Actuation of thin nematic elastomer sheets with controlled heterogeneity

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Abstract

Nematic elastomers and glasses deform spontaneously when subjected to temperature changes. This property can be exploited in the design of heterogeneously patterned thin sheets that deform into a non-trivial shape when heated or cooled. In this paper, we start from a variational formulation for the entropic elastic energy of liquid crystal elastomers and we derive from it an effective two-dimensional metric constraint, which links the deformation and the heterogeneous director field. Our main results show that satisfying the metric constraint is both necessary and sufficient for the deformation to be an approximate minimizer of the energy. We include several examples which show that the class of deformations satisfying the metric constraint is quite rich.

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1 Introduction and main results

1.1 Model for nematic elastomers with controlled heterogeneity

We consider a thin sheet of nematic liquid crystal elastomer of thickness $h \ll 1$. Initially, the sheet occupies a flat region in space,

$$\Omega_h := \omega \times (-h/2, h/2), \qquad \omega \subset \mathbb{R}^2,$$

where ω is an open, connected and bounded Lipschitz domain, which we call the midplane of the sheet. We envision that the liquid crystal molecules within the elastomer sheet are programmed so that there average alignment can be described macroscopically by a director field $N_0^h \colon \Omega_h \to \mathbb{S}^2$ at the initial temperature T_0. Upon changing the temperature from T_0 to the final temperature T_f, the sheet will spontaneously deform by a deformation $Y^h : \Omega_h \to \mathbb{R}^3$ which we assume minimizes the entropic elastic energy

$$\mathcal{I}^h_{N_0^h}(Y^h) := \int_{\Omega_h} W^e\left(\nabla Y^h, \frac{(\nabla Y^h)N_0^h}{|(\nabla Y^h)N_0^h|}, N_0^h\right) dx. \tag{1.1}$$

(If $(\nabla Y^h)N_0^h = 0$ on a set of positive measure, then we set $\mathcal{I}^h_{N_0^h}(Y^h) = \infty$.) Following Bladon et al. [11] (see also Warner and Terentjev [43]), we take the *entropic elastic energy density* W^e : $\mathbb{R}^{3\times 3} \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ as

$$W^e(F, N, N_0) := \frac{\mu}{2}\begin{cases} \mathrm{Tr}(F^T(\ell^f_N)^{-1}F\ell^0_{N_0}) - 3 & \text{if } \det F = 1, N, N_0 \in \mathbb{S}^2 \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Here, $\mu > 0$ is the shear modulus, F is the deformation gradient, and F^T denotes the transpose matrix of F. Moreover, $\ell^0_{N_0}, \ell^f_N \in \mathbb{R}^{3\times 3}_{sym}$ are the step length tensors at the initial temperature T_0 and final temperature T_f respectively. They are defined by

$$\ell^0_{N_0} := r_0^{-1/3}\left(I_{3\times 3} + (r_0 - 1)N_0 \otimes N_0\right), \tag{1.3}$$

$$\ell^f_N := r_f^{-1/3}\left(I_{3\times 3} + (r_f - 1)N \otimes N\right). \tag{1.4}$$

The parameters $r_0, r_f \geq 1$ quantify the degree of anisotropy at the initial and final temperature respectively. They describe the extent to which the material tends to deform in the directions N_0 and N respectively. We envision that r_0, r_f arise from evaluating some underlying, monotone decreasing function $\bar{r}(T)$ at the temperatures $T = T_0$ and $T = T_f$.

Remark 1.1. (i) The constant -3 in (1.2) is chosen so that $\min W_e = 0$.

(ii) The elastic energy $\mathcal{I}^h_{N_0^h}$ is defined without any displacement or traction boundary conditions as we are dealing actuation only.

(iii) The temperature-dependent function $\bar{r}(T)$ satisfies $\lim_{T\to\infty} \bar{r}(T) = 1$ because nematic materials behave isotropically at large temperature. Indeed, setting $r_0 = r_f = 1$ in the formulae above, one recovers the standard incompressible neo-Hookean energy for isotropic materials.

(iv) In the definition (1.1) of $\mathcal{I}^h_{N_0^h}(Y^h)$ we imposed the kinematic constraint $N^h = \frac{(\nabla Y^h)N_0^h}{|(\nabla Y^h)N_0^h|}$. There are nematic elastomers which do not satisfy this kinematic constraint (i.e. where the director N is allowed to vary more freely). Those materials can show macroscopic deformations which arise from the fine-scale microstructure produced by oscillations of N [13],[14],[15],[21] (see also the experiments by Kundler and Finkelmann [28]).

In the present paper, we are interested in actuating complex, yet predictable, shape by programming an initial heterogeneous anisotropy N_0^h in the nematic elastomer. It would be difficult to control actuation for a material that is capable of freely forming microstructure, which competes with the shape change driven by the programmed anisotropy, even at low energy. Thus, in order to produce "controlled heterogeneity" in actuation, we impose the kinematic constraint $N^h = \frac{(\nabla Y^h)N_0^h}{|(\nabla Y^h)N_0^h|}$. The constraint is similar to one that was imposed by Modes et al. [31] in their prediction for conical and saddle like actuation in nematic glass sheets with radial and azimuthal heterogeneity (in fact, both constraints are equivalent for zero energy/stress free states; see Proposition A.4).

