac equations for λ_L and λ_R are equivalent to

$$d^*F^{\{n\}} = 0, \quad dF^{\{n\}} = 0.$$

They are the equations of motion and Bianchi identities for antisymmetric tensors fields $A^{\{n-1\}}$ such that $F^{\{n\}} = dA^{\{n-1\}}$. Note that $A^{\{n-1\}}$ and $A^{\{9-n\}}$ are related by electric-magnetic duality, which exchanges equations of motion and Bianchi identities. The way they arise out of string theory places them on equal footing.

There is also an antisymmetric tensor field B in NS-NS sector, but the way it is coupled to the string is very different from the R-R fields. Recall from lecture one that the vertex operator for it couples directly to the VEV of its potential $B_{\mu\nu}$. Its contribution to the string action is just the integral of the pullback of B over the worldsheet. By analogy with the minimal coupling of the usual 1-form potential A_μ to the worldline of a charged point particle, we see that this means a string carries unit "electric" charge with respect to B. However, the coupling of R-R fields with string involves only the field strength. This means elementary string states cannot carry any charge with respect to the R-R fields. However, it was discovered by Polchinski that there are solitonic objects called *D-branes* which do carry such charges [10]. These are discussed extensively in his lectures at this school.

3.4 Dilaton and Antisymmetric Tensor Fields

The low energy effective action for the NS-NS fields is the same as that of the bosonic string:

$$S = \frac{1}{2\kappa^2}\int d^{10}X\,\sqrt{-G}e^{-2\Phi}\{R - \frac{1}{12}H^2 + 4(\nabla\Phi)^2 + O(\alpha')\},$$

where $H = dB$. The variation of S with respect to B gives

$$e^{2\Phi}\nabla^\mu\left(e^{-2\Phi}H_{\mu\nu\rho}\right) = (\nabla^\mu - 2\partial^\mu\Phi)H_{\mu\nu\rho} = 0.$$

The origin of the coupling between H and Φ can be traced to the way the dilaton couples to the string worldsheet, $\sqrt{g}R\Phi$. Since $T \sim \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{zz}}$, if the dilaton is not constant, the energy-momentum tensor T is modified as

$$T \sim -\frac{1}{2}(\partial X)^2 + \partial_\mu\Phi\,\partial_z^2 X^\mu.$$

The equation of motion for H can then be obtained from the Virasoro constraint (22) on physical states, which receives the additional contribution from Φ.

Now let us find out what happens to the antisymmetric tensor fields in the R-R sector. The dilaton field also modifies the supercurrent as

$$G \sim \imath\psi_\mu\partial X^\mu + \psi^\mu\partial_\mu\Phi.$$

As we recall, the zero mode of the super-Virasoro constraint yields the massless Dirac equation in the constant dilaton background. If the dilaton is not constant, the Dirac operator is modified as

$$G_0 \sim \partial\!\!\!/ - \partial\!\!\!/\Phi = e^{\Phi}\ \partial\!\!\!/ e^{-\Phi}.$$

Correspondingly, the equations of motion for the R-R fields are

$$d^*(e^{-\Phi}F^{\{n\}}) = 0,\quad d(e^{-\Phi}F^{\{n\}}) = 0.$$

Therefore it is the rescaled fields

$$\hat{F}^{\{n\}} \equiv e^{-\Phi}F^{\{n\}}$$

which obey the usual Bianchi identity and equations of motion for an antisymmetric tensor. We can then write $\hat{F}^{\{n\}} = d\hat{A}^{\{n-1\}}$ and their spacetime action is

$$\int d^{10}X \, \hat{F}^{\{n\}} \wedge *\hat{F}^{\{n\}},$$

without the usual $e^{-2\Phi}$ factor. Thus, we find that the R-R fields do not couple to the dilaton if they are suitably defined. This is contrary to the case of the NS-NS B field, for which such rescaling is not possible. This has far reaching consequences in string dualities, which are discussed extensively by other lecturers in this school.

3.5 T-Duality

To end this lecture, let us briefly discuss how the T-duality $R \to \frac{2}{R}$ acts on superstring compactified on $M^9 \times S^1$. Recall from the last lecture that this duality involves the isomorphism $\partial X^9_L \leftrightarrow \partial X^{9\prime}_L$ and $\partial X^9_R \leftrightarrow -\partial X^{9\prime}_R$. This same clearly carries over to superstring, but we also have to respect the worldsheet supersymmetry. It is clear that the isomorphism for the worldsheet fermions should be $\psi^9_L \leftrightarrow \psi^{9\prime}_L$ and $\psi^9_R \leftrightarrow -\psi^{9\prime}_R$. In particular, the zero mode of ψ^9 in R sector, which acts as γ^9 on the right movers, changes its sign. This means that the relative chirality between the left and right movers is flipped. Therefore $R \to \frac{2}{R}$ maps type II A superstring compactified on a circle of radius R to type IIB superstring on a circle of radius $\frac{2}{R}$. This is an identification of two different types of theories, rather than a gauge symmetry as in the case of bosonic string. What happened is that the operators responsible for the enhancement of gauge symmetry, $e^{\pm\imath\sqrt{2}X_L}$, are removed by the GSO projection, as are the physical states corresponding to them.

