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Duality invariance of all free bosonic and fermionic gauge fields

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Abstract

We give a simple general extension to all free bosonic and fermionic massless gauge fields of a recent proof that spin 2 is duality invariant in flat space. We also discuss its validity in (A)dS backgrounds and the relevance of supersymmetry.

Recently [1], it was shown that the duality invariance originally established [2] for Maxwell theory in D=4, extends to free massless spin 2. It was also conjectured there that higher spin gauge bosons, and perhaps General Relativity (GR) might enjoy a similar property. In this Note, a simple method is used to exhibit the duality invariance of all free gauge fields, bosonic and fermionic; some implications of supersymmetry and the extent of flat space duality in (A)dS are also discussed.

We begin with bosons. The key to our derivation is the remark that since free gauge field actions are (abelian) gauge invariant, they can be uniformly expressed–after elimination of constraints–in terms of the fundamental spatial gauge invariant symmetric transverse-traceless (TT) conjugate variables [3] ($\pi^{ij\cdots}_{TT}$, $q^{TT}_{ij\cdots}$),

$$\partial_i \pi^{ij\cdots}_{TT} \equiv \partial_i q^{TT\ i}{}_{j\cdots} \equiv 0\ ; \qquad \pi_{TT\ i}{}^{i\cdots} \equiv q^{TT\ i}{}_{i\cdots} \equiv 0\ . \tag{1}$$

In this connection, we recall that only the above dynamical variables can be meaningfully varied; neither constraint, gauge nor Lagrange multipliers play any role here. Only spatial indices appear (s of them for spin s) and we drop the "TT" henceforth. The obvious extension of the Maxwell action is

$$I_s = \int d^4x \left[\pi^{ij\cdots}\ \dot{q}_{ij\cdots} - H(\pi, q)\right] \tag{2a}$$

$$H = \frac{1}{2}\left[\pi^{ij\cdots} \quad \pi_{ij\cdots} + B_{ij\cdots} \quad B^{ij\cdots}\right] \equiv \frac{1}{2}\left(\pi^2 + B^2\right) , \tag{2b}$$

$$B_{ij\cdots} = \frac{1}{2s}\left(\epsilon_i{}^{lk}\partial_l q_{kj\cdots} + \epsilon_j{}^{lk}\partial_l q_{ik\cdots} + \cdots\right) \equiv (\Theta q)_{ij\cdots} \tag{2c}$$

Note the extended curl operation[1] on each index, suitably symmetrized and normalized. Clearly, in order to preserve the Hamiltonian, the desired transformation must be a rotation:

$$\delta\pi = B, \qquad \delta B = -\pi . \tag{3}$$

But one must first show that δB is indeed a transformation that can be implemented by some δq and that the resulting change in (π, q) is canonical, namely symplectic ($\int \pi\dot{q}$) form-preserving. As in Maxwell, this is exhibited by

$$\delta q = -\nabla^{-2}(\Theta\pi) \qquad \Rightarrow \qquad \delta B = -\pi \tag{4}$$

where Θ is the generalized curl of (2c). It is easy to check that (3,4) is indeed canonical: because the operator $\nabla^{-2}\Theta$ is hermitian, the integrands in $\int B\dot{q}$ and $\int \pi\nabla^{-2}(\Theta\dot{\pi})$ are total time derivatives. This completes our proof of flat space free field bosonic duality for all spin ≥ 1.

The fermionic case is quite analogous. The lowest gauge field is the spin 3/2 Rarita–Schwinger ψ_μ; fortunately its duality invariance properties were derived in [4], to which we refer for details. Briefly, the basic gauge-invariant variable is now the transverse, gamma-traceless spatial vector-spinor ψ_i^{Tt}

$$\partial^i\psi_i^{Tt} = 0 \, , \quad \gamma^i\psi_i^{Tt} = 0 \, , \quad \gamma^i\gamma^j\psi_{ij}^{Tt} \equiv \psi_{ii}^{Tt} = 0 \, . \tag{5}$$