(v) We have neglected Frank elasticity and related effects in our model, as these are expected to be small in comparison to the entropic elasticity (see discussion in Chapter 3 in Warner and Tarentjev [43]). However, to derive the key metric constraint (introduced below) as a necessary feature of low energy deformations, we add to the energy (1.1) a small contribution from Frank elasticity for technical reasons. This is discussed in Section 1.6.

Notation. So far, all the vector fields we considered were maps $\Omega_h \to \mathbb{R}^3$ and we denoted them by capital letters, e.g. N_0^h, Y^h. Throughout the paper, we will also consider vector fields defined on the midplane $\omega \subset \mathbb{R}^2$ and we denote these by lowercase letters, e.g. $n_0, y : \omega \to \mathbb{R}^2$. Moreover, we use $\tilde{(\cdot)}$ to distinguish the two-dimensional midplane variables from the three-dimensional ordinary variables, that is

$$x = (x_1, x_2, x_3), \qquad \tilde{x} = (x_1, x_2) \qquad \tilde{\nabla} = (\partial_{x_1}, \partial_{x_2}).$$

We will only consider $x \in \Omega_h$ and so $x_3 \in (-h/2, h/2)$ is small.

1.2 The metric constraint and overview of results

We have the following situation in mind: An initial director field N_0^h is heterogeneously programmed on the nematic sheet Ω_h at the initial temperature T_0. As T is varied from T_0 to the final temperature T_f, the sheet undergoes a non-uniform spontaneous deformation. We postulate that the spontaneous deformations are those that minimize the elastic energy $\mathcal{I}^h_{N_0^h}$.

Our goal is to characterize the class of director fields N_0^h and corresponding deformations Y^h which yield small elastic energy $\mathcal{I}^h_{N_0^h}(Y^h)$. This characterization comes in the form of a two-dimensional effective metric constraint (1.6). To see how this arises, we first consider a naive approach by requiring $\mathcal{I}^h_{N_0^h}(Y^h) = 0$ (recall that $\min W^e = 0$). By Proposition A.4, $\mathcal{I}^h_{N_0^h}(Y^h) = 0$ is equivalent to

$$(\nabla Y^h)^T \nabla Y^h = r^{-1/3}(I_{3\times 3} + (r-1)N_0^h \otimes N_0^h) =: \ell_{N_0^h} \quad \text{a.e. on } \Omega_h, \tag{1.5}$$

where $r = r_f/r_0$ so that $r \in (0,1)$ for heating and $r > 1$ for cooling. However, (1.5) is too strong of a condition to be useful, meaning that there are only few choices of N_0^h for which a Y^h satisfying (1.5) exists. We explain this in Remark 1.2 below.

Given that (1.5) is too restrictive, we relax the problem and study *approximate minimizers of the elastic energy* $\mathcal{I}^h_{N_0^h}(Y^h)$. The key observation is that by making use of the thinness of the sheet Ω_h and assuming that N_0^h does not vary too much as a function of x_3, one can show that

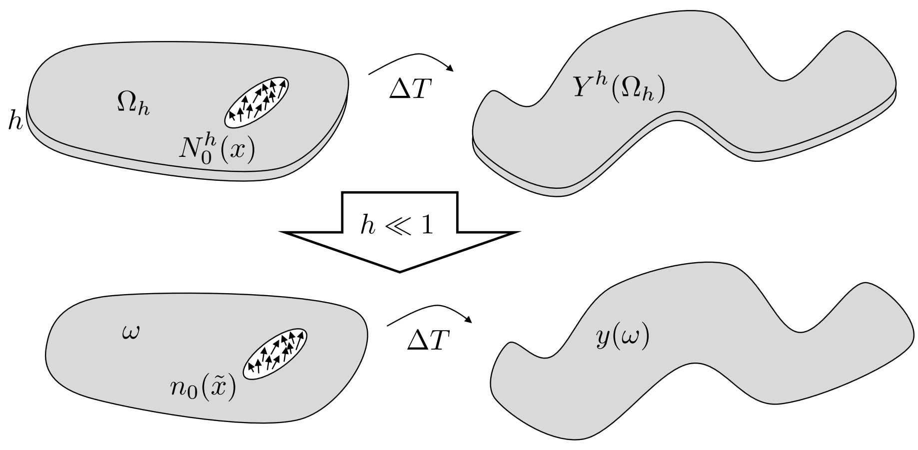


Figure 1: Actuation for thin sheets is characterized by the midplane fields.

approximate minimizers are characterized (in a sense to be made precise) by the following *effective metric constraint* (1.6). It is a two-dimensional reduction of the three-dimensional constraint (1.5) and reads

$$(\tilde{\nabla} y)^T \tilde{\nabla} y = r^{-1/3}(I_{2\times 2} + (r-1)\tilde{n}_0 \otimes \tilde{n}_0) =: \tilde{\ell}_{n_0} \qquad \text{a.e. on } \omega \tag{1.6}$$

Here we used the notation introduced above, so $y, n_0 : \omega \to \mathbb{R}^3$ are essentially the restrictions of Y^h, N_0^h to the midplane ω (i.e. $y(x_1, x_2) \approx Y^h(x_1, x_2, 0)$) and $\tilde{(\cdot)}$ refers to the projection onto midplane variables, i.e. $\tilde{\nabla} y$ is a 3×2 matrix and $\tilde{n}_0 \in B_1(0) \subset \mathbb{R}^2$ is the projection of n_0 onto ω.

