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Stability of Massive Cosmological Gravitons

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ABSTRACT

We analyze the physics of massive spin 2 fields in (A)dS backgrounds and exhibit that: The theory is stable only for masses $m^2 \geq 2\Lambda/3$, where the conserved energy associated with the background timelike Killing vector is positive, while the instability for $m^2 < 2\Lambda/3$ is traceable to the helicity 0 energy. The stable, unitary, partially massless theory at $m^2 = 2\Lambda/3$ describes 4 propagating degrees of freedom, corresponding to helicities $(\pm 2, \pm 1)$ but contains no 0 helicity excitation.

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1 Introduction

Massive higher spin fields in cosmological, AdS ($\Lambda < 0$) or dS ($\Lambda > 0$) backgrounds have recently been shown to exhibit a novel structure in the (m^2, Λ) plane [1], as compared to their flat space counterparts where only the $m^2 = 0$ theory is distinguished. The background space, with its added parameter Λ affects the lower helicity (in flat space language) modes in such a way that they disappear entirely along lines in the (m^2, Λ) (half-)plane. Underlying the appearance of these partially massless theories are new gauge invariances. The lower helicities also flip from unitary to nonunitary as the relevant lines are traversed. In particular, for massive spin 2 fields, it is known that the norm of the helicity zero mode changes sign [2, 1] across the dS line $m^2 = 2\Lambda/3$, along which a new local invariance appears [3]. The region $m^2 < 2\Lambda/3$ is therefore unitarily forbidden.

In this Letter we give a concrete proof of these results, *i.e.* that the $m^2 = 2\Lambda/3$ partially massless spin 2 theory describes 4 propagating degrees of freedom (PDoF) corresponding to helicities $(\pm 2, \pm 1)$ (but *not* 0). We then show that massive gravitons are only stable in the unitarily allowed region $m^2 \geq 2\Lambda/3$. Stability in (A)dS is defined just as for massless cosmological gravitons [4], in terms of positivity of the conserved energy associated with the timelike Killing vector within the physically accessible spacetime region, the intrinsic dS horizon.

In our Hamiltonian (3+1) approach, the behavior of the various helicity modes in the unitarily allowed and forbidden (m^2, Λ) regions and along the partially massless line is manifest. Since massive spin 2 is described by small oscillations of the cosmological Einstein theory about its vacuum, deformed by an explicit mass term that breaks the linearized coordinate invariance of the former, we utilize known aspects of the massless model [4]. The constraint structure and rich behavior in the (m^2, Λ) plane of the massive model are, however, very different. [Our stability analysis is carried out in a dS background, but applies to AdS as well.]

In outline, we begin in Section 2 by writing down the 3+1 Hamiltonian representation of the massive spin 2 theory in a dS background. Away from the strictly massless $m^2 = 0$ (linearized cosmological graviton) line, helicities $(\pm 2, \pm 1)$ are stable and unitary since they are immune to the helicity 0 (scalar) constraint. We derive their actions in Section 3. The renegade 0 helicity, responsible for the non-unitary, unstable region is analyzed in Section 4; we show both that a novel constraint banishes this excitation from the spectrum at $m^2 = 2\Lambda/3$ and that the helicity 0 action goes from stable to unphysical as this line is crossed. In Section 5, we map the stability regions of the models, and conclude with a brief discussion in Section 6.

2 The Action

We begin with the 3+1 form [4] of the cosmological Einstein action,

$$\begin{aligned} I_{E\Lambda} &= -\int d^4x \sqrt{-{}^{(4)}g}\,\Big[\, {}^{(4)}R - 2\Lambda \,\Big] \\ &= \int d^4x \,\Big[\, \pi^{ij}\, \dot{g}_{ij} + N\,\mathcal{R}^0 + N_i\,\mathcal{R}^i \,\Big]\,, \end{aligned} \tag{1}$$

$$\mathcal{R}^0 = \sqrt{g}\,\Big[\, R - 2\Lambda \,\Big] + \pi^{ij}\pi^{lm}\Big[\, \frac{1}{2}\, g_{ij}g_{lm} - g_{il}g_{jm} \,\Big]\Big/\sqrt{g}\,,$$

$$\mathcal{R}^i = 2D_j\pi^{ij}\,, \qquad N \equiv \Big(-g^{00}\Big)^{-1/2}\,, \quad N_i \equiv g_{0i}\,.$$

