Spectrum of Large N Gauge Theory from Supergravity

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Recently, Maldacena proposed that the large N limit of the $\mathcal{N} = 4$ supersymmetric gauge theory in four dimensions with U(N) gauge group is dual to the type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. We use this proposal to study the spectrum of the large N gauge theory on $\mathbb{R} \times S^3$ in a low energy regime. We find that the spectrum is discrete and evenly spaced, and the number of states at each energy level is smaller than the one predicted by the naive extrapolation of the Bekenstein-Hawking formula to the low energy regime. We also show that the gauge theory describes a region of spacetime behind the horizon as well as the region in front. [S0031-9007(98)06050-5]

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The number of states in a d-dimensional conformal field theory (CFT) grows as a function of the energy E as

$$S \sim E^{(d-1)/d} V^{1/d}, \tag{1}$$

where V is the volume of the $(d-1)$-dimensional space. For example, for $d = 2$, the precise formula is

$$S_{d=2} = \sqrt{\frac{2\pi}{3} cEV},$$

where c is the central charge of the Virasoro algebra [1]. For $d = 4$, the entropy of the $\mathcal{N} = 4$ U(N) gauge theory in the strong coupling regime is believed to be [2]

$$S_{d=4} = (\tfrac{2}{3})^{3/4} \sqrt{2\pi N}\, E^{3/4} V^{1/4} \tag{2}$$

because of the Bekenstein-Hawking entropy formula for the near extremal 3-brane solution of type IIB theory [3]. In general (1) can be derived by assuming that the entropy is an extensive quantity and that it is invariant under the dilatation.

Recently Maldacena made an interesting proposal in [4] that compactifications of M theory and string theory on a sphere to anti–de Sitter space (AdS) are dual to various conformal field theories. In particular, the large N limit of the $\mathcal{N} = 4$ U(N) gauge theory in four dimensions is claimed to be described by the type IIB string theory on $\mathrm{AdS}_5 \times S^5$. This implies that spectra of the two theories and, hence, the entropies are the same.

The purpose of this paper is to use this proposal to learn about the spectrum of the large N gauge theory on $\mathbb{R} \times S^3$. We will establish a correspondence between states in this gauge theory and states in string theory on $\mathrm{AdS}_5 \times S^5$. In the supergravity approximation to string theory, the energy levels are quantized in the units of the AdS radius $R = (4\pi g N)^{1/4} l_s$ where g is the string coupling constant, and l_s is the string length,

$$E = \frac{n}{(4\pi g N)^{1/4} l_s}, \qquad (n = 0, 1, 2, \ldots).$$

The supergravity approximation is valid, when E is less than both the string scale and the Planck scale,

$$E \ll l_s^{-1}, \qquad l_p^{-1}.$$

Since $l_p \sim g^{1/4} l_s$ in ten dimensions, we can trust the supergravity computation for

$$n \ll (gN)^{1/4}. \tag{3}$$

In this regime, we find that the entropy of the type IIB string theory scales as a function of $E = n/R$ as

$$S_{\mathrm{AdS}_5 \times S^5} \sim n^{9/10}, \tag{4}$$

for $n \gg 1$.

We will show that each supergravity state with energy $E = n/R$ corresponds to a state with energy $E = n$ in the gauge theory on $\mathbb{R} \times S^3$ with a unit radius sphere. Thus, Maldacena's proposal leads to the prediction that the entropy of the large N gauge theory on $\mathbb{R} \times S^3$ for $1 \ll n \ll (gN)^{1/4}$ is

$$S_{\mathrm{gauge}} \sim n^{9/10}. \tag{5}$$

In particular, the large N spectrum is independent of N.

On the other hand, the Bekenstein-Hawking formula (2) at this energy reads

$$S_{d=4} \sim \sqrt{N}\, n^{3/4}. \tag{6}$$

Since this formula was originally derived for the 3-branes wrapped on T^3, it is a prediction for the density of states for the gauge theory on T^3. However, for sufficiently large n, finite size effects are irrelevant, and we expect that the density of states is independent of the topology of the 3-manifold. In such a case, we can use the formula (6) for the gauge theory on S^3 also. Comparing this with (5), the power of n is different, and there is no factor of $\sqrt{N}$ in (5). Therefore, in the low energy regime (3), S_{gauge} is smaller than the Bekenstein-Hawking entropy (2). This does not mean that the proposal of [4] is wrong, since the finite size effects may become relevant for $n \ll N^2$, which includes the low energy regime we study in this paper. [The finite size effects are negligible when the wavelength corresponding to temperature T is

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much shorter than the size of the 3-manifold. For the 3-branes, $T \sim (n/N^2)^{1/4} V^{-1/3}$, so the finite size effects are negligible for $n \gg N^2$.]