4 Lecture Four: Heterotic Strings

In lecture one we studied the bosonic string which lives in $(25+1)$-dimensional spacetime. It contains only spacetime bosons, in particular a tachyon. In lecture three we studied the superstring, which includes spacetime fermions in its spectrum, and which, after GSO projection, loses the unwanted tachyon and exhibits spacetime supersymmetry. At first sight it seems hardly feasible to combine such two drastically different theories into one without running into disastrous inconsistencies. However, one important property of 2d (super)conformal field theories that we have used often in the last three lectures is the decoupling of left and right movers. The decoupling even extends to the zero modes — momentum and position — if we consider compactification on torus and take into account the winding sectors. In this lecture we will exploit this feature again and consider a theory with the right movers being those of a critical superstring and the left movers being those of a critical bosonic string. This is the heterotic string of Gross, Harvey, Martinec and Rohm (refs. 235, 236 and 237 in [3], Vol 1).

4.1 Marrying Bosonic String and Superstring

When we say the left movers of the heterotic string are those of the bosonic string, we mean that they possess the same diffeomorphism and Weyl rescaling invariance. The central charge for the ghost action is fixed to be -26. Anomaly cancellation or equivalently absence of ghosts thus requires there to be 26 left moving bosons in the matter sector. By similar reasoning the right moving sectors must consist of 10 matter bosons and fermions. To have genuine target spacetime interpretation as a coordinate, a boson must have both left and right movers, therefore an "uncompactified" heterotic string lives in 10 spacetime dimensions. The additional 16 left movers can be thought of as parametrizing an internal 16-dimensional torus.

When a theory discriminates between being left and right — when it violates parity invariance — it is liable to incur a gravitational anomaly. This could be an especially acute problem on the $(1+1)$-dimensional worldsheet, where the scalars can be chiral and where a chiral fermion and its CPT conjugate have the same chirality. It would be a disaster for the heterotic string, a manifestly left-right asymmetric theory, to develop some gravitational anomaly. Fortunately this does not happen for the critical heterotic string theory we are discussing. In fact, there is a relation between the gravitational anomaly and the Virasoro anomaly. Details can be found in §3.2.2–3.2.3 of [3]. Very briefly, from (33) one can deduce that the contributions from the left and right movers to the gravitational anomaly are proportional to their respective central charges. As shown in the reference mentioned above, if and only if they are equal, one can introduce local counterterms so that the total gravitational anomaly vanishes. This is certainly true for the critical heterotic string theory, where the total central charges are 0 for both the left and right movers.

4.2 Lattice and Gauge Group

Let us recall from lecture two that an affine Lie algebra $\hat{g}$ of level k can give rise to spacetime symmetries $\mathcal{G}$. When the affine Lie currents are present in the physical spectrum for, say, the left movers, we can pair it with $\bar{\partial}X^\mu$ of the right movers to make a physical vertex operator. Its tree level scattering amplitudes reproduce those of a Yang-Mill theory with gauge group $\mathcal{G}$. If such vertex operators are not in the physical spectrum, say due to GSO projection, then $\mathcal{G}$ cannot be a gauge symmetry for the lack of gauge fields. However, the worldsheet SCFT still possess the symmetry, and the physical states and operators fall into representations of $\mathcal{G}$. So $\mathcal{G}$ appears as a global symmetry for the perturbative string theory. Now just what kind of group $\mathcal{G}$ can be obtained from string theory in this way?

To answer this question, we need to make a detour to the representation theory of Lie groups and algebras. We will not focus on the mathematical details but only sketch the necessary ideas.

Given a finite dimensional Lie algebra, we can always find a maximum set of mutually commuting generators, the *Cartan subalgebra*. We call the commuting generators H_i $(i = 1, \ldots, n)$; n is the rank of the Lie algebra. All H's can be simultaneously diagonalized in a given representation, and every state can therefore be labeled by its eigenvalues for each of the H's, which we call charges or quantum numbers. We may naturally associate to each set of charges a point in $\mathbb{R}^n$, a *weight vector*. If we plot all of them, they form a lattice in $\mathbb{R}^n$. The reason is that the charges are additive. When you multiply two representations, the charge of the product of two states is the sum of those of each of them. As every finite dimensional representation can be obtained from finite products of a finite set of "basic" representations, their charge vectors form a lattice, the *weight lattice* Λ_W. By the same token, weight vectors of representations that can be obtained from products of the adjoint representation form a sublattice of the weight lattice, called the *root lattice* Λ_r. The quotient of the weight lattice by the root lattice gives rise to the *conjugacy classes* of representations of $\mathcal{G}$, where the conjugation is multiplication with the adjoint representation. Between the weight and root lattices there can be intermediate lattices. They and the weight and root lattices, are collectively known as *Lie algebra lattices*. Starting from the Lie algebra g, one can construct its universal covering Lie group $\mathcal{G}$. The subgroup of $\mathcal{G}$ whose elements commute with all of $\mathcal{G}$ is known as its *center* $C_{\mathcal{G}}$. Every element of $C_{\mathcal{G}}$ acts nontrivially on some representations in the weight lattice, but clearly they all act trivially on those in the root lattice. For representations on a Lie algebra lattice, they act as the quotient of $C_{\mathcal{G}}$ by some subgroup of it.

Every Lie algebra has an adjoint representation. Applying the above construction to this particular case, we obtain the *Cartan-Weyl* basis: H_i from the Cartan subalgebra and the remainder, denoted by A, that are eigenstates of the H,

$$\left[H^i, A\right] = a^i A.$$

Thus each A is associated with a root vector a^i in the weight space. One can show that each root vector is associated with only one generator.

