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Closed Form Effective Conformal Anomaly Actions in D≥4

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Abstract

I present, in any D≥4, closed-form type B conformal anomaly effective actions incorporating the logarithmic scaling cutoff dependence that generates these anomalies. Their construction is based on a novel class of Weyl-invariant tensor operators. The only known type A actions in D≥4 are extensions of the Polyakov integral in D=2; despite contrary appearances, we show that their nonlocality does not conflict with general anomaly requirements. They are, however, physically unsatisfactory, prompting a brief attempt at better versions.

1 Introduction

Conformal (Weyl) anomalies reflect the loss of classical scale invariance caused by unavoidable regularization of conformally invariant matter closed loops. Their properties, in all dimensions, are by now well understood. In particular [1] there are two, explicitly known, types that can be conveniently expressed in terms of an external metric $g_{\mu\nu}$ coupled to the matter. These local gravitational scalar densities $\mathcal{A}(g_{\mu\nu})$ differ in their separate IR/UV origins and in their behavior under Weyl variations. Both can be represented as responses of (nonunique) nonlocal effective actions $I[g_{\mu\nu}]$ under metric conformal variations

$$\delta g_{\mu\nu} = 2\phi(x) g_{\mu\nu} \ . \tag{1}$$

The built-in integrability condition on variations of these anomalies $\mathcal{A}(x) \equiv \delta I[g_{\mu\nu}]/\delta\phi(x)$,

$$\delta\mathcal{A}(x)/\delta\phi(x') = \delta^2 I/\delta\phi(x')\delta\phi(x) = \delta\mathcal{A}(x')/\delta\phi(x) \ , \tag{2}$$

serves as a useful check on candidate $\mathcal{A}$'s and on allowed forms of the type A actions; type B anomalies, are necessarily Weyl-invariant, satisfying (2) trivially.

The origin of the anomalies in closed loop graphs imposes constraints on the actions' dimensionality and nonlocality. These seem to clash with the only known closed form actions, essentially the obvious D≥4 generalizations of the D=2 Polyakov action in both cases (type B starts at D=4). On the other hand, since effective actions are not unique – nonlocal Weyl invariant dimensionless gravitational functionals will be exhibited – some choices will be better behaved physically than others, reflecting more accurately the underlying loop properties or being obtainable through integrating out a compensating field in an action that is physically acceptable, in particular ghost-free.

I will present here new complete closed form type B actions that correctly reflect their cutoff dependence and origins. Their construction is based on new tensor differential operators (generalizing existing scalar ones) that are conformal invariant when acting on Weyl-like tensors. For the existing type A actions, we will resolve the paradox that their explicit $\Box^{-2}$ nonlocalities violate the single pole ($\Box^{-1}$) behavior at lowest order about flat space required by dimensionality and general anomaly analysis. Quite apart from the above problems, however, they have long been known [2, 1], to be unsatisfactory in *e.g.*, long-distance behavior, failing to correctly represent the underlying stress tensor correlators. While I have not succeeded in constructing more suitable actions beyond the lowest order one given in [1], some remarks on this open problem are appended.

2 Type B

The type B anomalies $\mathcal{A}_B$ have two hallmarks: they arise from the UV behavior of the underlying matter loops, with consequent logarithmic cutoff dependence, and are themselves Weyl invariant. They only start at D=4, being in fact the first anomalies discovered there [3]; the unique local D=4 conformal invariant is the square of the Weyl tensor,

$$\mathcal{A}_B \equiv \sqrt{-g}\, tr\, C^2 \ , \quad \delta \mathcal{A}_B(x)/\delta\phi(x') \equiv 0 \ . \tag{3}$$