Throughout, latin indices are spatial as are all derivatives and index operations. Our signature is mostly plus, and the intrinsic spatial Ricci tensor $R_{ij} \sim +\partial_k\Gamma^k_{ij}$. We expand (1) about its dS vacuum, using the synchronous (if not fully covering) gauge

$$ds^2 = -dt^2 + f^2(t)\,dx^i\,dx^i\,, \tag{2}$$
$$f(t) \equiv e^{Mt}\,, \quad M \equiv \sqrt{\Lambda/3}\,.$$

In this frame, we will be almost able to remove all explicit time dependence due to f. Denoting the full metric by $g_{\mu\nu}$ and its above background value by $\overline{g}_{\mu\nu}$, the deviations are defined by

$$\begin{aligned} g_{ij} &= \overline{g}_{ij} + h_{ij} = f^2\,\delta_{ij} + h_{ij}\,, \\ \pi^{ij} &= \overline{\pi}^{ij} + p^{ij} = -2M\,f\delta^{ij} + p^{ij}\,, \\ N &= 1 + n\,, \quad g_{00} \equiv -1 + h_{00} = -1 + n/2 + \mathcal{O}(n^2)\,. \end{aligned} \tag{3}$$

Here $\overline{\pi}^{ij}$ is (essentially) the second fundamental form in our gauge; p^{ij} is of course the (independent) momentum conjugate to h_{ij}; with respect to the background, p^{ij} is a contravariant tensor density, while h_{ij} is a covariant tensor. The shift N_i needs no expansion since its background value vanishes.

Before expanding the action (1) to quadratic order in the deviations (p^{ij}, h_{ij}, N_i, n), we introduce the mass term. It maintains the background coordinate invariance but breaks the linearized diffeomorphism symmetry of the small oscillations,

$$\begin{aligned} I_m &= -\frac{m^2}{4}\int d^4x\sqrt{-{}^{(4)}\overline{g}}\;h_{\mu\nu}\;h_{\rho\sigma}\left[\,\overline{g}^{\mu\rho}\overline{g}^{\nu\sigma} - \overline{g}^{\mu\nu}\overline{g}^{\rho\sigma}\right] \\ &= -\frac{m^2}{4}\int d^4x\,f^{-1}\left[\,h_{ij}^2 - h_{ii}^2 - 2f^2N_i^2 - 4f^2nh_{ii}\,\right]. \end{aligned} \tag{4}$$

In the last line (and from now on) we indicate the time dependence f^{-1} explicitly and contract spatial indices with Kronecker deltas. The massive spin 2 action in a dS background is, therefore,

$$I = I^Q_{E\Lambda} + I_m\,, \tag{5}$$

where

$$I^Q_{E\Lambda} = p^{ij}\,\dot{h}_{ij} + n\,\mathcal{R}^0_L + \mathcal{R}^0_Q + N_i\mathcal{R}^i_L\,. \tag{6}$$

We denote an expression's linear and quadratic parts in the fluctuations (p^{ij}, h_{ij}, N_i, n) by L and Q, respectively. We also drop all integration signs and integrate freely by parts.

In absence of the mass term (but with $\Lambda \neq 0$), the four familiar constraints $\mathcal{R}^0_L = 0 = \mathcal{R}^i_L$, imposed by the (lapse and shift) Lagrange multipliers n and N_i, leave only the top, helicity ± 2, linearized graviton excitations, consonant with the four gauge invariances of the system [4]. Addition of the mass term alters this counting: the n^2 term is still absent, but (for $m \neq 0$) an N_i^2 term is present. Therefore only the n-constraint remains, generically reducing the 6 canonical pairs (p^{ij}, h_{ij}) to the five physical helicities $(\pm 2, \pm 1, 0)$ of massive spin 2. However, as we shall demonstrate, along the line $m^2 = 2\Lambda/3 \equiv 2M^2$ a further constraint appears and excises the scalar helicity 0 mode.