What kind of construction can realize these structures in the context of string theory? The additivity of charges gives us a hint — we can represent them as momenta. Consider a Lie algebra lattice Λ of some Lie algebra g. That the charges take values on the lattice Λ reminds us of compactification over a torus of the same dimension as the rank, namely n. Denote the left moving bosons parameterizing the "torus" as ϕ^i. The Cartan generator H^i is realized by the zero mode of the current $\partial\phi^i_L$, as they measure the charges — momenta. Therefore this "torus" is nothing but the maximal Abelian subgroup of $\mathcal{G}$, generated by the H's, known as the *maximal torus* of $\mathcal{G}$. Let Λ be the charge lattice for the left moving bosons. The momentum carried by a state in the lattice is simply equal to its weight vector w. It is created by the vertex operator $:\exp(\imath w \cdot \phi_L):$. We see now why Λ must be a Lie algebra lattice: it must contain the adjoint representation so that the A's can also be represented as vertex operators. Furthermore, those in the adjoint should have the same conformal weight of $(1,0)$ as $\partial\phi^i$, so they can together form the affine Lie algebra (58). This requires all the generators A^w to have $w^2 = 2$. Lie algebras satisfying this requirement are called *simply-laced*. They are $so(2n)$, $su(n+1)$, and e_n[27], and the products thereof. The $SU(2)$ enhanced symmetry at self-dual radius encountered in lecture two is their simplest example. If Λ is the weight lattice, the symmetry group is the universal covering group $\mathcal{G}$. Otherwise, it is the quotient of $\mathcal{G}$ by some subgroup of $C_{\mathcal{G}}$. To be precise, for this construction to satisfy the OPE for the affine Lie algebra, we need to introduce additional factors known as *cocyles*. Details can be found in §6.4.4–6.4.5 of [3]. Moreover we should always remember there is a crucial additional requirement from string theory itself — modular invariance. Therefore the lattice must be even and self-dual.

4.3 E_8 Lattice

For the heterotic string, the left movers do not suffer the GSO projection. Therefore the vertex operators for the non-Abelian generators A's remain in the spectrum and we conclude that the theory has non-Abelian gauge symmetry with gauge group determined by the left components of the lattice. For the heterotic string in 10 dimensions, the appropriate lattice is 16-dimensional. However, it is instructive to start with the 8-dimensional even self-dual lattices.

Let us first state some facts about even self-dual lattices $\hat{\Lambda}$ in $(D, D+n)$ spaces. It is known mathematically that such objects exist only for $n \equiv 0 \pmod 8$. They are unique up to $O(D, D+n)$ isomorphism for $D \neq 0$, and even so for $D = 0$ if $n = 8$. In $(0, 8)$, the lattice can be chosen to be Γ_{E_8}, generated by

$$\begin{aligned} e_1 &= (1, -1, 0, \ldots, 0) \\ e_2 &= (0, 1, -1, 0, \ldots, 0) \\ &\vdots \\ e_7 &= (0, \ldots, 0, 1, -1) \\ e_8 &= (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}). \end{aligned}$$

[27] e_n exits for $n = 6, 7, 8$

The associated *theta function*

$$\theta_{\hat{\Lambda}}(q) \equiv \sum_{p\in\hat{\Lambda}} q^{p^2}$$

is invariant under the modular group $\mathrm{SL}(2,\mathbb{Z})$. The first seven vectors are root vectors of $so(16)$. The eighth is a weight vector for the chiral spinor representation of $so(16)$. Together they generate all the weight vectors for the adjoint and chiral spinor representations of $so(16)$. Therefore Λ_{E_8} is a $so(16)$ Lie algebra lattice. Weight vectors for the vector representation take the form $\pm v_i \pm v_j$, where v_i is the 8-vector with the i-th component 1 and the rest 0. Those for the chiral spinor representation are $(\pm 1/2, \pm 1/2, \ldots, \pm 1/2)$ with an even number of minuses. Corresponding to them we have vertex operators

$$e^{\pm \imath\phi_L^a \pm \imath\phi_L^b}, \qquad a,b = 1\ldots 8 \tag{85}$$

and

$$e^{\frac{\imath}{2}(\pm\phi_L^1 \pm \phi_L^2 \cdots \pm \phi_L^8)}. \tag{86}$$

This suggests us to fermionize these left moving bosons. Recall from lecture two that the operators

$$\psi^a \equiv e^{\imath\phi_L^a}, \qquad a = 1,\ldots,8$$

are 8 complex Weyl (worldsheet) fermions. We can decompose them into 16 Majorana-Weyl fermions:

$$\Psi^a \equiv \frac{1}{2}(\psi^{2a-1} + \imath\psi^{2a}).$$

Then the operators in (86) are just the spin fields of $SO(16)$ with a definite chirality.

Based on our discussion in lecture two, it is easy to write down the partition function for these fermions:

$$Z_{E_8} = \frac{1}{2}q^{-1/3}\left\{\prod_{n=1}^{\infty}(1+q^{n-1/2})^{16} + \prod_{n=1}^{\infty}(1-q^{n-1/2})^{16} + 2^8\prod_{n=1}^{\infty}(1+q^n)^{16}\right\}.$$

Here we choose the projection so that the vacuum is *not* projected out since the origin is certainly in Λ_{E_8}. If this were part of a "compactification" of a bosonic string, its contribution of massless states would be from those with weight 1. From NS sector there are $16 \times 15/2 = 120$ of them, corresponding to the antisymmetrized product of two Majorana-Weyl worldsheet fermions $\psi^\mu \psi^\nu$. From R sector there are $2^8/2 = 128$ of them, corresponding to the R sector vacuum of definite $SO(16)$ chirality.