The number of independent $\mathcal{A}_B$ rises rapidly with dimension; for example there are [4] three varieties at D=6: two independent index traces of $\sqrt{-g}\, C^3$ and a third of the schematic form $\sqrt{-g}\, trC(\Box + R)C$. An effective D=4 action that reflects the required logarithmic behavior was already introduced in [3] at lowest, cubic, order in an expansion about flat space, $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$:

$$I^4_B \approx \int d^4x\, tr\, C \ln(\Box/\lambda^2) C + \mathcal{O}(h^4) \ , \tag{4}$$

each C and $\Box$ being effectively $\mathcal{O}(h)$, since the quadratic part is Weyl invariant. While not strictly correct, this approximation reproduced some of the desired scaling characteristics, including the logarithmic dependence on the cutoff λ of the closed loops. What we really want of course is to retain that behavior, but with $\Box$ replaced by an argument $\tilde{\Delta}$ that produces the proper variation $\delta \ln \tilde{\Delta} = \phi$ to all orders; $\tilde{\Delta}$ must be a scalar (covariance forbids densities from being arguments of logs) and of dimension 4, to bring in a λ^{-4}. The problem of obtaining a physical effective action I_B thus reduces to finding dimension 4 operators $\tilde{\Delta}_4$ that, at least when acting on a specific tensorial class Z such as scalar or 4-tensors, are themselves Weyl invariant, $(\delta\tilde{\Delta}_4)Z = 0$. One such operator has long been know and indeed underlies the type A construction to be discussed in Sec. 3: the self-adjoint Paneitz (scalar density) operator [5] acting on scalars,

$$\Delta_P = \sqrt{-g}[\Box^2 + 2D_\mu(R^{\mu\nu} - \tfrac{1}{3}\, g^{\mu\nu} R)D_\nu] \ . \tag{5}$$

It is the unique D=4 generalization of the D=2 invariant $\Delta_2 \equiv \sqrt{-g}\, \Box$; its extra terms complete the merely constant-scale invariant $\sqrt{-g}\,\Box^2$. Unfortunately, because it acts on scalars, Δ_p is useless here:

$$\tilde{I}^4_B \sim \int d^4x \sqrt{-g}\, \ln(\Delta_P/\sqrt{-g}\,\lambda^4) tr\, C^2 \tag{6}$$

is a total divergence, with vanishing variation.

From the above lesson, it is clear that we must abandon invariant Δ's acting on scalars and instead seek one that begins at D=4 and acts invariantly on 4-tensors T, $\delta(\tilde{\Delta})T = 0$. More specifically, it suffices that $\tilde{\Delta}$ be invariant when acting on the Weyl tensor, for concreteness in its Weyl invariant $C^\mu{}_{\alpha\beta\gamma}$ index configuration, and to reproduce the latter's tensorial rank and algebraic properties:

$$\tilde{C}^\mu{}_{\nu\alpha\beta} \equiv (\tilde{\Delta} C)^\mu_{\nu\alpha\beta} \equiv \tilde{\Delta}^{\mu\nu'\alpha'\beta'}_{\mu'\nu\alpha\beta}\, C^{\mu'}{}_{\nu'\alpha'\beta'} \ . \tag{7}$$

Preserving constant scale invariance already requires $\tilde{\Delta}$ (like Δ_p) to be a $4^{\rm th}$ derivative tensor density; so if $\tilde{\Delta}$ obeys $\delta(\tilde{\Delta})C = 0$, it will follow that

$$\delta\tilde{C} = 0\ , \quad \delta(\tilde{\Delta}\,\frac{1}{\sqrt{-g}}\,\tilde{C}) = 0\ , \quad \delta[\tilde{\Delta}\frac{1}{\sqrt{-g}}\,\tilde{\Delta}\ ...\ \frac{1}{\sqrt{-g}}\,\tilde{\Delta}C)] = 0\ , \tag{8}$$

where the intermediate $1/\sqrt{-g}$ factors must be included to keep subsequent $\tilde{\Delta}$ acting on tensors rather than on densities. The underlying physics clearly demands that such a $\tilde{\Delta}$ exist, and it has now indeed been found [6]; while its form is unfamiliar (*e.g.*, it has no $\Box^2$ part at all), it is only necessary for our purposes to know that it exists since its only role is to allow for the presence of the "compensator field" $\ln\sqrt{-g}$. In terms of $\tilde{\Delta}_4$ the desired action is simple:

$$I^4_B = -\tfrac{1}{4}\int d^4x\sqrt{-g}\,C_\mu{}^{\nu\alpha\beta}[\ln(\tilde{\Delta}_4/\lambda^4\sqrt{-g}\,)C]^\mu_{\nu\alpha\beta}\ . \tag{9}$$