Assuming henceforth that $m^2 \neq 0$ (since the stability of linearized cosmological gravitons is understood [4]), integrating out the shift function N_i yields the action

$$I = p^{ij}\,\dot h_{ij} + n\,(\mathcal{R}^0_L + f^{-1}\,m^2 h_{ii}) + \mathcal{R}^0_Q - \frac{1}{2}\,f^{-1}\,\Big(\frac{1}{m}\,\mathcal{R}^i_L\Big)^2 - \frac{m^2}{4}\,f^{-1}\,\Big(h^2_{ij} - h^2_{ii}\,\Big)\,. \tag{7}$$

It is very convenient to minimize the explicit time dependence (due to $f(t)$) of the action, by making the simple field redefinition

$$h_{ij} \longrightarrow f^{1/2} h_{ij}\,, \qquad p^{ij} \longrightarrow f^{-1/2} p^{ij}\,, \qquad n \longrightarrow f^{-3/2} n\,. \tag{8}$$

The symplectic terms then become

$$p^{ij}\,\dot h_{ij} \longrightarrow f^{-1/2} p^{ij}\,\frac{d}{dt}\Big(f^{1/2} h^{ij}\Big) = p^{ij}\,\dot h_{ij} + \frac{M}{2}\,p^{ij}\,h_{ij}\,. \tag{9}$$

It is easy to verify that the only remaining explicit time dependence of the action is through the Laplacian

$$\nabla^2 = f^{-2}\,\partial_i^2\,. \tag{10}$$

Our analysis makes essential use of the familiar flat 3-space orthogonal decomposition of symmetric 2-tensors,

$$\begin{aligned} T_{ij} &= T^{Tt}_{ij} + 2\,\partial_{(i} T^t_{j)} + \frac{1}{2}\,(\delta_{ij} - \widehat{\partial}_{ij})\,T^t + \widehat{\partial}_{ij}\,T^l\,, \\ T^{Tt}_{ii} &= 0 = \partial_i T^{Tt}_{ij} = \partial_i T^t_i\,, \quad \widehat{\partial}_{ij} \equiv \partial_i \partial_j / \partial_k^2\,, \end{aligned} \tag{11}$$

which, of course, commutes with $\partial/\partial t$. The constraint $\mathcal{R}^0_L$, being a scalar linear in the fluctuations, can only depend on the helicity 0, (T^t, T^l) parts of (h_{ij}, p^{ij}).

Furthermore, since the action is of quadratic order, there is no interaction between distinct helicities, schematically

$$I = I_{\pm 2}(T^{Tt}_{ij}) + I_{\pm 1}(T^t_i) + I_0(T^t, T^l)\ , \qquad (12)$$

the hallmark of the orthogonal decomposition (11). We now derive and examine each helicity term in turn.

3 Safe Helicities $(\pm 2, \pm 1)$

Helicities $(\pm 2, \pm 1)$ are the easiest part of the calculation since they are unconstrained (for $m^2 \neq 0$). Let us begin with the helicity ± 2 part, where there is never a constraint. We denote $(p^{Tt}_{ij}, h^{Tt}_{ij})$ by $(p_{\pm 2}, q_{\pm 2})$ respectively because, thanks to the transverse-traceless property, indices can only contract in an obvious way[1]. By explicitly writing out the helicity ± 2 dependence of the action (7) [note that the linearized Einstein tensor gives ${G_L^{Tt}}_{ij} = -\frac{1}{2}\,\nabla^2 q^{Tt}_{ij}$] we find

$$I_{\pm 2} = p_{\pm 2}\,\dot q_{\pm 2} - \Big[\,\Big(p_{\pm 2} + \frac{5M}{4}\,q_{\pm 2}\Big)^2 \,+\, \frac{1}{4}\,q_{\pm 2}\Big(-\nabla^2 + m^2 - \frac{9M^2}{4}\Big)\,q_{\pm 2}\,\Big]\,. \qquad (13)$$