The metric constraint (1.6) appeared in our earlier short paper [39] with a view towards applications. In the present paper, we discuss the metric constraint from a mathematical perspective.

Remark 1.2. Let us explain why (1.5) is too restrictive. Assuming that N_0^h is sufficiently smooth (the non-smooth case is treated in Lewicka and Pakzad [30]), there exists a Y^h satisfying (1.5) if and only if the components of the Riemann curvature tensor of $\ell_{N_0^h}$ vanish. This condition is well-known in the physics literature (e.g. Efrati et al. [22]), and in the language of continuum mechanics; it gives compatibility of the Right Cauchy-Green deformation tensor (e.g. Blume [12]). As a consequence, N_0^h has to satisfy a certain nonlinear partial differential equation, and so it must come from a very restricted set of functions.

We will now discuss the precise context in which (1.6) arises; that is under which circumstances the midplane vector fields characterize approximate minimizers of the strain energy (and thus give a recipe for design). The intuition is expressed by Figure 1.

It is clear that we have to assume that the full director field N_0^h does not vary too much away from the midplane. For the following, recall our notation $n_0(x_1, x_2) \approx N_0^h(x_1, x_2, 0)$ and $\tilde{x} = (x_1, x_2)$.

Assumption 1.3. We assume

$$N_0^h(x) = n_0(\tilde{x}) + O(h), \quad \text{for a.e. } x \in \Omega_h, \qquad \text{i.e., } \|N_0^h - n_0\|_{L^\infty(\Omega_h)} \le \tau h \text{ for some } \tau > 0. \tag{1.7}$$

The $O(h)$ term accounts for the following two possible deviations from the midplane field. For definiteness, we have fixed the maximum tolerance $\tau > 0$ for these non-idealities.

(a) It accounts for deviations of the director field through the thickness, which are of the same order as the thickness. Note that this excludes twisted or splay-bend nematic sheets [25],[42], for which one prescribes the director field on the top surface of the sheet and then differently then on the bottom surface, so that the director field has to vary by an $O(1)$ amount through the thickness.

(b) It accounts for the possibility of planar deviations. In the synthesis techniques employed by Ware et. al. [41], the director field is prescribed in voxels or cubes whose characteristic length is similar to the thickness and we expect the experimental error to be of this order.

From now on we make Assumption 1.3. We would like to use the metric constraint (1.6) to characterize approximate minimizers of the strain energy as $h \to 0$. Such a characterization would ideally contain two parts; namely (1.6) should be a sufficient condition and a necessary condition for being an approximate minimizer. (Ideally, this corresponds to the two statements which would comprise Γ-convergence to a two-dimensional energy functional, though we do not quite prove such a convergence.) The *main results* in our paper make progress in both of these directions. To put the results in context, we note that *generic deformations* have energy $\mathcal{I}^h_{N_0^h}(Y^h) = O(h)$, so strain energies of order $\ll h$ can be reasonably considered small.

The *main results* are as follows.

(i) *Sufficiency of the metric constraint.* Given midplane fields (y, n_0) satisfying the metric constraint (1.6) and for sheets of sufficiently small thickness h, we can construct global deformations $Y^h \colon \Omega_h \to \mathbb{R}^3$ which approximately minimize the strain energy and satisfy $Y^h(\tilde{x}, 0) \approx y(\tilde{x})$ in the appropriate Sobolev norm. It turns out that the correct energy scale of an "approximate minimizer" is related to the smoothness of the design. That is, for nonisometric origami (piecewise constant design) as in Figure 13, our constructions satisfy

$$\mathcal{I}^h_{N_0^h}(Y^h) \leq O(h^2) \quad (\textbf{Theorem 1.4}), \tag{1.8}$$

and for sufficiently smooth surfaces such as the lifted surfaces in Figure 2, our constructions satisfy

$$\mathcal{I}^h_{N_0^h}(Y^h) \leq O(h^3) \quad (\textbf{Theorem 1.8} \text{ and } \textbf{Corollary 1.9}).$$

The techniques employed here are akin to those of Conti and Dolzmann [16],[17] for incompressibility and Conti and Maggi [18] for nonisometric origami.

(ii) *Optimality of nonisometric origami.* We also have a result concerning the optimality of (1.8) for nonisometric origami: We consider a two dimensional analog to the entropic elasticity ($\tilde{\mathcal{I}}^h_{n_0}$ in equation (1.18) below), and show in this setting that interfaces separating regions of distinct constant director necessarily incur an elastic penalty of $O(h^2)$ upon actuation (**Theorem 1.6**).

