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Conformal Invariance of Partially Massless Higher Spins

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Abstract

We show that there exist conformally invariant theories for all spins in $d = 4$ de Sitter space, namely the partially massless models with higher derivative gauge invariance under a scalar gauge parameter. This extends the catalog from the two known gauge models – Maxwell and partially massless spin 2 – to all spins.

1 Introduction

In this Letter we present new conformally invariant gauge theories in dimension four constant curvature spaces generalizing the well known conformally improved scalar and Maxwell fields. Conformal flatness of these spaces then also implies lightcone propagation [1]. An obvious mechanism to achieve lightlike excitations is gauge invariance, and indeed we recently presented new partially massless gauge invariant higher spin theories that propagate on the lightcone in constant curvature backgrounds. While gauge invariance alone is certainly not sufficient to ensure conformal invariance, it turns out that in dimension four, there exists a series of gauge invariant higher spin theories whose first two elements are the improved scalar and Maxwell fields. These theories enjoy higher derivative (s derivatives at spin s) gauge invariances, with a scalar gauge parameter, just like their Maxwellian cousin. The "maximal depth" partially massless theories describe $2s$ lightlike and unitary (for $\Lambda > 0$ – de Sitter space) physical degrees of freedom. Unlike previous attempts [2] to generalize conformal invariance to Maxwell-like theories involving higher derivative actions, we maintain the physical requirement of actions quadratic in derivatives and instead increase the number of derivatives appearing in gauge variations.

The results are organized as follows: Section 2 contains a brief review of massive higher spins in constant curvature backgrounds while Section 3 deals with action principles for these fields. Conformal symmetry in de Sitter backgrounds is discussed in Section 4 and conformally improved scalars are revisited in Section 5. The main result is contained in Section 6 which derives the conformal invariance of maximal depth partially massless fields. Lightlike propagation is displayed in Section 7. Our conclusions may be found in the final Section.

2 Higher Spins in Constant Curvature Spaces

Constant curvature

$$R_{\mu\nu}{}^{\rho\sigma} = -\frac{2\Lambda}{n}\ \delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} \tag{1}$$

spaces in $d \equiv n+1$ dimensions are a consistent background for higher spin propagation. The cosmological constant Λ is positive for de Sitter space in our conventions. We will often work in the steady state patch

$$ds^2 = -dt^2 + e^{2Mt} d\vec{x}^{\,2} \,, \tag{2}$$

where

$$M^2 \equiv \frac{\Lambda}{n} \,. \tag{3}$$

The field equation for physical fields is [3]

$$\Big(D_\mu D^\mu + \Big[(s-1)^2 + (s-2)(n-3) - 3 \Big] M^2 - m^2 \Big) \, \varphi^{\rm phys}_{\mu_1 \ldots \mu_s} = 0 \,. \tag{4}$$

Physical fields are classified by the mass parameter m. In steady state coordinates they obey the following criteria for all $m^2 > 0$

$$\varphi_{0 \mu_2 \ldots \mu_s} = 0 = \varphi^\rho{}_{\rho \mu_2 \ldots \mu_s} \,, \tag{5}$$

as well as additional conditions

$$\partial^{\mu_1} \ldots \partial^{\mu_t} \varphi_{\mu_1 \ldots \mu_s} = 0 \,, \tag{6}$$

when

$$m^2 = (t-1)(2s - t + n - 3) M^2 \,, \qquad t = 1, \ldots, s \,. \tag{7}$$

The integer t is called the "depth" of a partially massless field subject to a mass tuning (7). Depth $t = 1$ corresponds to the on-shell condition for a strictly massless spin s field whose physical fields are spatially traceless-transverse symmetric tensors. For all values of the depth $1 \leq t \leq s$, the theory enjoys a higher derivative gauge invariance

$$\delta \phi_{\mu_1 \ldots \mu_s} = D_{(\mu_1} \cdots D_{\mu_t} \xi_{\mu_{t+1} \ldots \mu_s)} + \cdots \,. \tag{8}$$

Non-tuned values of the mass

$$m^2 < (s-1)(s + n - 3) M^2 \,, \tag{9}$$

correspond to massive fields described by spatially traceless tensors. (Unitarity is violated for mass values not obeying (7) or (9) [4].) Strictly massless and massive fields survive in the Minkowski limit $M = 0$. Peculiar to de Sitter space are physical fields for values of $t \neq 1$ [4]. These are partially massless fields of depth t. Like their strictly massless counterparts they propagate at speed of light. Their physical degrees of freedom are spanned by intermediate helicity counts as demanded by (6). These theories are also unitary in de Sitter ($\Lambda > 0$) backgrounds and obey an energy positivity condition within the horizon [5].