The only non-vanishing variation of (9) stems entirely from $\ln\sqrt{-g}$,

$$\delta I^4_B = \tfrac{1}{4}\int d^4x\sqrt{-g}\,C^2\delta\ln\sqrt{-g} = \int d^4x(\sqrt{-g}C^2)\delta\phi\ , \tag{10a}$$

all the rest of (9), including the left factor $(\sqrt{-g}\,g^{\cdot\cdot}g^{\cdot\cdot}C^{\cdot}...)$, being manifestly invariant. In more detail, since it is the density $\tilde{\Delta}_4$ that is Weyl invariant, any power in the log's expansion, $(\frac{1}{\sqrt{-g}}\,\tilde{\Delta}...\frac{1}{\sqrt{-g}}\tilde{\Delta})C$ correctly avoids having $\tilde{\Delta}_4$ act on densities and only the "outer" $1/\sqrt{-g}$ factor contributes in each term. Hence we may indeed conclude from (10a) that

$$\delta I^4_B/\delta\phi = \sqrt{-g}\,C^2\ . \tag{10b}$$

The various possible $\mathcal{A}_B(x)$ in higher dimension will similarly be expressible in terms of the corresponding $\tilde{\Delta}_D$, which are also sure to exist:

$$I^d_B \sim \int d^dx\sqrt{-g}\,Z_\mu^{\nu\alpha\beta}[\ln(\tilde{\Delta}_D/\lambda^D\sqrt{-g}\,)C]^\mu_{\nu\alpha\beta} \tag{11}$$

where Z is the "rest" of the local invariant in question, *e.g.*, $Z \sim (CC)$ or $C(\Box + R)$ in D=6 and similarly for D>6.

I close this section with an object lesson on ambiguities in effective actions; it is an apt introduction to the type A problem, being modeled on the only closed form action known there and being even more unphysical for type B (because it totally violates the logarithmic dependences) than for type A. It is based on the fact that (as explained below) the quantity $(\bar{\mathcal{E}}_4\Delta_p^{-1})$, where $\bar{\mathcal{E}}_4$ is essentially the D=4 Euler invariant, Weyl transforms as a compensator field (21). Consequently [7],

$${}^*I^4_B = \int d^4x\,\bar{\mathcal{E}}_4\Delta_p^{-1}(\sqrt{-g}\,C^2)\ , \quad \delta\,{}^*I^4_B/\delta\phi = \sqrt{-g}\,C^2\ . \tag{12}$$

Note the complete contrast between the actions (9) and (12), even though both succeed in the limited requirement of correctly yielding $\mathcal{A}_B$ under Weyl variation.

3 Type A

To understand this family, it is useful to review D=2, where type A is the only possible anomaly. By power counting, the anomaly $\mathcal{A}_2(x)$ must have dimension 2; the only local diffeo-invariant is the Euler density $\mathcal{E}_2(x)$, a total divergence:

$$\mathcal{A}_2(x) = \sqrt{-g}\,R(x) = \tfrac{1}{2}\,\sqrt{-g}\,\epsilon^{\mu\nu}\epsilon^{\alpha\beta}R_{\mu\nu\,\alpha\beta} \equiv \mathcal{E}_2(x)\ . \tag{13}$$

Unlike type B, this quantity is not Weyl invariant; rather $\delta\mathcal{E}_2(x) \equiv 2\sqrt{-g}\,\Box\phi$. The indicated Weyl variation of $\mathcal{A}_2$ guarantees the integrability condition:

$$\delta\mathcal{E}_2(x)/\delta\phi(x') = 2\sqrt{-g}\,\Box\delta^2(x-x') = \delta\mathcal{E}_2(x')/\delta\phi(x)\;. \tag{14}$$