(iii) *Necessity of the metric constraint.* We prove that the metric (1.6) is necessary for pure bending if we augment the entropic elasticity $\mathcal{I}^h_{N_0^h}$ in (1.1) with an approximation of Frank elasticity which is natural to nematics (**Theorem 1.12**). Specifically, if $\mathcal{I}^{h,\varepsilon}_{N_0^h}$ is the sum of a nondimensionalized version of $\mathcal{I}^h_{N_0^h}$ and an additional term approximating Frank elasticity with lengthscale ε (equation (1.28) below) which is comparable to thickness (i.e. $\varepsilon \equiv O(h)$), then using the results of Friesecke, James and Müller [23] on Geometric Rigidity, we show that bending configurations satisfying $\mathcal{I}^{h,\varepsilon_h}_{N_0^h}(Y^h) \leq O(h^3)$ must be characterized by sufficiently smooth midplane fields satisfying (1.6).

We note that there are, in general, multiple deformations y which satisfy (1.6) for a given n_0 (e.g. the sheet can actuate upward or downwards in different places). We imagine that one can distinguish between these by appropriately breaking additional symmetries, but we do not investigate this further.

The constraint (1.6) generalizes a metric constraint that has been proposed by Aharoni et al. [1] for actuation of nematic sheets. Indeed, (1.6) is more general in that (a) it need only hold almost everywhere, allowing for piecewise constant director designs and (b) the director can be programmed out of plane. At the same time, it is easy to see that (1.6) reduces to the contraint [1] for smooth planar director fields. (With $n_0 \equiv \tilde{n}_0$, we can write $\tilde{n}_0 \cdot e_1 = \cos(\theta)$ and $\tilde{n}_0 \cdot e_2 = \sin(\theta)$ for a Cartesian basis $\{e_1, e_2\} \subset \mathbb{R}^2$ on the plane. It follows that $(\tilde{\nabla} y)^T \tilde{\nabla} y = \tilde{\ell}_{n_0} = \tilde{R}(\theta)\text{diag}\ (r^{2/3}, r^{-1/3})\tilde{R}(\theta)^T$ for $\tilde{R}(\theta) \in SO(2)$ a rotation of θ about the normal to the initially flat sheet as required by [1].)

1.3 Nonisometric origami constructions under the metric constraint

We consider continuous, piecewise affine deformations (nonisometric origami) and we prove that if these satisfy the effective metric constraint (1.6), then their strain energy scales at most like h^2.

By definition a *nonisometric origami* is a union of a finite number of polygonal regions ω_α which each have a constant director field, i.e.,

$$\omega = \bigcup_{\alpha=\{1,\ldots,N\}} \omega_\alpha, \qquad \omega_\alpha \text{ mutually disjoint and polygonal,} \tag{1.9}$$

$$n_0 \colon \omega \to \mathbb{S}^2 \quad \text{satisfies} \quad n_0(\tilde{x}) \equiv n_{0\alpha}, \qquad (\tilde{x} \in \omega_\alpha, \forall \alpha \in \{1,\ldots,N\}), \tag{1.10}$$

$$\tilde{n}_{0\alpha} \neq \pm \tilde{n}_{0\beta} \quad \text{when a } g_{\alpha\beta} \in \mathbb{S}^1 \text{ interface connects } \omega_\alpha \text{ and } \omega_\beta,\ \alpha \neq \beta. \tag{1.11}$$

The last condition is only there to ensure that each interface corresponds to a non-trivial change of the director (otherwise that interface would be superfluous). Examples of nonisometric origami are presented in Figure 13 in the appendix.

Our main result for nonisometric origami says that, given a (continuous, piecewise affine) midplane deformation $y \colon \omega \to \mathbb{R}^3$ satisfying the effective metric constraint (1.6), we can construct map $Y^h : \Omega_h \to \mathbb{R}^3$ which has the strain energy $\mathcal{I}^h_{N_0^h}(Y^h) \leq O(h^2)$. (We think of Y^h as an approximate minimizer of the strain energy, especially given the optimality result of the $O(h^2)$ scaling proved later.) Moreover, the restriction of the map Y^h to the midplane is close to the original map y (in the appropriate Sobolev norm) and in this sense Y^h is an approximate extension of y. We comment further on the additional technical assumption that y has a δ-smoothing after the theorem; for now we just note that it is satisfied in several exemplary cases.

Theorem 1.4. *Let ω and n_0 be a nonisometric origami as defined above, and let $N_0^h : \Omega_h \to \mathbb{S}^2$ be any vector field that is close to n_0 in the sense of (1.7). Let $y \in W^{1,\infty}(\omega, \mathbb{R}^3)$ be a piecewise affine midplane deformation such that*

$$y(\tilde{x}) = \tilde{F}_\alpha \tilde{x} + c_\alpha, \tag{1.12}$$

$$(\tilde{F}_\alpha)^T \tilde{F}_\alpha = \tilde{\ell}_{n_{0\alpha}}, \tag{1.13}$$

for all $\tilde{x} \in \omega_\alpha$ and all $\alpha \in \{1,\ldots,N\}$. Suppose further that for all small enough $\delta > 0$, y has a δ-smoothing y^δ in the sense of Definition 1.5 below.