3 Actions

Defining

$$\varphi^{\mathrm{phys}}_{i_1\dots i_s} \equiv \sqrt{-g}^{\;\frac{s}{n}-\frac{1}{2}} q_\varepsilon \,, \tag{10}$$

where ε stands for a helicity labeling[1] of the spatial indices $i_1 \dots i_s$ subject to (5) and possibly (6), the field equation (4) becomes

$$\Big(-\frac{d^2}{dt^2} + \Delta + \Big[\frac{M(2s+n-4)}{2}\Big]^2 - m^2 \Big) q_\varepsilon = 0 \,. \tag{11}$$

Here the spatial Laplacian $\Delta \equiv e^{2Mt}\vec{\partial}^2$ carries the only explicit time dependence and we will raise and lower spatial indices with impunity. An action principle follows immediately

$$S = \int dt \Big(\sum_\varepsilon p_\varepsilon \dot{q}_\varepsilon - H \Big) \,, \tag{12}$$

where the Hamiltonian

$$H \equiv \sum_\varepsilon \int d^n x \, \frac{1}{2} \Big[p_\varepsilon^2 + e^{-2Mt} [\vec{\partial} q_\varepsilon]^2 + \mu^2 q_\varepsilon^2 \Big] \,. \tag{13}$$

Here the "effective mass" is

$$\mu^2 \equiv m^2 - \Big[\frac{M(2s+n-4)}{2}\Big]^2 \,. \tag{14}$$

The action (12) may also be obtained by a detailed constraint analysis of covariant, higher spin actions [5].

4 Conformal Symmetry

De Sitter spacetime, being conformally flat, enjoys not only an $so(d,1)$ isometry algebra but also an $so(d,2)$ conformal algebra. Perhaps the simplest construction of explicit conformal Killing vectors is to express the metric in a manifestly conformally flat frame

$$ds^2 = \frac{-d\tau^2 + d\vec{x}^2}{M^2\tau^2} \,, \tag{15}$$

[1] See [5] for a details of this Hamiltonian analysis.

where $\tau = -M^{-1}\exp\Big(Mt\Big)$ is the conformal time coordinate. One then writes down the flat conformal generators in the coordinates $(\tau, \vec{x})$,

$$
\begin{aligned}
iP_i = \partial_i\,, \qquad iL = \frac{d}{d\tau}\,, \\
iD = \tau\frac{d}{d\tau} + \vec{x}\cdot\vec{\partial}\,, \qquad iM_{ij} \;=\; x_i\partial_j - x_j\partial_i\,, \qquad iN_i = x_i\frac{d}{d\tau} + \tau\partial_i\,, \\
iK_i = 2ix_iD + \Big[-\tau^2 + \vec{x}^2\Big]\,\partial_i\,, \quad iJ = -2i\tau D + \Big[-\tau^2 + \vec{x}^2\Big]\,\frac{d}{d\tau}\,.
\end{aligned}
\tag{16}
$$

The generators (P_i, D, M_{ij}, K_i) obey the $so(d,1)$ de Sitter isometry algebra while the remaining generators enlarge this to the conformal $so(d,2)$ algebra with corresponding conformal Killing vectors

$$
[iL, ds^2] = -\frac{2}{\tau}\,ds^2\,, \quad [iN_i, ds^2] = -\frac{2x_i}{\tau}\,ds^2\,, \quad [iJ, ds^2] = -\frac{2(-\tau^2+\vec{x}^2)}{\tau}\,ds^2\,. \tag{17}
$$