As already noted, the scalar density operator $\Delta_2 \equiv \sqrt{-g}\,\Box$ is Weyl invariant at D=2, when (and only when) acting on a scalar

$$\delta\Delta_2 \equiv \delta[\partial_\mu(\sqrt{-g}\,g^{\mu\nu})\partial_\nu] = 0\;. \tag{15}$$

Hence the nonlocal scalar operator

$$\delta(\mathcal{E}_2/\Delta_2) = 2\phi(x) \tag{16}$$

transforms like a Weyl compensator field, leading to the Polyakov [8] construction,

$$I_2 = \tfrac{1}{4}\int d^2x\;\mathcal{E}_2\Delta_2^{-1}\mathcal{E}_2\;,\quad \delta I_2/\delta\phi(x) = \mathcal{E}_2(x)\;. \tag{17}$$

Note that although Δ_2^{-1} acts on the density $\mathcal{E}_2$, its variation vanishes because we must first write $\delta\Delta_2^{-1}\mathcal{E}_2 = -\Delta_2^{-1}\,\delta(\Delta_2)(\Delta_2^{-1}\mathcal{E}_2)$ and $(\Delta_2^{-1}\,\mathcal{E}_2)$ is a scalar. The pole behavior of the action is clearly $\sim p^{-2}$, in accord with the power counting of the 2-point closed loop $\sim (\int d^2p/p^4)R_L^2$ where $R_L \sim (pph)$ are the linearized scalar curvatures representing external gravitons coupled to the matter $T_{\mu\nu}$: The underlying correlator, $< T^{\mu\nu}(p)T^{\alpha\beta}(-p) >$, is multiplied by $h_{\mu\nu}h_{\alpha\beta}$ and the four factors of momentum in the $< TT >$ numerator convert them to curvatures. However, this counting is true only to leading order in $h_{\mu\nu}$: while "dressings" of the curvatures from expanding (17) in powers of h but keeping the flat space Δ_2^{-1} do indeed maintain the p^{-2} overall behavior (as they should diagrammatically since this is still effectively a 2-point function) expanding the denominator Δ_2^{-1},

$$\begin{aligned}\Delta_2^{-1} &\equiv [\Box_0 + \partial_\mu\mathcal{H}^{\mu\nu}\partial_\nu]^{-1} = (1 - \Box_0^{-1}\partial_\mu\mathcal{H}^{\mu\nu}\partial_\nu + ...)\Box_0^{-1}, \qquad \Box_0 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu \\ \mathcal{H}^{\mu\nu} &\equiv \sqrt{-g}\,g^{\mu\nu} - \eta^{\mu\nu} = -(h^{\mu\nu} - \tfrac{1}{2}\,\eta^{\mu\nu}h) + ...\end{aligned} \tag{18}$$

gives rise to increasing powers of p^{-2}, in total agreement with the diagrammatics; a 3-point closed loop generically acquires another p^{-2} from the extra propagator and so on for the n-point expansion. Indeed, the folklore that anomalies must have only a $\Box_0^{-1}$ nonlocality, applies only to their leading terms. This fact will be essential in D=4. Despite these seemingly unpleasant higher poles, the Polyakov action (unlike its D>2 extensions) is perfectly physical as attested by its derivability through integrating out a physical ghostfree compensator field's action, $I_2[\sigma] = \int d^2x[\frac{1}{2}\sigma\Delta_2\sigma + \sigma\mathcal{E}_2]$; it is also vouched for by being the covariantization of the matter loop integrals $\int d^2x h < TT > h$, as noted above.