A field redefinition

$$p_{\pm 2} \longrightarrow \Big(p_{\pm 2} - \frac{5M}{4}\,q_{\pm 2}\Big)\Big/\sqrt{2}\,, \qquad q_{\pm 2} \longrightarrow \sqrt{2}\,q_{\pm 2}\,, \qquad (14)$$

yields the diagonal action

$$I_{\pm 2} = p_{\pm 2}\,\dot q_{\pm 2} - \Big[\,\frac{1}{2}\,p^2_{\pm 2} \,+\, \frac{1}{2}\,q_{\pm 2}\Big(-\nabla^2 + m^2 - \frac{9M^2}{4}\Big)\,q_{\pm 2}\,\Big]\,. \qquad (15)$$

We will explain and meet again the effective mass $(m^2 - 9M^2/4)$ later, and at present just reassure the reader that this action ensures stable, unitary propagation for *all* m^2. Likewise, the string of field redefinitions (8) and (14) is valid for any m^2. Therefore the helicity ± 2 modes propagate according to (15) for all models in the (m^2, Λ) half-plane.

Next consider the transverse vector action, $I_{\pm 1}$. The decompositions (11,12) implies that the result takes the form $\partial_i T^t_j\,\partial_{(i}T'_{j)}{}^t = -\frac{1}{2}\,T^t_i\,\partial^2_j\,T'^t_i$, which begs for the field redefinition

$$h^t_i \longrightarrow q_{\pm 1}\Big/\sqrt{-\partial^2_j}\,, \qquad p^t_i \longrightarrow p_{\pm 1}\Big/\sqrt{-\partial^2_j}\,, \qquad (16)$$

[1] For example $p_{\pm 2} q_{\pm 2} \equiv \sum_{\varepsilon=\pm 2} p_\varepsilon q_\varepsilon = p^{Tt}_{ij} q^{Tt}_{ij}$.

(again we will suppress the sums over helicities ± 1). Returning to the action (7) and extracting its helicity ± 1 dependence, after a somewhat lengthy computation[2] we find

$$I_{\pm 1} = 2\,p_{\pm 1}\,\dot q_{\pm 1} - \Big[2\,p_{\pm 1}\Big(\frac{-\nabla^2+m^2}{m^2}\Big)p_{\pm 1} - M\,p_{\pm 1}\Big(\frac{-8\,\nabla^2+5m^2}{m^2}\Big)q_{\pm 1} + \frac{1}{2}\,q_{\pm 1}\Big(m^2+4\,M^2\,\frac{-4\,\nabla^2+m^2}{m^2}\Big)q_{\pm 1}\Big]\,. \quad (17)$$

The field redefinition

$$q_{\pm 1} \longrightarrow \frac{3M\,q_{\pm 1}+2\,p_{\pm 1}}{2m}\,, \qquad p_{\pm 1} \longrightarrow \frac{4M\,p_{\pm 1}-(m^2-6M^2)\,q_{\pm 1}}{2m}\,, \quad (18)$$

yields the desired –stable and unitary– action

$$I_{\pm 1} = p_{\pm 1}\,\dot q_{\pm 1} - \Big[\frac{1}{2}\,p_{\pm 1}^2 + \frac{1}{2}\,q_{\pm 1}\Big(-\nabla^2+m^2-\frac{9M^2}{4}\Big)q_{\pm 1}\Big]\,. \quad (19)$$

The helicity ± 1 action is identical to its ± 2 counterpart (15) with one important difference: The field redefinition (18) is singular at $m^2 = 0$ (and complex for $m^2 < 0$). This reflects the gauge invariance at $m^2 = 0$ (and instability of the theory for $m^2 < 0$). The vector constraint, imposed by the shift functions N_i, is reincarnated in the $m^2 = 0$ theory and removes the above helicity ± 1 states.

4 Dangerous Helicity 0

For $m^2 \neq 0$, helicities $(\pm 2, \pm 1)$ are unaffected by constraints. The physical helicity 0 state leads a more interesting life as it can be (i) stable and unitary when $m^2 > 2\Lambda/3 \equiv 2M^2$, (ii) absent when $m^2 = 2M^2$ or (iii) unstable and nonunitary for $m^2 < 2M^2$.