In general we will denote the function multiplying $2ds^2$ on the right hand side (equaling the divergence of the corresponding conformal Killing vector divided by the dimensionality of spacetime) as α_X:

$$
[iX, ds^2] = 2\alpha_X\;ds^2\,. \tag{18}
$$

It is important to note that the conformal $so(d,2)$ algebra is obtained by requiring closure of the generator L and the $so(d-1,1)$ isometry algebra under commutation. This has the pleasant consequence that L and de Sitter invariance are sufficient for a theory to be conformal. In the steady state coordinates (2) we have

$$
iL = e^{Mt}\frac{d}{dt}\,. \tag{19}
$$

5 Scalars

The improved scalar action

$$
S = -\frac{1}{2}\int dx\sqrt{-g}\,\left(\partial_\mu\varphi g^{\mu\nu}\partial_\nu\varphi + \frac{1}{6}R\,\varphi^2\right)\,, \tag{20}
$$

is invariant under conformal transformations

$$\delta\varphi = X\varphi + \left(\frac{d}{2} - 1\right)\alpha_X\varphi\,, \tag{21}$$

where $X \equiv \xi^\rho\partial_\rho$, $D_{(\mu}\xi_{\nu)} = \alpha_X\ g_{\mu\nu} = \frac{1}{d}D.\xi\ g_{\mu\nu}$ and $\frac{d}{2} - 1$ is the conformal weight of the field φ.

It is useful for our purposes to spell out this invariance explicitly in a de Sitter background. In the steady state coordinates, the action reads

$$S = -\frac{1}{2}\int dtd^nx\ e^{nMt}\left(-\dot{\varphi}^2 + e^{-2Mt}[\vec{\partial}\varphi]^2 + \frac{M^2}{4}\ (n^2-1)\varphi^2\right)\,. \tag{22}$$

As discussed in the previous Section, it suffices to consider the generator $iL = e^{Mt}\frac{d}{dt}$. By either computing the right hand side of (21) or examining the gradient terms in the action, which must be separately conformally invariant, we find

$$\delta_L\varphi = e^{(3-n)Mt/2}\frac{d}{dt}\Big(e^{(n-1)Mt/2}\varphi\Big)\,. \tag{23}$$

It is then easy to verify that the δ_L variation of the Lagrangian in (22) is a total time derivative.

Let us now perform this computation yet again in a first order formulation. Making the field redefinition (10) and a Legendre transformation, the action reads

$$S = \int dtdx^n\left(p\dot{q} - \frac{1}{2}\left[p^2 + e^{-2Mt}[\vec{\partial}q]^2 - \frac{1}{4}M^2q^2\right]\right)\,. \tag{24}$$

It enjoys the conformal invariance

$$\delta_Lq = e^{3Mt/2}\frac{d}{dt}\Big(e^{-Mt/2}q\Big)\,,\qquad \delta_Lp = \frac{d}{dt}\Big(e^{3Mt/2}\frac{d}{dt}\Big(e^{-Mt/2}q\Big)\Big)\,. \tag{25}$$

Although the above calculation is a triviality, the conformally invariant action (24) plays an archetypal rôle in what follows.

6 Conformal Partially Massless Higher Spins

Our remaining task is to determine which, if any, of the partially massless theories are also conformal. A Herculean, but perhaps noble task would be to write down covariant actions at arbitrary values of the spin[2] and determine conformal invariance explicitly. To arrive quickly at an answer, however, we proceed as follows. Firstly, we need only look for invariance with respect to "de Sitter dilations" δ_L. Secondly, a notable feature of free higher spin fields is that despite the complexity of their covariant actions, first order action principles in terms of physical degrees of freedom are extremely simple–see equation (12)!

Now, the crux of our argument: compare actions (12) and (24). Since the dilation δ_L necessarily has no spin dependence, we can ignore the helicity summation $\sum_\varepsilon$. Therefore conformal invariance is guaranteed by choosing the effective mass

$$\mu^2 = -\frac{1}{4}M^2\,. \tag{26}$$

Setting the mass parameter m^2 to its depth t partially massless value we therefore require

$$-\frac{1}{4}M^2(2s-2t+n-2)^2 = -\frac{1}{4}M^2\,, \tag{27}$$

which is solved via

$$t = s + \frac{n\pm 1}{2} - 1\,. \tag{28}$$

However, the depth t must be both integer and no greater than s. The only the solutions therefore are $n = 3, 1$, *i.e.* spacetime dimensions four and two! In dimension two, there are no helicities so we find nothing new.