As shown in [1], the D=2 anomaly (13) extends uniquely to any D=$2n$: the same "infrared" type is given by the Euler density at D=$2n$,

$$\mathcal{A}_{2n} = \mathcal{E}_{2n} \equiv \epsilon^{1..2n}\epsilon^{1'..2n'}R_1...R_{...2n'}/\sqrt{-g}\;; \tag{19}$$

note that $\mathcal{E}_{2n}$ and hence its variation vanishes identically in lower D (since the Levi–Civita $\epsilon^{1...2n}$ symbol does). Integrability is always satisfied because $\mathcal{E}_{2n}$ varies according to

$$\delta\mathcal{E}_{2n}(x)/\delta\phi(x') = \mathcal{G}_{2n}^{\mu\nu}(x)D_\mu D_\nu\delta(x-x') = \delta\,\mathcal{E}_{2n}(x')/\delta\phi(x)\;. \tag{20}$$

where $\mathcal{G}^{\mu\nu}_{2n}$ is an identically conserved tensor (as it must be, since $\mathcal{E}_{2n}$ and its variations are total divergences); it is essentially the "Einstein tensor" of the Euler action at dimension 2(n-1). For concreteness we will work in D=4, then indicate the generalization to arbitrary D. Here, $\mathcal{G}^{\mu\nu}_4$ is of course the true Einstein tensor (that indeed vanishes at D=2) and $\mathcal{E}_4 \equiv \sqrt{-g}\,(R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2)$ is the usual Gauss–Bonnet combination. It is therefore tempting to follow the form of the D=2 action (17), in terms of the suitable generalization of Δ_2, namely the Paneitz operator (5) and replacing $\mathcal{E}_2$ by $\mathcal{E}_4$. This was the proposal of [7], with the minor modification of using $\bar{\mathcal{E}}_4 \equiv \mathcal{E}_4 + \frac{2}{3}\sqrt{-g}\,\Box R$ rather than $\mathcal{E}_4$, to achieve the extension of (16), to

$$\delta(\bar{\mathcal{E}}_4/\Delta_P) = \phi\ , \tag{21}$$

and therefore of (17) to

$$I^A_4 = \int d^4x\, \bar{\mathcal{E}}_4 \Delta_p^{-1} \bar{\mathcal{E}}_4\ , \qquad \delta I^A_4/\delta\phi = \bar{\mathcal{E}}_4\ , \tag{22}$$

by the same reasoning as in D=2. Since $\Box R$ itself derives from a local (and hence irrelevant, removable) action, $\delta \frac{1}{18}\int d^4x\sqrt{-g}\,R^2/\delta\phi = \frac{2}{3}\sqrt{-g}\,\Box R$, we see that while (22) literally varies into $\bar{\mathcal{E}}_4$, it effectively also varies into $\mathcal{E}_4$.

The representation (22) presents a paradox: Δ_p^{-1} contains a double ($\Box_0^{-2}$) pole, whose presence is incompatible with the leading, 3-point, function. Just by momentum counting around a matter loop with three external curvatures, the latter's leading $\mathcal{O}(h^3)$ term has to be $\Box_0^{-1}$, and not $\Box_0^{-2}$, nonlocal. To understand the conflict in detail, first simply expand the Δ_p of (5) in (22):

$$I^A_4[h^3] = \int d^4x(\mathcal{E}_4 + \tfrac{2}{3}\Box_0 R)[1 - 2\Box_0^{-2}\partial_\mu(R^{\mu\nu} - \tfrac{1}{3}\,\eta^{\mu\nu}\,R)\partial_\nu + ...]\Box_0^{-2}(\mathcal{E}_4 + \tfrac{2}{3}\Box_0 R)\ . \tag{23}$$

Here curvatures are needed only to their linearized $\mathcal{O}(h)$ order and derivatives are also flat space ones; all corrections to those quantities either lead to $\mathcal{O}(h^4)$ or are $\mathcal{O}(h^3)$ but harmless, $\sim \Box_0^{-1}$. The same is true of the unity part of the Δ_4^{-1} expansion: the quadratic terms are the local $\int d^4x\, R^2$, the cubics are $\sim \int d^4x[\mathcal{E}_4 \Box_0^{-1} R + \Box_1 R \Box_0^{-1} R]$ where $\Box_1$ is the $\mathcal{O}(h)$ part of $\Box$; they are single-pole. Now pass to the correction term which seems to have a $\Box_0^{-4}$. However, being linear, it only multiplies the quadratics $(\Box_0 R)\Box_0^{-2}\Box_0 R \sim R^2$, so it is $\Box_0^{-2}$ at worst. Before proceeding further, note two useful properties [1] of our cubic integrals: first, the position of $\Box_0^{-1}$ among the three factors is irrelevant; second, integration by parts rules are very useful, *e.g.*, (for any S) $\int d^4x\, S\,\partial_\mu R \partial^\mu R = -\frac{1}{2}\int d^4x\, S \Box_0 R^2$. Both are used implicitly below. The dangerous cubic terms in (23) reduce to the form