The "field strength" is

$$f_{\mu\nu} \equiv \partial_\mu\psi_\nu - \partial_\nu\psi_\mu \, , \tag{6}$$

which obeys the (first order) field equations

$$f_{\mu\nu} + \gamma_5\tilde{f}_{\mu\nu} = 0 \, , \qquad \gamma_5^2 = -1 \, . \tag{7}$$

The theory is invariant under two separate transformations: "bosonic" duality with respect to the world index ($f_{\mu\nu} \to \tilde{f}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}f_{\alpha\beta}$) and "fermionic" – chiral-invariance under γ_5-transformations on the spinor index. The spin ($3/2{+}n$) extension scarcely needs elaboration, since it merely increases the number of bosonic, "Tt" indices, $\psi_i \to \psi_{ij\ldots}$, and their tracelessness is guaranteed by (5). Since the gauge invariant action is again of "bosonic" form (apart from being first order), world index duality follows as for bosons.

We conclude our free field considerations with brief remarks on two topics: supersymmetry and duality in constant curvature (rather than merely flat) backgrounds. Supersymmetry (SUSY) links the duality invariances[2] of adjoining $(s, s+1/2)$ systems. Hence the ladder that starts at Maxwell (or Rarita–Schwinger) is extended to all higher rungs by SUSY; this is hardly surprising since all massless spins in D=4 have exactly 2 helicities and duality just expresses their mutual

[1] Actually, the integral of B^2 is–as in Maxwell–equivalent to that of $(\nabla q)^2$, but we retain the historical "magnetic" $(\nabla \times q)^2$ notation, which preserves the index ranks of the fields in (and only in) D=4 and points directly to duality.

[2] Not all invariances "propagate" like this, conformal invariance being the simplest counterexample: only some "tuned" higher spins retain it in constant curvature spaces [5]. The generic criterion is commutation of the invariance with the supercharges.

rotation. Can free fields duality be extended to constant curvature, (A)dS, spaces? That of Maxwell is obvious, since it is conformally invariant, and (A)dS is conformally flat. For $s = 3/2$, duality invariance in AdS ($\Lambda < 0$) was already proven in [4]. There, the Rarita–Schwinger action necessarily acquires a "mass" term $\sim \sqrt{-\Lambda}\, \psi_\mu \sigma^{\mu\nu} \psi_\nu$, in order to retain gauge invariance [6]. The field strength is now defined as the covariant curl

$$f_{\mu\nu} = \mathcal{D}_\mu \psi_\nu - \mathcal{D}_\nu \psi_\mu \;, \qquad \mathcal{D}_\mu \equiv D_\mu + \frac{1}{2}\sqrt{-\Lambda}\, \gamma_\mu \;, \qquad [\mathcal{D}_\mu, \mathcal{D}_\nu] = 0 \;. \tag{8}$$

Commutation of the $\mathcal{D}_\mu$ essentially reduces the process to the flat space one and duality (but not chirality) invariance is maintained. Since supersymmetry is still present, for example in linearized cosmological supergravity, the appropriate bosonic linearization of the latter, (massless) $s = 2$ in AdS, should also be duality invariant. However, this result may not extend to higher spins (some recent reviews of higher spins in (A)dS are given in [7]), where (8) may not be applicable.

In summary, we have extended to all spins the duality invariance established in [1, 2, 4] for gauge fields of spins (2,1,3/2) respectively. Our main simplification has been to formulate the gauge invariant actions solely in terms of the two time local "TT" bosonic, and corresponding fermionic, variables. The form of the field variations is then uniformly seen to keep the $\frac{1}{2}\int(\pi^2 + \mathbf{B}^2)$ Hamiltonians ($\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{q}$) rotation-invariant, without affecting the symplectic $\int \pi \dot{q}$ form. We then used SUSY arguments to back up these results and to extend them to AdS backgrounds, where possible. Our results may also be of interest for higher spin fields, where a vast literature already exists on related topics [8].

We have not touched here on the important question of whether GR shares (deformed) duality invariance with its linearized spin 2 limit, a problem we hope to discuss elsewhere.

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