Which symmetry group would this lattice generate? The first thought might be $Spin(16)$ or its quotient by some center. However, $so(16)$ only has 120 generators, accounted for the massless states in the NS sector. The R sector ground states which transform as chiral spinor of $Spin(16)$ also have weight (1,0) and hence correspond to affine Lie currents as well. In fact they enlarge $so(16)$ to E_8[28], which has $120+128 = 248$ generators. We now construct it explicitly.

Let us start with $so(N)$. The generators are $J^{\mu\nu} = -J^{\nu\mu}$, $\mu \neq \nu$ ranging between 1 and N. Their commutation relations are well known:

$$[J^{\mu\nu}, J^{\rho\sigma}] = \delta^{\mu\sigma} J^{\nu\rho} + \delta^{\nu\rho} J^{\mu\sigma} - \delta^{\mu\rho} J^{\nu\sigma} - \delta^{\nu\sigma} J^{\mu\rho}.$$

To this, let us add a generator σ_α with spinor index α. Because there exist Majorana-Weyl spinors in (16+0) dimension, we may consider Hermitian operators with definite chirality. Their commutation relation with the J's, if nonzero, must be

$$[J^{\mu\nu}, \sigma_\alpha] \sim (\gamma^{\mu\nu})_{\alpha\beta} \sigma_\beta.$$

The normalization is fixed by demanding Jacobi identities on $[[\sigma, J], J]$. The commutators among the σ's, after proper normalization, must take the form

$$[\sigma_\alpha, \sigma_\beta] = (\gamma^{\mu\nu})_{\alpha\beta} J^{\mu\nu}.$$

However, one can then check that the Jacobi identity for $[[\sigma, \sigma], \sigma]$ holds only if

$$(\gamma^{\mu\nu})_{\alpha\beta} (\gamma^{\mu\nu})_{\gamma\delta} + \text{cyclic permutation in}(\alpha, \beta, \gamma) = 0.$$

For $so(N)$, this "Fierz" type identity holds only for N= 8, 9, 16. For $N = 8$, it extends $so(8)$ to $so(9)$. For $N = 9$, it extends $so(9)$ to f_4. For the relevant

[28] It is customary to denote with E_8 both the Lie group and the Lie algebra associated with it. There is no ambiguity as E_8 has only one conjugacy class of representations, which means that there is only one group (i.e. E_8) associated with this Lie algebra.

case of $N = 16$, it extends $so(16)$ to E_8. For more details and other interesting facts about E_8, the students are referred to appendix 6.A of [3].

4.4 $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$

Now let us consider 16-dimensional self-dual even lattices. Mathematically, it is known that there are two of them up to $SO(16)$ rotations. One of them is simply the direct product of 2 copies of Λ_{E_8}. Its generators, in one-to-one correspondence with weight one vertex operators, are simply generators of either of the two E_8's. The associated partition function

$$Z_{E_8\times E_8} = Z_{E_8}^2.$$

But there is another lattice, unrelated to the one above by any $SO(16)$ rotation yet equally simple to describe. It is generated by

$$\pm w_i \pm w_j, \quad i \neq j,$$

where w_i is now a $\mathbb{R}^{16}$ vector with the i-th components 1 and the rest 0, and

$$(\pm 1/2, \pm 1/2, \ldots, \pm 1/2),$$

with an even number of minuses. By analogy with Λ_{E_8}, it contains the root vectors of $so(32)$ and the weight vectors of its chiral spinor representation. It is the $so(32)$ Lie algebra lattice. The difference between this and the last case is that the chiral spin fields now have weight $(2, 0)$ so do not form the currents. The weight $(1, 0)$ operators all correspond to the roots of $so(32)$. The lattice does not include the vector and anti-chiral spinor representations. So the gauge group is not quite $Spin(32)$, but rather its quotient by a $\mathbb{Z}_2$ subgroup of its $\mathbb{Z}_2 \times \mathbb{Z}_2$ center. It is usually written as $Spin(32)/\mathbb{Z}_2$ to distinguish it from $SO(32)$[29]. It is simple to check that $so(32)$ has the same number of generators as $E_8 \times E_8$, namely 496. We can also calculate the partition function

$$Z_{SO(32)/\mathbb{Z}_2} = \frac{1}{2}q^{-2/3} \left\{ \prod_{n=1}^{\infty}(1+q^{n-1/2})^{32} \right.$$

[29] $SO(32)$ is the quotient of $Spin(32)$ by the other $\mathbb{Z}_2$ in its $\mathbb{Z}_2 \times \mathbb{Z}_2$ center. It would have been the gauge group if the Lie algebra lattice had included both the adjoint and the vector representations but neither of the two spinor representations.

$$+\prod_{n=1}^{\infty}(1-q^{n-1/2})^{32}+2^{16}q^2\prod_{n=1}^{\infty}(1+q^n)^{32}\Bigg\}.$$

By using Jacobi's abstruse identity, introduced in lecture three, it is easy to show that[30]

$$Z_{SO(32)/\mathbb{Z}_2}=Z_{E_8\times E_8}=Z_{E_8}^2.$$

4.5 Particle Spectrum

We now study the low lying particle spectrum for the two heterotic string theories in 10 dimensions. The procedure for the left movers is identical to that of bosonic string; for the right movers it is identical to that of the type II string. Therefore we will be very brief.