Then, there exists an $m > 0$ such that if we set $\delta_h = mh$, then for all small enough $h > 0$ there exists a map $Y^h : \Omega_h \to \mathbb{R}^3$ with

$$Y^h(\tilde{x}, 0) = y^{\delta_h}(\tilde{x}), \quad (\tilde{x} \in \omega), \tag{1.14}$$

$$\mathcal{I}^h_{N_0^h}(Y^h) \leq O(h^2). \tag{1.15}$$

Moreover, Y^h is an approximate extension of y in the sense that $\|y^{\delta_h} - y\|_{W^{1,2}(\omega,\mathbb{R}^3)} \leq O(h)$.

Theorem 1.4 is proved in Section 3.3. In the statement we assumed that y has a δ-smoothing in the following sense:

Definition 1.5. We say that $y : \omega \to \mathbb{R}^3$ has a δ-smoothing, if there exists a map $y^\delta \in C^3(\bar{\omega}, \mathbb{R}^3)$ and a subset $\omega_\delta \subset \omega$ of area less than $C\delta$ such that

$$\begin{gathered} y^\delta = y \text{ on } \omega \setminus \omega_\delta, \qquad \|\partial_1 y^\delta \times \partial_2 y^\delta\|_{L^\infty} \geq c > 0, \\ \|\tilde{\nabla} y^\delta\|_{L^\infty} \leq C, \qquad \|\tilde{\nabla}\tilde{\nabla} y^\delta\|_{L^\infty} \leq C\delta^{-1}, \qquad \|\tilde{\nabla}^{(3)} y^\delta\|_{L^\infty} \leq C\delta^{-2} \end{gathered} \tag{1.16}$$

for some constants $C, c > 0$ which can depend on y and ω but not on δ.

The existence of such a δ-smoothing (of the only Lipschitz continuous midplane deformation y) is an important technical tool. It is needed because the global deformation Y^h has to satisfy the *incompressibility constraint* $\det \nabla Y^h = 1$. (Essentially, the non-degeneracy of the derivatives of y^δ allows one to employ the inverse function theorem to derive a sufficiently well-behaved ordinary differential equation as described in section 3.)

This technical issue has appeared in previous works on incompressibility (also, a $\det F > 0$ constraint) in thin sheets. It was first appreciated by Belgacem [5] and later addressed in some generality by Trabelsi [40] and Conti and Dolzmann [16]. However, their methods are very geometrical in nature (they are largely based on Whitney's ideas on the singularities of functions $\mathbb{R}^n \to \mathbb{R}^{2n-1}$) and it is not obvious how to extract from them the $\delta-$dependent control of the higher derivatives which we need in the present context.

We discuss several examples of nonisometric origami in an extensive **appendix C**. The examples are depicted in Figure 13 and include a construction which will fold into a box, originally due to [33], as well as further examples which previously appeared in a companion paper to this one [39] that was geared towards a physics audience. We also discuss in some detail an equivalent formulation of the metric constraint (1.13) for nonisometric origami in terms of *compatibility conditions*. These are akin to the rank-one condition studied in the context of fine-scale twinning during the austenite martensite phase transition and actuation active martensitic sheets [2],[8],[9] and to the recently studied compatibility conditions for the actuation for nematic elastomer and glass sheets using planar programming of the director [32],[33].

Importantly, we prove that all of the examples of nonisometric origami considered in this work *indeed have a δ-smoothing*, in the sense of Definition 1.5, by using explicit constructions (see Sections 2.1-2.4). We leave it as an interesting open problem to prove that any nonisometric origami possesses a δ-smoothing.

1.4 On the optimality of nonisometric origami

From Theorem 1.4, we can construct approximations to nonisometric origami (under the hypothesis (1.16)) with energy $O(h^2)$. Thus, it is natural to ask whether these constructions are energetically optimal for prescribed director field. Following the work of Kohn and Müller [27] and others [7], [18], optimality in our context would be established by

$$\inf \left\{ \mathcal{I}^h_{N_0^h}(Y^h) \colon Y^h \in W^{1,2}(\Omega_h, \mathbb{R}^3) \right\} \geq c_L h^2 \tag{1.17}$$

for all small enough $h > 0$. Here ω and n_0 describe a nonisometric origami (defined in the last section) and N_0^h is close to n_0 in the sense of (1.7).