Before writing down an action for the helicity 0 excitations (analogously to the helicity $(\pm 2, \pm 1)$ ones in (15) and (19)), we analyze the constraint imposed by integrating out the lapse Lagrange multiplier n. Using $h_{ii} = h^t + h^l$ and writing out the linearization of $\mathcal{R}^0_L$ explicitly[3] we obtain

$$(-\nabla^2+\nu^2)\,h^t + \nu^2\,h^l - 2M\,(p^t+p^l) = 0\,,$$

[2]Since $\sqrt{g}\,R$ is the usual Einstein action, its quadratic part is $-\frac{1}{2}\,h_{ij}\,G_L^{ij}$, and hence does not contribute in this sector, by the linearized Bianchi identity $\partial_i G_L^{ij} = 0$.

[3]We use the linearizations

$$\Big(\sqrt{g}\,R\Big)^L = -\nabla^2 h^t\,, \quad \Big(\sqrt{g}\Big)^L = (h^t+h^l)/2\,,$$

$$\nu^2 \equiv (m^2 - 2M^2)\ . \tag{20}$$

The sign of the parameter ν^2 controls the stability, unitarity and PDoF count of the model; negative values will yield non-unitary, unstable helicity 0 excitations.

Let us now examine the effect of the constraint (20) on the symplectic terms in the helicity 0 action

$$I_0 = p^l\, \dot{h}^l + \frac{1}{2}\, p^t\, \dot{h}^t - H_0(p^l, h^l; p^t, h^t)\,. \tag{21}$$

We choose (with no loss of generality in curved backgrounds) to eliminate the variable p^t via (20)

$$p^t = -p^l + \frac{1}{2M}\Big((-\nabla^2 + \nu^2)\, h^t + \nu^2\, h^l\Big)\ , \tag{22}$$

which leads to

$$I_0 = \Big(p^l - \frac{\nu^2}{4M}\, h^t\Big)\, \Big(\dot{h}^l - \frac{1}{2}\, \dot{h}^t\Big) - \frac{1}{4}\, h^t\, \nabla^2\, h^t - H_0(p^l, h^l; h^t)\ . \tag{23}$$

Diagonalizing the kinetic terms by the field redefinition

$$p^l \longrightarrow p_0 + \frac{\nu^2}{4M}\, h^t\ , \qquad h^l \longrightarrow q_0 + \frac{1}{2}\, h^t\ , \tag{24}$$

we are finally ready to display the full helicity 0 action

$$\begin{aligned} I_0 \ &= \ p_0\, \dot{q}_0 - \Bigg[-\frac{3\,\nu^2\, m^2}{32M^2}\, (h^t)^2 \ - \ \frac{1}{2M}\, h^t \Big(\nabla^2 \Big[p_0 - Mq_0\Big] + \frac{\nu^2\, m^2}{4M}\, q_0 \Big) \\ &\qquad - \frac{2}{m^2}\Big[p_0 - Mq_0\Big]\, \nabla^2\, \Big[p_0 - Mq_0\Big] + \ \frac{3}{2}\Big[p_0 - Mq_0\Big]\Big(p_0 - \frac{m^2}{3M}\, q_0\Big) \Bigg]\ . \end{aligned} \tag{25}$$

The denominators M in this expression do not represent a genuine singularity, but arise from choosing to solve the constraint (20) in terms of p^t. In contrast, the denominators m^2 are due to integrating out the shift N_i and are a reminder (as we have seen already) of the strictly massless $m^2 = 0$ gauge theory. The key point is to notice that the coefficient of $(h^t)^2$ vanishes on the critical line $\nu^2 = 0$ (as well as at $m^2 = 0$, concordant with the previous remark). At criticality, the field h^t appears only linearly and is a Lagrange multiplier for a new constraint, whereas for $\nu^2 \neq 0$, we can integrate out h^t by its algebraic field equation and there are no further constraints. Let us deal with each of these cases in turn.

$$\Big(\Big[\,\frac{1}{2}\,{\pi^i}_i{}^2 - \pi_{ij}\pi^{ij}\Big]\Big/\sqrt{g}\Big)^L = -2M(p^t + p^l) + M^2(h^t + h^l)\ .$$