For the right movers, the mass shell condition is $L_0=1/2$ or

$$m^2=-p^2=2\tilde{N}$$

where p is the 10-dimensional spacetime momentum, and $\tilde{N}$ is the measure of oscillator excitation defined as in (80) and (81). The ground state is projected out by GSO projection, so the lowest lying physical states are either

$$e_\mu\tilde{\psi}^\mu_{-1/2}\left|k\right\rangle, \qquad \mu=0,\ldots,9$$

in the NS sector, satisfying the massless Klein-Gordon equation

$$k^2=0,\quad k\cdot e=0,$$

or the ground states in the R sector with definite chirality:

$$\xi^\alpha\left|k\right\rangle_\alpha,$$

satisfying the massless Dirac equation

$$\not{k}\xi=0.$$

For the left movers, the mass-shell condition is $L_0=1$ or

$$m^2=-p^2=2(N-1)+p_L^2,$$

[30]This is not a coincidence. Mathematically it is known that there is a unique modular form of modular weight 8.

where p_L is the internal momentum living on the 16-dimensional even self-dual lattice, and N the measure of left moving oscillator excitation as defined in (25). The ground state is

$$|k\rangle\,, \quad k^2 = 2.$$

Because of the left and right asymmetry, the level matching condition for the heterotic string is modified[31]:

$$N + p_L^2/2 = \tilde{N} + 1. \tag{87}$$

Note that this means p_L^2 and hence the internal lattice must be even. Note also that for $N = 0$, p_L^2 must be at least 1. This means that although the left movers have no GSO projection, the tachyon is still projected out. The first excited states are massless. They include the usual

$$e^\mu \alpha_{-1}^\mu |k\rangle \qquad \mu = 0, \ldots 9 k \cdot e = 0$$

and contribution from the internal bosons:

$$J_{-1}^a |k\rangle\,,$$

J_n^a being the Fourier modes of the current J^a.

Putting the left and right movers together, the massless spectra of the heterotic strings include the usual spacetime bosons $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ coming from

$$\alpha_{-1}^\mu \tilde{\psi}_{-1/2}^\nu |k\rangle\,,$$

and spacetime fermions — gravitinos — coming from

$$\alpha_{-1}^\mu |k\rangle_\alpha\,.$$

These are similar to what one would get from NS-NS and NS-R sectors of superstring, but the additional 16 left moving bosons or, equivalently, 32 left

[31] One can again understand this by looking at the partition function. The integration over the twist moduli $\mathrm{Im}\,\tau$ enforces the level matching condition (87). The constant 1 and 1/2 originate from the different central charges of the left and right movers in the light-cone gauge.

handed fermions give rise to something quite new. We have now gauge fields from

$$J^a_{-1}\tilde{\psi}^{\mu}_{-1/2}\left|k\right\rangle ,$$

and gauginos from

$$J^a_{-1}\left|k\right\rangle_{\alpha} .$$

Therefore the low energy approximation to a heterotic string theory would be a theory with $N=1$ supergravity and $N=1$ super-Yang-Mills. It is anomaly free only when the gauge symmetry algebra is $E_8\times E_8$, $so(32)$, $u(1)^{248}\times E_8$ or $u(1)^{496}$. We have thus explained the "existence" of the first two as being low energy approximation to the two heterotic string theories.

4.6 Narain Compactification

Recall that in lecture two, we considered generalized compactification over T^D by letting the internal left and right momenta to live on a $(D+D)$-dimensional lattice $\hat{\Lambda}$. The requirement of modular invariance then places stringent restrictions on $\hat{\Lambda}$. This construction can be carried over for the heterotic string, in which case the left moving bosons have 16 more "dimensions" than the right moving ones. Thus in toroidal compactification down to $10-D$ dimensions, the left and right momenta lives on a $(16+2D)$-dimensional lattice $\hat{\Lambda}_H$. Modular invariance again requires $\hat{\Lambda}_H$ to be even and self-dual with respect to a metric of signature $(16+D,D)$. Such a $\hat{\Lambda}_H$ is known as *Narain lattice* (ref. 340 in [3], Vol 1).

Following the discussion earlier, the non-Abelian gauge symmetry of the compactified heterotic theory is determined by the special points in the lattice of the form $(p_L,0)$ with $p_L^2=2$, and the global symmetry determined by those with charge vector $(0,p_R)$ with $p_R^2=2$. Generically, there will be no points like those, and the gauge symmetry of the theory is Abelian $U(1)^{16+D}\times U(1)^D$. The $U(1)^D\times U(1)^D$ are just the Kaluza-Klein gauge fields. The $U(1)^{16}$ is what remains of the original gauge symmetry of the heterotic string. The breaking of the gauge symmetry $E_8\times E_8$ or $Spin(32)/\mathbb{Z}_2$ down to products of $U(1)$ is achieved by turning on *Wilson lines*, which we will discuss presently. Let us note, however, that there are also nongeneric lattices where such special points do exist. The self-dual radius is again the simplest example. As in that case, we would have an enhancement of gauge and/or global symmetries. The existence of such points plays an important role in understanding the string-string duality between the heterotic string on T^4 and the type II A string on K_3. It is discussed in Aspinwall's lectures in this school.