We also point out that Maldacena's conjecture implies that the four-dimensional gauge theory describes a region of spacetime *behind* the horizon as well as the region in front.

Conformal symmetry and spectrum.—AdS_{d+1} has an isometry group of $SO(2,d)$. In [4], this group was identified with the conformal symmetry of a gauge theory on $\mathbb{R}^{1,d-1}$ for $d = 3,4,6$. The conformal symmetry is generated by the momentum P_μ, the Lorentz generators $L_{\mu\nu}$, the dilatation D, and the special conformal generators K_μ $(\mu, \nu = 0,1,\ldots,d-1)$. To simplify the following analysis, the generators are normalized so that the AdS radius R does not show up in the structure constants.

Minkowski space $\mathbb{R}^{1,d-1}$ can be conformally embedded in the Einstein static universe, which has the topology of $\mathbb{R} \times S^{d-1}$ as follows. Introducing the null coordinates $u = t - r$ and $v = t + r$, the Minkowski metric is

$$ds^2 = -du dv + \tfrac{1}{4}(v - u)^2 d\Omega_{d-2}.$$

Multiplying this by the conformal factor, $4(1+u^2)^{-1}(1+v^2)^{-1}$, and introducing new coordinates

$$u = \tan\left(\frac{\tilde{t} - \chi}{2}\right), \qquad v = \tan\left(\frac{\tilde{t} + \chi}{2}\right),$$

the rescaled metric becomes

$$d\tilde{s}^2 = -d\tilde{t} + d\chi^2 + \sin^2 \chi \, d\Omega_{d-2},$$

which can be recognized as that of the Einstein universe with unit radius. Although the original Minkowski space is mapped to a subset of this space, it was shown by Lüscher and Mack [5] that correlation functions of CFT on $\mathbb{R}^{1,d-1}$ can be analytically continued to the full Einstein universe. Moreover, since

$$\frac{\partial}{\partial \tilde{t}} = \frac{1}{2}(1+u^2)\frac{\partial}{\partial u} + \frac{1}{2}(1+v^2)\frac{\partial}{\partial v},$$

the generator H of the global time translation on $\mathbb{R} \times S^{d-1}$ is given by

$$H = \tfrac{1}{2}(P_0 + K_0). \tag{7}$$

Students of two-dimensional CFT would recognize that this is a higher dimensional generalization of the fact that the Virasoro generator L_0 is the translation generator on $\mathbb{R} \times S^1$, while the momentum on $\mathbb{R}^{1,1}$ is L_{-1}.

AdS_{d+1} is a globally static space, and $H = \frac{1}{2}(P_0 + K_0)$ is also its global time translation generator. Therefore, the conjecture of [4] implies that each state of the CFT on $\mathbb{R} \times S^{d-1}$ is identified with a state in the M theory and string theory on AdS_{d+1} times a sphere.

The Hilbert space of the semiclassical supergravity is constructed from a free gas of local fluctuations of the fields. Since AdS_{d+1} is not globally hyperbolic (infinity is a timelike boundary), we need to impose appropriate boundary conditions on the supergravity fields. The requirement that the local fluctuations should give unitary representations of $SO(2,d)$ severely limits the choice of boundary conditions [6,7]. This is a reasonable requirement in the present case as we expect that the CFT discussed in [4] has a unitary spectrum.

Let us examine the case of the IIB theory on $\mathrm{AdS}_5 \times S^5$. The compactification of the supergravity on S^5 creates a tower of massive particles on AdS_5, and their spectrum is classified in [8,9]. Curiously they found that all the eigenvalues of H are integers, including excitations on AdS_5. Even though AdS is a noncompact space, the curvature introduces an effective infrared cutoff, which makes the spectrum discrete. Since our H is normalized so that the AdS radius R does not show up in the structure constants of $SO(2,4)$, this means that all the wave modes are periodic in time with the period $2\pi R$, and the supergravity theory is well defined in the single cover of AdS_5. (To be precise, the fermions have half odd integral modes and therefore are antiperiodic on AdS_5.) We will discuss implications of this observation to the near horizon geometry of the 3-brane later.