4.1 $\nu^2 = 0$: The Partially Massless Theory

Consider the models with mass tuned to the cosmological constant via $m^2 = 2\Lambda/3$. As is clear from (25), the Lagrange multiplier h^t imposes the new constraint

$$p_0 - Mq_0 = 0\,. \tag{26}$$

Eliminating q_0 (say) and since $(p_0\dot{p}_0)$ is a total derivative, the 0 helicity action (25) vanishes exactly,

$$I_0 = 0\,. \tag{27}$$

It is known [3] that the critical theory possesses a local scalar gauge invariance,

$$\delta h_{\mu\nu} = (D_{(\mu}D_{\nu)} + \frac{\Lambda}{3}\,\bar{g}_{\mu\nu})\,\xi(x)\,. \tag{28}$$

Thus, our result establishes that its effect is to remove the lowest helicity excitation. Therefore, the total action is

$$I_{\nu^2=0} = \sum_{\varepsilon=(\pm 2,\pm 1)} \left\{ p_\varepsilon\,\dot{q}_\varepsilon - \Big[\,\frac{1}{2}\,p_\varepsilon^2 \,+\, \frac{1}{2}\,q_\varepsilon\Big(-\nabla^2 - \frac{M^2}{4}\Big)\,q_\varepsilon\,\Big] \right\}\,. \tag{29}$$

[The effective mass $-M^2/4$ is the same one as in (15) and (19), evaluated at $\nu^2 = 0$.] These remaining helicity $(\pm 2, \pm 1)$ excitations are both unitary [2, 1] and, as we will show, stable.

4.2 $\nu^2 \neq 0$: The Massive Theory

We may now eliminate h^t by its algebraic field equation

$$h^t \;=\; -\frac{8M}{3\,\nu^2 m^2}\,\nabla^2\,\Big[p_0 - Mq_0\Big] - \frac{2}{3}\,q_0\;, \tag{30}$$

which yields the action

$$\begin{aligned} I_0 \;&=\; p_0\dot{q}_0 - \Big[\,\frac{1}{24M^2}\,\nu^2 m^2\,q_0^2 + \frac{1}{6M}\Big[p_0 - Mq_0\Big]\Big(2\nabla^2 - 3m^2 + 9M^2\Big)\,q_0 \\ &\quad + \frac{1}{6\nu^2 m^2}\Big[p_0 - Mq_0\Big]\Big(4\nabla^4 - 12\nu^2\nabla^2 + 9\nu^2 m^2\Big)\Big[p_0 - Mq_0\Big]\,\Big]\,. \end{aligned} \tag{31}$$

A penultimate field redefinition/canonical transformation

$$p_0 \longrightarrow p_0 + M\Big[q_0 + \frac{2M}{\nu^2 m^2}\Big(-2\nabla^2 + 3\nu^2 - 3M^2\Big)\,p_0\Big]\ , \tag{32}$$

$$q_0 \longrightarrow q_0 + \frac{2M}{\nu^2 m^2}\Big(-2\nabla^2 + 3\nu^2 - 3M^2\Big)\,p_0\ , \tag{33}$$

diagonalizes the action (31),

$$I_0 = p_0\,\dot q_0 - \Bigg[\ \frac{1}{2}\left[\frac{\nu^2\ m^2}{12M^2}\right]\ q_0^2\ +\ \frac{1}{2}\left[\frac{12M^2}{\nu^2\ m^2}\right]\ p_0\Big(-\nabla^2 + m^2 - \frac{9M^2}{4}\Big)\,p_0\ \Bigg]\ . \tag{34}$$

Before we present the final, complete, action, some important comments on its penultimate form (34) are needed:

- The sign of the parameter ν^2 controls the positivity of the Hamiltonian (and consequently the energy). Therefore we find that the (m^2, Λ) plane is divided into a stable region $m^2 \geq 2\Lambda/3$ and an unstable one $m^2 < 2\Lambda/3$.

- A final field redefinition,

  $$p_0 \longrightarrow -\frac{\nu m}{2\sqrt{3}\ M}\,q_0\ , \qquad q_0 \longrightarrow \frac{2\sqrt{3}\ M}{\nu m}\,p_0 \tag{35}$$

  brings the helicity 0 action into the same form as its helicity $(\pm 2, \pm 1)$ counterparts (15) and (19), but this is only legal in the stable massive region $m^2 > 2\Lambda/3$.