In the present paper, we prove the analogous lower bound for a two-dimensional analogue of the three-dimensional entropic strain energy,

$$\tilde{\mathcal{I}}^h_{n_0}(y) = h\int_\omega \left(|(\tilde{\nabla}y)^T\tilde{\nabla}y - \tilde{\ell}_{n_0}|^2 + h^2|\tilde{\nabla}\tilde{\nabla}y|^2\right) d\tilde{x}. \tag{1.18}$$

The first term here represents membrane stretching part and is minimized exactly when the metric constraint (1.6) is satisified. The second term approximates bending. Such a two-dimensional energy is a widely used proxy to describe the elasticity of non-Euclidean plates (e.g. Efrati et al. [22] and Bella and Kohn [6]). In a broader context, these proxies often agree in h-dependent optimal energy scaling with that of the three dimensional elastic energy, and deformations which achieve this scaling in this two dimensional setting tend to form the midplane deformations for optimal three dimensional constructions (e.g. Bella and Kohn [7] and the single fold approximation of Conti and Maggi [18]).

For the effective two-dimensional theory, we prove a lower bound with the expected $O(h^2)$ scaling.

Theorem 1.6. *Let $r > 0$ and $\neq 1$, ω satisfy (1.9), and n_0 as in (1.10) and (1.11). For $h > 0$ sufficiently small*

$$\inf\left\{\tilde{\mathcal{I}}^h_{n_0}(y) \colon y \in W^{2,2}(\omega,\mathbb{R}^3)\right\} \geq c_L h^2.$$

Here, $c_L = c_L(n_0, r, \omega) > 0$ is independent of h.

Theorem 1.6 is proved in Section 4.

Remark 1.7. (i) If, in addition to the assumptions of the theorem, there exists a $y \in W^{1,\infty}(\omega,\mathbb{R}^3)$ satisfying (1.12) and each $\tilde{F}_\alpha$ and $n_{0\alpha}$ in these relations satisfies (1.13), then for $h > 0$ sufficiently small, there exists a $y^h \in C^3(\bar{\omega},\mathbb{R}^3)$ such that

$$\|y^h - y\|_{W^{1,2}} \leq O(h) \quad \text{and} \quad \tilde{\mathcal{I}}^h_{n_0}(y^h) \leq O(h^2). \tag{1.19}$$

Hence, nonisometric origami is optimal in this two dimensional setting. The proof of this is straightforward. Indeed, the estimates (1.16), with exception to the full-rank condition, can be obtained by standard mollification (for more details, see Section 2). Setting $\delta = h$ for these estimates yields a y^h satisfying (1.19).

(ii) Let us discuss some of the heuristics behind the lower bound in Theorem 1.6 and the conjectured lower bound (1.17). At an interface separating two regions of distinct constant director, an energetic penalty associated with membrane stretching at $O(h)$ drives the deformation to be piecewise affine with a fold precisely at the interface connecting the two regions, whereas an energetic penalty associated with bending at $O(h^3)$ cannot accommodate sharp folds, and thus a smoothing is necessitated. This interplay gives rise to an intermediate energetic scaling between $O(h)$ and $O(h^3)$. For isometric origami, folds can be smoothed to mostly preserve the isometric condition, leading to approximate constructions and (under suitable hypothesis) lowerbounds which scale as $O(h^{8/3})$ (see, for instance, Conti and Maggi [18]). For nonisometric origami, the preferred metric jumps across a possible fold and this leads to a larger membrance stretching term.

1.5 Examples of pure bending actuation under the metric constraint

We turn now to the case of smooth or sufficiently smooth surfaces and programs satisfying the metric constraint (1.6). For these configurations, we will show the actuation is pure bending, i.e. $O(h^3)$ in the entropic strain energy after actuation.

Below, we give a general result that gives $O(h^3)$ scaling for sufficiently smooth y, n_0. Before we state that result (Corollary 1.9), we present a large class of y, n which *automatically satisfy the two-dimensional metric constraint* (1.6). These surfaces are given as the graph of a function, combined with an appropriate contraction (here we consider cooling, so $r > 1$). We call these "lifted surfaces". They are defined by

$$y(\tilde{x}) = r^{-1/6}(x_1 e_1 + x_2 e_2) + \varphi(r^{-1/6}\tilde{x})e_3. \tag{1.20}$$

where the function φ is from the following set

$$\big\{\phi \in W^{2,\infty}(r^{-1/6}\omega, \mathbb{R})\colon \|\tilde{\nabla}\phi\|_{L^\infty} < \lambda_r := r - 1, \quad \text{supp}\phi \subset r^{-1/6}\omega_m\big\}. \tag{1.21}$$

Here, we set $\omega_m := \{\tilde{x} \in \omega\colon \text{ dist}(\tilde{x}, \partial\omega) > m > 0\}$ (recall that $\omega \subset \mathbb{R}^2$ is the midplane of the sheet, a bounded Lipschitz domain). The corresponding director field of a lifted surface is

$$n_0(\tilde{x}) = \frac{1}{\lambda_r^{1/2}} \begin{pmatrix} \partial_1\varphi(r^{-1/6}\tilde{x}) \\ \partial_2\varphi(r^{-1/6}\tilde{x}) \\ (\lambda_r - |\tilde{\nabla}\varphi(r^{-1/6}\tilde{x})|^2)^{1/2} \end{pmatrix}. \tag{1.22}$$

We emphasize again that any such choice of y, n_0 satisfies (1.6). This fact can be proved by rewriting (1.6) in an equivalent form, which is in fact more practical from the perspective of design and we discuss this below. The first main result is then that lifted surfaces have entropic energy of $O(h^3)$ (and therefore they are good candidates for designable actuation).