The discussion in lecture two on the moduli spaces for toroidal compactification can be carried over to the present case. As expected, they are

$$O(D+16, D; \mathbb{Z})\backslash O(D+16, D)/O(D+16) \times O(D)$$

for $D > 0$. By arguments similar to those given in lecture two, these *Narain moduli* are VEV's for the massless fields $\partial X^M \bar{\partial} X^N$ and $\partial \phi^i \bar{\partial} X^N$, where M are indices tangent to the compactification torus T^D and i are labels in the Cartan subalgebra of either $Spin(32)/\mathbb{Z}_2$ or $E_8 \times E_8$. The first type are just the familiar Kaluza-Klein scalars G_{MN}, B_{MN}. The latter are components A^i_M of the gauge fields in the Cartan. For $D = 0$, the moduli space consists of two discrete points, corresponding to $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$. However, as mentioned earlier, for $D > 0$ the moduli space is connected. This has the interesting implication that one can continuously interpolate between the two heterotic string theories compactified over T^D. We now sketch one such interpolation for compactification on S^1. Starting with $SO(32)$, we give some constant VEV's to A^i_9 in the Cartan. This is known as "turning on the Wilson line" around S^1. It is so called because it lets the Wilson loop around S^1, i.e. the path ordered exponential

$$P \exp^{\left(i \int_{S^1} A_9 dx^9 \right)},$$

develop a nontrivial VEV, which can be chosen to break $SO(32)$ down to $SO(16) \times SO(16)$. After an appropriate $O(17, 1; \mathbb{Z})$ T-duality transformation, it becomes a Wilson line configuration for the $E_8 \times E_8$ heterotic string compactified on S^1.

- **Exercise** 4.1
  Another way to obtain gauge symmetry in string theory is to consider open strings. This subject is discussed extensively in Polchinski's lectures. For this exercise, reconsider bosonic string on a worldsheet Σ with a *boundary* $\partial\Sigma$. To solve the Cauchy problem,

one must impose boundary conditions along $\partial\Sigma$. This leads to new constraints on the phase space. Repeat the classical and quantum analysis of lecture one for this case, assuming the Neumann boundary condition

$$\partial_{normal} X = 0|_{\partial\Sigma}$$

for all X's. Find the Virasoro constraints and determine the massless spectrum. What happens if instead we use Dirichlet boundary condition

$$\partial_{tangential} X = 0\Big|_{\partial\Sigma}$$

for some X's?

5 Lecture Five: Orbifold Compactifications

Although simple and interesting, toroidal compactifications cannot give rise to realistic theories because they have a rather large number of unbroken spacetime supersymmetries for the uncompactified spacetime. To see this, consider the compactification over T^6. Both heterotic string theories have $N = 1$ spacetime supersymmetry in 10 dimensions, corresponding to $2^4 = 16$ real components of supercharges forming a constant Majorana-Weyl spinor in $(9 + 1)$-dimensions. Because T^6 is flat, all of them survive as unbroken supersymmetry for M^4. $N = 1$ supersymmetry in M^4 has 4 real components of supercharge. Thus the heterotic string compactified on T^6 gives rise to an $N = 4$ theory in 4 dimensions. The number of supersymmetries is doubled for type II theories, because they start with $N = 2$ in 10 dimensions.

To obtain realistic models one has to consider compactifications on more complicated manifolds known as Calabi-Yau spaces or more general superconformal field theories as the internal part. These are discussed extensively in Greene's lectures at this school. Here we will discuss the simplest type of Calabi-Yau spaces, known as *orbifolds* [11].

5.1 $S^1/\mathbb{Z}_2$

This is the simplest illustration of the idea of orbifold compactification. As you recall from lecture two, $T^1 \sim S^1$ can be defined as the quotient of $\mathbb{R}^1$ by $2\pi R\mathbb{Z}$. Now let us consider a further $\mathbb{Z}_2$ equivalence relation:

$$X \sim -X.$$

This defines the quotient $S^1/\mathbb{Z}_2$. What does the resulting space look like? To find out, note that it has two fixed points: 0 and πR. The latter is a fixed point because $-\pi R \sim \pi R$ on the S^1. $S^1/\mathbb{Z}_2$ therefore looks like a line segment (fig. 11).

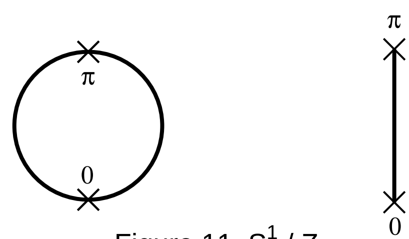


Figure 11, S^1 / Z_2

Recall that in toroidal compactifications, requiring the spacetime wavefunction to be single valued results in the quantization of center of mass momentum. We could alternatively say that we project out all the states which are not invariant under the equivalence relation defining the torus (44) with the operator

$$\sum_{\Delta X \in \Lambda} e^{ip\cdot\Delta X}$$

where $e^{ip\cdot\Delta X}$ is the operator that performs a translation by the lattice vector ΔX. It is clear that this operator is simply a periodic delta function in momentum space singling out the correctly quantized momenta. Similarly, for orbifold compactification we should project out states which are not invariant under the $\mathbb{Z}_2$ operation with the projection operator

$$P = (1 + \Omega)/2$$

where Ω is the operator that perform the appropriate $\mathbb{Z}_2$ on X:

$$\Omega^{-1} X(z, \bar{z}) \Omega = -X(z, \bar{z}).$$

This is very similar to the action of $(-1)^F$ introduced in lecture two, so it is easy to see that the partition function is

$$Z_u \equiv \mathrm{Tr}\left(P\, q^{L_0 - 1/24} \bar{q}^{\tilde{L}_0 - 1/24}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{|\eta|^2}\sum_{p,\tilde{p}} q^{p^2/2}\bar{q}^{\tilde{p}^2/2} + \frac{1}{\left|q^{1/24}\prod_n(1+q^n)\right|^2}\right\}. \quad (88)$$