$$\int d^4x\, R_{,\mu}\,\Box_0^{-1}(R^{\mu\nu} - \tfrac{1}{3}\,\eta^{\mu\nu} R)\Box_0^{-1}\, R_{,\nu}\ . \tag{24}$$

The pure R^3 part, $\sim \int \eta^{\mu\nu} R_{,\mu}\Box_0^{-1} R \Box_0^{-1} R_{,\nu} = -\frac{1}{2}\int R^3/\Box_0$ is obviously safe. This leaves the first, $R^{\mu\nu}$-dependent one,

$$\int d^4x R_{,\mu}\Box_0^{-1} R^{\mu\nu}\Box_0^{-1} R_{,\nu}\ , \tag{25}$$

which is certainly $\sim \Box_0^{-2}$ as it stands. Note that there is no dimensional contradiction: the extra $\Box_0^{-1}$ is compensated for by the extra $\partial_\mu\partial_\nu$ in the numerator, but these are not mutable into a $\Box_0$ by parts integration as long as we write everything in terms of curvatures alone. This impasse disappears by relaxing the latter requirement and expressing the Ricci tensor in terms of its metric definition,

$$2R_{\mu\nu} = \Box_0 h_{\mu\nu} - (\partial^2_{\mu\alpha} h^\alpha_\nu\ + (\nu\mu)) + h^\alpha_{\alpha,\mu\nu}\ , \quad R = \Box_0 h^\alpha_\alpha - \partial^2_{\alpha\beta} h^{\alpha\beta}\ . \tag{26}$$

The $\Box_0 h_{\mu\nu}$ term is manifestly $\sim \Box_0^{-1}$; the remaining ones also provide an additional $\Box_0$, after integration by parts. The result of a simple calculation yields the equality

$$\int d^4x\, R_{,\mu}R_{,\nu}\Box_0^{-2}R^{\mu\nu} = \tfrac{1}{2}\int d^4x \left[R_{,\mu}R_{,\nu}\Box_0^{-1}h^{\mu\nu} - \tfrac{1}{4}\,R^2 h^\alpha_\alpha + \tfrac{1}{2}R^2\Box_0^{-1}R\right] \ . \tag{27}$$

whose right side, although its first term is irreducible to "curvatures/$\Box_0$" is of course just as gauge-invariant under $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, $\delta R = 0$ as the left, after using partial integration. These steps demonstrate that the leading, h^3, terms (23) have only a simple pole, which is all one can demand of them on dimensional or general anomaly grounds. Before proceeding to higher order, it should be emphasized that the $\Box_0^{-1}$ nonlocality is irreducible, and cannot be removed even by expressing all curvatures in terms of $h_{\mu\nu}$. More important, their failure to have $\int R^3\Box_0^{-1}$ form is not unavoidable, but rather symptomatic of their physical defects: that form has been achieved [1].

Beyond leading order there will clearly appear higher and higher poles in the $h_{\mu\nu}$-expansion of Δ_p^{-1}. Indeed, as explained previously, each successive additional vertex insertion into the loop diagram involves an extra propagator and so, generically an (acceptable) extra power of $\Box_0^{-1}$. For higher D, the Δ_{2n} will go as $\Box^n$, but again the leading, $(n+1)$-point function must go as $\int(d^{2n}p)/(p^2)^{n+1} \sim p^{-2}$, and it will, by similar considerations as for D=4, with

$$I_A = \int d^{2n}x\, \bar{\mathcal{E}}_{2n}\Delta_{2n}^{-1}\, \bar{\mathcal{E}}_{2n} \tag{28}$$

and $\Delta_{2n} \sim \Box^n + ..$, $\bar{\mathcal{E}}_{2n} \sim \mathcal{E}_{2n} + ...$ where the additional terms in Δ_n and $\bar{\mathcal{E}}_{2n}$ are of lower/higher derivative order respectively, and as in D=2,4 with $\delta\bar{\mathcal{E}}_{2n} = \Delta_{2n}\phi,\ \ \delta\Delta_{2n} = 0$.