- The $\nu^2 = 0$ singularity signals the onset of a gauge invariance where the constraint analysis of Section 4.1 must be applied.

- The apparent singularity at $M = 0$ is spurious and reflects our (arbitrary) choice of solution to the constraint (20).

The final action for massive spin 2 in the region $m^2 > 2\Lambda/3$ is

$$I_{\nu^2>0} = \sum_{\varepsilon=(\pm 2,\pm 1,0)} \Bigg\{p_\varepsilon\,\dot q_\varepsilon - \Big[\ \frac{1}{2}\,p_\varepsilon^2\ +\ \frac{1}{2}\,q_\varepsilon\Big(-\nabla^2 + m^2 - \frac{9M^2}{4}\Big)\,q_\varepsilon\ \Big]\Bigg\}\ , \tag{36}$$

and describes stable, unitary, helicity $(\pm 2, \pm 1, 0)$ excitations.

5 Stability: Positivity of the Energy

We are now ready to demonstrate the stability of the model in the allowed region $m^2 \geq 2\Lambda/3$. The dS background possesses a Killing vector

$$\overline{\xi}^{\mu} = (-1, Mx^i)\,, \qquad \overline{\xi}^2 = -1 + \Big(fMx^i\Big)^2\,, \tag{37}$$

timelike within the intrinsic horizon $(fMx^i)^2 = 1$. Therefore, in this region of spacetime, it is possible to define a conserved energy whose positivity guarantees the stability of the model.

Let us consider helicity ε (omitting 0 at criticality) described by the conjugate pair $(p_\varepsilon, q_\varepsilon)$, whose time evolution is generated by the Hamiltonian

$$H_\varepsilon = \frac{1}{2}\, p_\varepsilon^2 \;+\; \frac{1}{2}\, q_\varepsilon \Big(-\nabla^2 + m^2 - \frac{9M^2}{4} \Big)\, q_\varepsilon\;. \tag{38}$$

Hence the field equations are[4]

$$\dot{q}_\varepsilon = p_\varepsilon\;, \qquad \dot{p}_\varepsilon = \Big(\nabla^2 - m^2 + \frac{9M^2}{4}\Big)\, q_\varepsilon\;. \tag{39}$$

However, the Hamiltonian (38) is not conserved, thanks to the explicit time dependence $f^{-2}(t)$ in ∇^2, which was to be expected since it generates time evolution $\frac{d}{dt}$ rather than along the Killing direction $\overline{\xi}^{\mu}\partial_\mu$. Instead, the conserved energy is defined in terms of the stress energy tensor

$$E_\varepsilon = T^{0}_{\varepsilon\;\mu}\, \overline{\xi}^{\mu} = -T_\varepsilon{}^0{}_0 + M\, x^i\, T_\varepsilon{}^0{}_i\,. \tag{40}$$

The momentum density $T_\varepsilon{}^0{}_i$ will be defined below and $-T_\varepsilon{}^0{}_0 = H_\varepsilon$ is the Hamiltonian in (38). For gravity, the momentum density T^0_i is the quadratic part of the coefficient of N_i, and a similar result holds here. Keeping track of our field redefinitions, we find (modulo irrelevant superpotentials)

$$T_\varepsilon{}^0{}_i = -p_\varepsilon\, \partial_i\, q_\varepsilon + \frac{1}{2}\, \partial_i\, \Big(p_\varepsilon\, q_\varepsilon\Big)\,. \tag{41}$$

It is not difficult (using (39) and spatial integrations by parts) to verify that the energy function

$$E_\varepsilon = H_\varepsilon - M\, x^i\, p_\varepsilon \partial_i q_\varepsilon - \frac{3}{2}\, M\, p_\varepsilon q_\varepsilon\,, \tag{42}$$

[4]The resulting second order field equation $-\ddot{q}_\varepsilon + \Big(\nabla^2 - m^2 + \frac{9M^2}{4}\Big)\, q_\varepsilon = 0$ agrees precisely with the covariant one, $(D^2 - m^2 - 2\Lambda/3)\phi_{\mu\nu} = 0$ (together with the usual onshell conditions $D.\phi_\nu = 0 = \phi_\rho{}^\rho$) when written out explicitly in this frame, remembering the field redefinition (8).

is indeed conserved, $\dot{E} = 0$.