There is an immediate problem with this partition function. We know the first term in (88) is modular invariant, because it is simply the internal part of the partition function for the string compactified on S^1, derived in lecture two. However, it can be checked that the second part is not modular invariant. In fact it is easy to figure out the modular transformation property of the second term since $q^{1/24}\prod_n(1+q^n)$ is exactly the partition function of the free fermion studied in lecture two. Under the modular transformation S

$$\left|q^{1/24}\prod_n(1+q^n)\right|^2$$

becomes

$$\frac{1}{2}\left|q^{-1/48}\prod_n(1-q^{n-1/2})\right|^2,$$

which then becomes

$$\frac{1}{2}\left|q^{-1/48}\prod_n(1+q^{n-1/2})\right|^2$$

after the T transformation. Therefore we must include all of them in the modular invariant partition function

$$\begin{aligned} Z \quad = \frac{1}{2}q^{-1/24}\bar{q}^{-1/24} \quad & \left\{\frac{\sum_{p,\tilde{p}} q^{p^2/2}\bar{q}^{\tilde{p}^2/2}}{\left|q^{1/24}\prod_n(1-q^n)\right|^2} + \frac{1}{\left|q^{1/24}\prod_n(1+q^n)\right|^2}\right. \\ & \left. + \frac{2q^{1/16}\bar{q}^{1/16}}{\left|\prod_n(1-q^{n-1/2})\right|^2} + \frac{2q^{1/16}\bar{q}^{1/16}}{\left|\prod_n(1+q^{n-1/2})\right|^2}\right\}. \quad (89) \end{aligned}$$

What is the meaning of the last two terms? Recall again the case of toroidal compactification. There not only do we quantize the center-of-mass momentum to ensure the single-valuedness of the wavefunction, but we also have to take into account the winding sectors, which represent strings wrapping around nontrivial loops on the torus:

$$X(\sigma^1 + 2\pi, \sigma^2) = X(\sigma^1, \sigma^2) + 2\pi m R.$$

On $S^1/\mathbb{Z}_2$ there are more sectors due to the identification $X \sim -X$. We should consider *twisted sectors*, which correspond to

$$X(\sigma^1 + 2\pi, \sigma^2) = -X(\sigma^1, \sigma^2) + 2\pi m R.$$

The minus sign in this boundary condition requires that the modding of X be half integral.

$$X = x + \sum_{n \in \mathbb{Z}+1/2} \frac{\imath}{n} (\alpha_n e^{-nz} + \tilde{\alpha}_n e^{-n\bar{z}}).$$

We cannot have nonzero momentum or winding number here since they are not consistent with the anti-periodic boundary condition. The boundary condition also restricts x to be 0 or πR. Therefore there are two twisted sectors, each centering on a fixed point of the $\mathbb{Z}_2$ action on S^1. This is a general feature of orbifold compactification.

The additional terms in the partition function can now be understood as contribution from the two twisted sectors. They both give the same contribution

$$\begin{aligned}
&\mathrm{Tr}_{\,twisted}\, \{P\, q^{L_0 - 1/24} \bar{q}^{\tilde{L}_0 - 1/24}\} \\
&\quad = \frac{1}{2} q^{-1/24} \bar{q}^{-1/24} \left\{ \frac{q^{1/16} \bar{q}^{1/16}}{\left|\prod_n (1 - q^{n-1/2})\right|^2} + \frac{q^{1/16} \bar{q}^{1/16}}{\left|\prod_n (1 + q^{n-1/2})\right|^2} \right\}.
\end{aligned}$$

Note that the formula (89) contains the factor 2, reflecting the fact that there are two fixed points of $\mathbb{Z}_2$. Modular transformation mixes the partition function for twisted and untwisted sectors, with or without the insertion of the operator Ω, in exactly the same fashion it mixes different spin structures as discussed in lecture two.

Recall that in the free fermion theory, the ground state of the periodic, Ramond, sector has a higher energy relative to the anti-periodic, Neveu-Schwarz sector. For the bosonic orbifold theory, however, the ground state of the anti-periodic, i.e. twisted sector, has a higher eigenvalue of L_0 and $\tilde{L}_0$. Its weight is (1/16, 1/16)[32] per twisted coordinate. This is the same as that of the R-R ground states per real fermion.

[32] This can be obtained by computing the OPE of the energy-momentum tensor with a twist field, which generates the twisted boundary condition of X, or by the ζ-function regularization. Here we derive it by requiring modular invariance.