Theorem 1.8 (Lifted Surfaces)**.** *Let $r > 1$ and $m > 0$. Given a midplane deformation y as in (1.20) with φ taken from the set (1.21), define the director field n_0 as in (1.22). Let N_0^h be close to n_0 in the sense of (1.7).*

Then, for every $h > 0$ sufficiently small, there exists a $y^h \in C^3(\bar{\omega}, \mathbb{R}^3)$ and an extension $Y^h \in C^1(\bar{\omega}, \mathbb{R}^3)$ such that

$$Y^h(\tilde{x}, 0) = y^h(\tilde{x}), \;\; \tilde{x} \in \omega, \qquad \|y^h - y\|_{W^{1,\infty}(\omega)} \leq O(h), \qquad \mathcal{I}^h_{N_0^h}(Y^h) \leq O(h^3).$$

The key reason why lifted surfaces satisfy the $O(h^3)$ scaling is that they satisfy the metric constraint and are sufficiently smooth. Thus, we can generalize the proof of Theorem 1.8 to obtain the following result. (Here y is assumed to be smoother than it has to be for lifted surfaces smooth and so one does not need to introduce the smoothing y^h as in the theorem above.)

Corollary 1.9 (Smooth Surfaces)**.** *Let $r \in (0, 1)$ or $r > 1$. Let n_0 satisfy (1.7). If $y \in C^3(\bar{\omega}, \mathbb{R}^3)$ and $n_0 \in C^2(\bar{\omega}, \mathbb{S}^2)$ such that $(\tilde{\nabla}y)^T\tilde{\nabla}y = \tilde{\ell}_{n_0}$ everywhere on ω, then for $h > 0$ sufficiently small, there exists a $Y^h \in C^1(\overline{\Omega}_h, \mathbb{R}^3)$ such that*

$$Y^h(\tilde{x}, 0) = y(\tilde{x}), \;\; \tilde{x} \in \omega \qquad \mathcal{I}^h_{N_0^h}(Y^h) \leq O(h^3).$$

The results stated here are proved in section 3.4.

The idea of lifted surfaces is based on an equivalent rewriting of the metric constraint $(\tilde{\nabla}y)^T\tilde{\nabla}y = \tilde{\ell}_{n_0}$. (This equivalent form also yields a concrete design scheme for the actuation of nematic elastomers sheets.) Essentially, we take the picture of y being a solution to $(\tilde{\nabla}y)^T\tilde{\nabla}y = \tilde{\ell}_{n_0}$ defined by a predetermined n_0 and turn it on its head. That is, we first identify the set of deformation gradients that are consistent with (1.6) for any director field and then we identify the director associated with that deformation gradient.

Proposition 1.10. *Let $r > 1$. The metric constraint (1.6) holds if and only if*

$$\tilde{\nabla} y(\tilde{x}) = (\partial_1 y|\partial_2 y)(\tilde{x}) \in \mathcal{D}_r, \quad n_0(\tilde{x}) \in \mathcal{N}^r_{\tilde{\nabla} y(\tilde{x})} \qquad a.e.\ \tilde{x} \in \omega. \tag{1.23}$$

Here,

$$\mathcal{D}_{r>1} = \Big\{ \tilde{F} \in \mathbb{R}^{3\times 2} \colon |\tilde{F}|^2 \leq r^{-1/3} + r^{2/3},\ \ r^{-1/3} \leq |\tilde{F}\tilde{e}_\alpha|^2 \leq r^{2/3}, \alpha = 1, 2, \\ (\tilde{F}\tilde{e}_1 \cdot \tilde{F}\tilde{e}_2)^2 = (|\tilde{F}\tilde{e}_1|^2 - r^{-1/3})(|\tilde{F}\tilde{e}_2|^2 - r^{-1/3}) \Big\} \tag{1.24}$$

and

$$\mathcal{N}^{r>1}_{\tilde{F}} = \Big\{ m \in \mathbb{S}^2 \colon (m \cdot e_\alpha)^2 = \frac{|\tilde{F}\tilde{e}_\alpha|^2 - r^{-1/3}}{r^{2/3} - r^{-1/3}} \alpha = 1, 2, \\ sign((m\cdot e_1)(m\cdot e_2)) = sign(\tilde{F}\tilde{e}_1 \cdot \tilde{F}\tilde{e}_2) \Big\} \tag{1.25}$$

for sign: $\mathbb{R} \to \{-1, 0, 1\}$ the sign function with sign$(0) = 0$. (For $r < 1$, the inequalities in (1.24) and the sign in (1.25) are reversed.)