The above "rehabilitation" of (22) in no way improves its problematic physical behavior. One illustration of its problems is supplied by the unphysical nature of the compensating field action that generates it when the field is integrated out:

$$I_A[\sigma; g_{\mu\nu}] = \int d^Dx[\tfrac{1}{2}\,\sigma\Delta_D\sigma + \sigma\bar{\mathcal{E}}_D] \ , \quad \delta I[\sigma+\phi;\ 2\phi g_{\mu\nu}]/\delta\phi = \bar{\mathcal{E}}_D \ . \tag{29}$$

This means that for D>2, the σ propagator becomes more and more ghostlike, with correspondingly worse long-distance behavior associated to the higher powers $\Box^n$ in Δ_D. This correlation is unavoidable: since σ must be dimensionless, its kinetic part has to be of D-derivative order. There are of course an infinite number of compensator actions, just as there are of purely gravitational ones. An example of the former whose variation yields $\mathcal{E}_4$ is [7, 1]

$$I'_A[\sigma; g_{\mu\nu}] = \int d^4x[8\sigma\mathcal{E}_4 + \sigma G^{\mu\nu}D_\mu D_\nu\sigma + \tfrac{1}{2}\Box\sigma(\partial_\mu\sigma)^2 + (\partial_\mu\sigma)^2(\partial_\nu\sigma)^2] \ , \tag{30}$$

each succeeding term correcting the residual variation of the previous ones; there is no kinetic σ term at all. Examples of ambiguities in the gravitational actions are furnished by polynomials in the dimensionless conformally invariant scalar building block $X \equiv \sqrt{-g}\,C^2\Delta_p^{-1}$, namely

$$I_{conf} = \sum_{m=0}^{\infty} a_n \int d^4x X^{m+1}\sqrt{-g}\,C^2 \ . \tag{31}$$

Each term begins at order $h^{2(m+2)}$, with corresponding poles $\sim \Box^{-2(m+1)}$.

What is clearly required at the purely gravitational action level is a way of finding (perhaps from the expression given in [1]) the covariantization (without distorting its behavior) of the lowest order $\int d^4x hhh < TTT >$ action dictated by the actual loop structure. The goal would be to generalize the D=2 action only by increasing the number of curvatures in the numerator, but keeping the denominator of second order, in terms of new second-derivative tensor operators $\tilde{\Delta}$ (not necessarily Weyl invariant) that would permit fully covariant actions of the form

$$\tilde{I}_A = \int d^4x \sqrt{-g}\, (RR)^{\mu\nu\alpha\beta} (\tilde{\Delta}_2^{-1} R)_{\mu\nu\alpha\beta} \; . \tag{32}$$

It may also be possible to obtain such actions by a descent procedure from conformal invariants in the regularized, D=4-2ϵ dimension.

4 Summary

Use of a novel class of tensorial conformal invariant operators has made possible compact closed form expressions for type B effective actions in any dimension; these retain all the physical "UV" characteristics of the underlying matter loop integrals from which they arise. By contrast the known type A actions are (beyond D=2) far from reflecting those origins. Nevertheless, it was possible to verify that, when properly reformulated, their leading terms have only single poles, as required by dimension and general anomaly considerations and therefore bound to be fulfilled by any action that yields the correct anomaly. Given the ongoing popularity of type A, in problems ranging from phonomenological gravitational actions to C-theorems and holography (for some recent work see *e.g.*, [9, 10] and references therein), improved versions, perhaps of the suggested form (32), of its effective action are eminently worth finding.

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