5.2 $T^4/\mathbb{Z}_2$

Now let us consider an only slightly more involved example which is nonetheless already a limiting case of Calabi-Yau compactification. The compact manifold is now $T^4/\mathbb{Z}_2 \sim (S^1)^4/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts on each of the four S^1 as in the last example:

$$X^i \to -X^i, \quad i = 1, \dots, 4.$$

As each S^1 has 2 fixed points, on this orbifold there are $2^4 = 16$ fixed points. An analysis similar to the one given above shows there is a twisted sector associated with each of them. The weight of their ground state is

$$\left(\frac{1}{4}, \frac{1}{4}\right) = 4 \times \left(\frac{1}{16}, \frac{1}{16}\right).$$

Since we want to discuss superstring compactified on this orbifold, we should include the worldsheet fermion ψ's as well. They transform as tangent vectors in spacetime. Now the $\mathbb{Z}_2$ map clearly acts on the tangent space as well, as it reverses spacetime direction:

$$\psi^i \to -\psi^i.$$

In fact this is also required by the superconformal invariance, which mixes between X^i and ψ^i. As the ψ's already have periodic and anti-periodic boundary condition, the $\mathbb{Z}_2$ action merely exchanges their assignment to R and NS sectors respectively. Previously we saw that each ψ^i increases the conformal weight of the ground state by $\frac{1}{16}$ when going from the NS to the R sector. Thus, in the twisted sector, the fermions should contribute $4 \times \frac{1}{16} = \frac{1}{4}$ to the conformal weight. The total conformal weight of the twisted sector is then (1/2, 1/2). In particular, they correspond to massless states in the physical spectrum of type II superstring.

In fact each fixed point gives rise to 4 massless scalar fields in the uncompactified $(5+1)$ dimensions. In order to change the boundary condition of the fermions, we may bosonize the 4 fermions into 2 bosons ϕ^1 and ϕ^2, and consider the spin operators

$$\sigma_{\pm\pm} = e^{\pm\frac{i}{2}\phi^1 \pm \frac{i}{2}\phi^2}.$$

The GSO projection forces the number of minuses to be even, so there are 2 choices. Since the left and the right movers can have different choices, there are $2 \times 2 = 4$ ways to change the boundary condition of the fermions. Since there are 16 fixed points, the type II superstring on $T^4/\mathbb{Z}_2$ gives $4 \times 16 = 64$ massless scalar fields from the twisted sector.

In addition, there are $4\times4 = 16$ massless scalars coming from the untwisted sector. They are constant modes of the metric G_{ij} and the NS-NS B_{ij} $(i, j = 1, ..., 4)$ and correspond to the Narain moduli of T^4. In fact, the 64 scalars from the twisted sector share a similar geometric interpretation. They are so-called *blow-up* modes, and their VEV's deform and resolve the orbifold singularity at the fixed points. When these singularities are fully resolved, one recovers a smooth Calabi-Yau manifold known as K_3. Combining the twisted and the untwisted sectors together, the moduli space of type II string compactification over K_3 is $16 + 64 = 80$. This is the same as that of the heterotic string compactified over T^4 since $4 \times (16 + 4) = 80$. This is not a mere coincidence, and its deeper reason will be uncovered during the school.

We hope you have acquired the necessary knowledge to cope with the more advanced lectures in this school. *Bon Voyage*!

Acknowledgments

We would like to thank the school organizer K.T. Mahanthappa and the program director B. Greene for their beautiful planning and organization of the school. We thank J. Harvey and J. Polchinski for their useful suggestions on the outline of the course and for J. Feng and M. Peskin for having a preliminary draft of their forthcoming book available to us. We are also grateful to C-S. Chu, Y. Oz and H. Steinacker for reading the draft. This work was supported in part by the National Science Foundation under grants PHY-951497 and in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098. Z.Y. is supported in part by a Graduate Research Fellowship of the U.S. Department of Education.

References

1. C. Vafa, "Conformal Algebra of Riemann Surfaces", in Proceedings of TASI 89, "Particle, Strings and Supernovae", Providence, RI, 1989, eds. A. Jevicki and C.-I. Tan, World Scientific (1989).
2. E. D'Hoker, "TASI Lectures on Critical String Theory", in Proceedings of TASI 92, "From Black Holes and Strings to Particles", Boulder, CO, 1992, eds. J. Harvey and J. Polchinski, World Scientific (1993).
3. M. Green, J. Schwarz and E. Witten, "Superstring Theory", Vol 1 & 2, Cambridge University Press, (1987).
4. S. Weinberg , "The Quantum Theory of Fields", Vol 1, Cambridge University Press (1995), §13.1 and reference 2 and 3 therein.
5. P. Ginsparg and G. Moore, "Lectures on 2-D Gravity and 2-D String Theory", in Proceedings of TASI 92, "From Black Holes and Strings to Particles", Boulder, CO, 1992, eds. J. Harvey and J. Polchinski, World Scientific (1993).
6. J. Polchinski, "What is String Theory?", in Proceedings of Les Houches Summer School, "Fluctuating Geometries in Statistical Mechanics and Field Theory, " Les Houches, France, 1994, eds. F. David and P. Ginsparg, http://xxx.lanl.gov/lh94.
7. P. Ginsparg, "Applied Conformal Field Theory", in Proceedings of Les Houches Summer School, "Fields, Strings, and Critical Phenomena", Les Houches, France, 1988, eds. E. Brezin and J. Zinn-Justin, North-Holland (1990).
8. D. Lüst and S. Theisen, "Lectures on String Theory", Springer-Verlag (1989).
9. A. Giveon, M. Porrati and E. Rabinovici, "Target Space Duality in String Theory" *Phys. Rept.* **244C** (1994) 77.
10. J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges", *Phys. Rev. Lett.* **75** (1995) 4724.
11. L. Dixon, "Some Worldsheet Properties of Superstring Compactifications, On Orbifolds and Otherwise", in Proceedings of the Summer Workshop in High Energy Physics and Cosmology, Trieste, Italy, 1987, World Scientific (1988).