Dimension four is more interesting, we are forced to take $t = s$. When $s = t = 0$, we obtain a conformally improved scalar. For $s = t = 1$, we have a single gauge invariance

$$\delta\phi_\mu = D_\mu\xi\,, \tag{29}$$

yielding Maxwell theory in four dimensional de Sitter space with its scalar gauge parameter. This theory is well known to be conformal. For $s = t = 2$, we have a double derivative gauge invariance

$$\delta\phi_{\mu\nu} = \Big(D_{(\mu}D_{\nu)} + \frac{\Lambda}{3}\Big)\,\xi \tag{30}$$

and we obtain the original spin 2 partially massless theory of [1]. Indeed, these authors arrived at this theory by demanding conformal invariance and

[2]Indeed, covariant actions can be determined from the work of [6].

found a higher derivative gauge transformation as a consequence, as opposed to the opposite logic which led to partially massless yet not necessarily conformal theories in [3, 4, 5, 8].

In general, our result is that in dimension four, partially massless theories with maximal depth gauge invariances

$$\delta\phi_{\mu_1\ldots\mu_s} = \Big(D_{(\mu_1}\ldots D_{\mu_s)} + \cdots\Big)\,\xi\,, \tag{31}$$

are conformally invariant.

7 Null Propagation

Another way to uncover the conformal invariance of dimension four maximal depth partially massless theories is to solve explicitly their higher spin wave equations. This computation has been carried out in [3]. In steady state coordinates the wave equations for these theories are of Bessel type. In general the solutions are, of course, Bessel functions, but Huygen's principle [7] implies that for half integer values of their index ν, these Bessel solutions become simply massless plane waves multiplied by slowly varying polynomials. There are cardinally infinitely many such special solutions but the index $\nu = 1/2$ is special, since it corresponds to a conformally improved scalar. Some details:

Let us work in four dimensional spacetime since that is where the interesting conformal theories live. Fourier transforming $\vec{\partial} \to i\vec{k}$ and rescaling the conformal time coordinate $\tau \to z = |\vec{k}|\tau$, the field equation for a massive spin s, helicity ε field in de Sitter space reads

$$\frac{d^2q}{dz^2} + \frac{1}{z} + \Big(1 - \frac{\nu^2}{z^2}\Big)q = 0, \tag{32}$$

where $q \equiv |\vec{k}/M|^{3/2-s}q_\varepsilon$ and the index

$$\nu^2 = \frac{1}{4} + s(s-1) - \frac{m^2}{M^2}\,. \tag{33}$$

Setting m^2 to its tuned values (7) yields

$$\nu^2 = \frac{1}{4}(2s - 2t + 1)^2\,. \tag{34}$$

The conformal value $\nu^2 = 1/4$ is obtained only for $s = t$ in agreement with the analysis of the previous Section. In this case the solution to the wave equation is $q(z) = z^{-1/2}\exp(iz)$ which amounts to massless plane wave propagation since the overall exponential behavior is $\exp(i|\vec{k}|\tau + i\vec{k}\cdot\vec{x})$.

8 Conclusions

In this Letter we have uncovered new conformal, gauge invariant theories in four dimensional de Sitter space generalizing Maxwell's vector theory. These new theories describe spin s, lightlike excitations with helicities $\pm s, \ldots, \pm 1$. As for Maxwell theory, elimination of the zero helicity mode via gauge invariance suffices to ensure conformal invariance.

Many pressing questions remain but as usual the main difficulty is interactions. As yet no obvious mechanism for partially massless interactions is available, although one might expect Strings to do the job [9]. Another speculation is that since the maximal depth partially massless theories are singled out by conformal invariance, perhaps the interaction problem is more tractable for these theories. The guiding principle would, of course, be conformal invariance.

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