Finally we come to positivity. Here we need only a simple extension of the method in [4]. Rewriting E as

$$E = \frac{1}{2}\left(\widehat{x}^i\,\widetilde{p}_\varepsilon\right)^2 + \frac{1}{2}\left(f^{-1}\partial_i\,q_\varepsilon\right)^2 - fM|x|\left(\widehat{x}^i\,\widetilde{p}_\varepsilon\right)\left(f^{-1}\,\partial_i\,q_\varepsilon\right) + \frac{1}{2}\,m^2\,q_\varepsilon^2\,,$$

$$\widetilde{p}_\varepsilon \equiv p_\varepsilon - \frac{3M}{2}\,q_\varepsilon\,, \qquad x^i \equiv |x|\,\widehat{x}^i\,, \tag{43}$$

the first three terms are positive by the triangle equality within the intrinsic dS horizon

$$f\,M\,|x| < 1\,, \tag{44}$$

and the fourth, mass term is manifestly positive[5] This concludes our stability proof.

The instability of the region $m^2 < 2\Lambda/3$ is also manifest: Consider helicity $\varepsilon = 0$. Recall that once $\nu^2 < 0$, we cannot make the rescalings with factors ν and ν^{-1} in the final field redefinition (35). This does not prevent us from constructing a conserved energy with a "triangle" form (43), but the *caveat* is that p_0^2 carries a factor ν^2 and likewise q_0^2 a factor ν^{-2}. Therefore the energy is negative and the theory is unstable in this region. Clearly, helicity 0 is the sole felon responsible for this behavior.

6 Discussion

Spin 2 excitations in (A)dS backgrounds have the following features in the (m^2, Λ) half-plane: (i) For generic m^2, there are 5 propagating helicities. (ii) The $m^2 = 0$ strictly massless theory retains only helicity ± 2 excitations thanks to the gauge invariance of small oscillations of Einstein gravity about its (A)dS vacuum. These are both stable and unitary. (iii) Between the vertical line $m^2 = 0$ and the dS line $m^2 = 2\Lambda/3$, all five helicities are present, but the theory is both unstable and non-unitary in the 0 helicity sector. (iv) At the $m^2 = 2\Lambda/3$ line, a scalar gauge invariance allows both ± 2, and ± 1, but removes the dangerous helicity 0

[5] The Killing energy of a massive scalar in dS also takes the form (43) and is therefore stable for $m^2 \geq 0$. [In this non-gauge example, there is no analog to the spin 2 instabilities at negative values of m^2 or ν^2 whose vanishing is associated with gauge invariances.] Scalars in AdS actually enjoy a somewhat wider stability range, extending to negative values of m^2 [5] due to a shift in the spectrum of the AdS 3-Laplacian. This broadening is unlikely for spin 2, since its stability is controlled entirely by the above gauge coefficients.

excitation. This partially massless model is unitary and stable. Furthermore, it is the unique spin 2 theory whose equation of motion implies propagation along the null cone of the conformally flat dS spacetime [3]. (v) As $(m^2 - 2\Lambda/3)$ turns positive beyond this line, all five excitations behave and propagate normally. This region includes all of AdS and Minkowski space.

The splitting of the (m^2, Λ) half-plane into forbidden and allowed regions separated by (partially) massless gauge lines occurs for all spins $s > 1$ [1] and it would be an amusing exercise to carry out a Hamiltonian analysis for spin 3/2, to exhibit the origin of the critical (AdS) line there; the required formalism already exists [6].

Another interesting question for higher spin theories is whether their propagation is causal [7]. Unitarity, classical stability and causality are all directly related. As shown in [8] (in a slightly different context), the failure of canonical commutators to support unitary representations also implies acausal propagation: The spin 2 theory is acausal[6] in the unstable, unitarily forbidden region.

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