The number of the Kaluza-Klein modes with $(\text{mass})^2 \sim (l/R)^2$ is of the order of l^4 for large l. The fluctuations of each Kaluza-Klein mode on AdS_5 give a representation of $SO(2,4)$ with the highest weight $H \sim l$. The action of $SO(2,4)$ creates states with energy $H \sim l + s$ $(s = 0,1,2,\ldots)$ with the asymptotic degeneracy of order s^3, except for a special representation called the singleton for which the degeneracy grows slower [10]. The singletons appear for special values of Kaluza-Klein masses and do not contribute to the leading behavior of the entropy. The number $\rho(\epsilon)$ of single particle states with the energy $H = \epsilon$ is therefore

$$\rho(\epsilon) \sim \sum_{(l,s);l+s=\epsilon} l^4 s^3 \sim \epsilon^8. \tag{8}$$

Following the standard procedure, the single particle spectrum $\rho(\epsilon)$ can be converted into the entropy of a free gas of particles on $\mathrm{AdS}_5 \times S^5$ with total energy $H = n$ to obtain

$$S \sim n^{9/10}. \tag{9}$$

Since we identify the AdS Hamiltonian $H = \frac{1}{2}(P_0 + K_0)$ with the CFT Hamiltonian on $\mathbb{R} \times S^3$ with a unit radius, (9) also gives the entropy of the CFT on $\mathbb{R} \times S^3$ at energy $E = n$.

It is straightforward to repeat the analysis for other supergravity backgrounds such as $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_4 \times S^7$. In both cases, the entropy behaves as

$$S \sim n^{10/11},$$

just as a free gas in 11 dimensions.

There is a possibility that the correspondence of the large N gauge theory and the supergravity on $\mathrm{AdS}_5 \times S^5$ require a nonstandard choice of boundary conditions. In the above analysis we only assumed that fluctuations of the fields make unitary representations of SO(2, 4), and the above estimate would not be sensitive to the precise choice of the boundary conditions having such a property.

The result (9) has an obvious interpretation. When the radius of $\mathrm{AdS}_5 \times S^5$ is large, the space looks almost like $\mathbb{R}^{1,9}$. The entropy $S \sim n^{9/10}$ simply reflects the bulk degrees of freedom in ten dimensions.

Spacetime beyond the horizon.—The role of spacetime beyond the horizon has been puzzling in the recent successful description of black hole microstates in terms of states of weakly coupled D-branes [11]. Maldacena's conjecture sheds light on this issue. The supergravity solution describing N extremal 3-branes consists of a completely nonsingular spacetime with an infinite number of asymptotically flat regions, each with a horizon [12]. One can periodically identify this space so that there are only two asymptotically flat regions (one on each side of the horizon), but then one introduces closed timelike curves. The region near the horizon is, of course, locally AdS_5. If one identifies to have only two asymptotically flat regions, the near horizon geometry is globally AdS_5.

Now consider the four-dimensional large N gauge theory. The conformal symmetries of this theory are globally either SO(2, 4) or its covering group. These symmetries must be reflected in the near horizon AdS geometry. The region to one side of one horizon is not invariant under this group. One must include at least the region on both sides of the horizon. Thus, *the gauge theory describes spacetime on both sides of the horizon.* The question of whether the infinite chain of horizons must be included depends on whether the conformal weights of the theory are all integer. If so, the conformal group is just SO(2, 4) (not the covering group), and the spacetime contains just one horizon. We have seen that supergravity on $\mathrm{AdS}_5 \times S^5$ has integer energies with respect to the global time, so it is well defined on the single cover of AdS_5.

The fact that the entire tower of Kaluza-Klein states of supergravity have integer energy levels can be interpreted in terms of the AdS supersymmetry algebra [13] (we thank Paul Townsend for pointing this out to us). They are Bogomolny-Prasad-Sommerfeld (BPS) states, forming short supersymmetry multiplets, and their levels are naturally integral. Assuming the correspondence of the AdS supergravity and the CFT [4], these Kaluza-Klein states should correspond to chiral states in the gauge theory without anomalous dimensions. By counting all chiral states of the gauge theory, one should be able to check the conjecture (5). On the other hand, there is no supersymmetry reason to expect that massive string states have integral energy levels. This suggests that the string theory should be defined on the universal cover of AdS_5.

After submitting this paper, we received a paper by Gubser, Klebanov, and Polyakov [14], where they suggested that the massive string states do not have integer energy levels, and the anomalous dimensions of the corresponding operators in the gauge theory are increased by the factor $(gN)^{1/4}$. We also received a paper by Witten [15], where he outlined the correspondence between some chiral states in the gauge theory and the Kaluza-Klein supergravity states, extending earlier work [16]. More recently this correspondence has been established for all supergravity states by Ferrara, Fronsdal, and Zaffaroni [17]. Assuming there are no other light gauge theory states, this confirms the prediction of this paper.

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