We prove this equivalence in the appendix, see Proposition B.1. There, we also show that for any $\tilde{F} \in \mathcal{D}_{\bar{r}}$, there exists an $m \in \mathcal{N}^r_{\tilde{F}}$. This means that for characterizing the geometry of surfaces which satisfy the metric constraint (1.6), we need only to consider the set of deformation gradients from a flat sheet ω which satisfy $\tilde{\nabla} y(\tilde{x}) \in \mathcal{D}_r$ a.e. $\tilde{x} \in \omega$. Unfortunately, such a broad characterization remains open. Of particular difficulty is the fact that this condition on the deformation gradient implies the equality

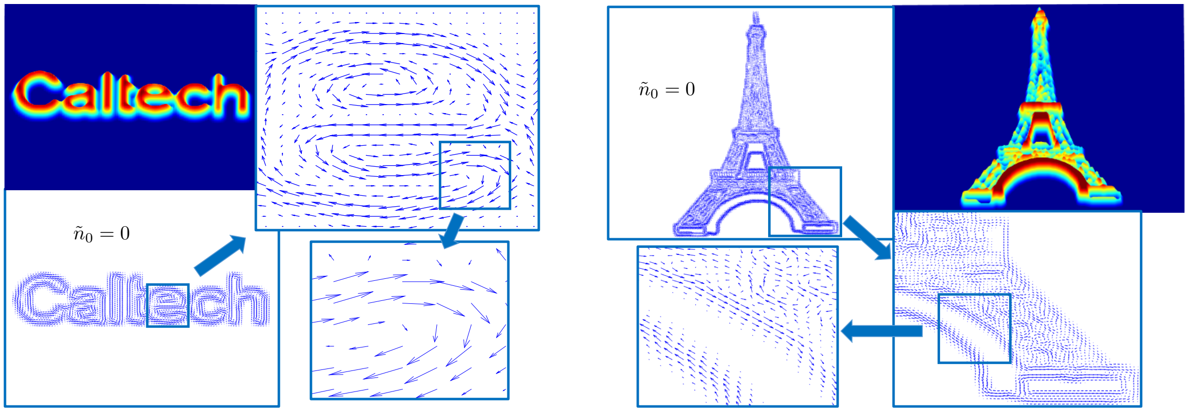
$$(\partial_1 y \cdot \partial_2 y)^2 = (|\partial_1 y|^2 - r^{-1/3})(|\partial_2 y|^2 - r^{-1/3}), \qquad \text{a.e. on } \omega. \tag{1.26}$$

Lifted surfaces constitute a broad class of deformations such that this constraint holds trivially.

Remark 1.11. (i) The surfaces of revolution in Aharoni et al. [1] and the designs exploring Gaussian curvature in Mostajeran [36] satisfy the conditions of Corollary 1.9. Thus, these designs and their predicted actuation are pure bending configurations in that they have entropic energy of $O(h^3)$ (which justifies that they are good candidates to be realized in actuation).

(ii) The lifted surfaces ansatz allows for actuation of a large variety of shapes, since the limitations imposed by (1.21) are not very restrictive. Since r can be significantly different from 1 in nematic elastomers, one can form shapes with significant displacement like spherical caps and sinusoidally rough surfaces. Figure 2 shows two additional examples with complex surface relief. These are but a small sample of the designs amenable to this framework. Indeed, given any arbitrary greyscale image $\mathcal{G}$, we can program a nematic sheet so that the surface of the sheet upon cooling corresponds to this image. We do this by smearing $\mathcal{G}$ (for instance by mollification or by averaging over a small square twice) and taking this as φ.

(iii) The key ingredient to the design of lifted surfaces is the ability to program the director three dimensionally. However, to our knowledge, experimental studies on nematic elastomer sheets such as Ware et. al. [41] have examined planar inscription of the director only. While we



(a) Caltech (b) Eiffel Tower

Figure 2: The deformed shape and designs for lifted surfaces. The planar part of the director $\tilde{n}_0$ is plotted.

hope that promising designs such as lifted surfaces will inspire future experimentation to realize three dimensional programming, in any case, the theory and design scheme are easily adapted to the planar case. Specifically, the metric constraint (1.6) reduces to the metric underlying Aharoni et al. [1] in the case of a planar program.

(iv) To arrive at the results presented in this section (fully detailed in Section 3.4), we employ techniques of Conti and Dolzmann [16], [17] to construct incompressible three dimensional deformations $Y^h \in C^1(\overline{\Omega}_h, \mathbb{R}^3)$. These techniques rely on the ability to approximate Sobolev functions by sufficiently smooth functions (see Section 3.1). In this direction, an important feature of lifted surfaces is that given any y as in (1.20) with φ as in (1.21), there exists a smooth y^h approximating y in the $W^{2,2}(\omega, \mathbb{R}^3)$ norm which additionally satisfies $\tilde{\nabla} y^h \in \mathcal{D}_r$ on ω. The space $\mathcal{D}_r$ can be thought of as the appropriate generalization to nematic anisotropy of the space of matrices representing isometries. Specifically, in the isotropic case $r = 1$, $\mathcal{D}_r$ reduces to $\mathcal{D}_1 = \{\tilde{F} \in \mathbb{R}^{3\times 2} \colon \tilde{F}^T \tilde{F} = I_{2\times 2}\}$. The corresponding function space

$$W^{2,2}_{iso}(\omega, \mathbb{R}^3) := \{y \in W^{2,2}_{iso}(\omega, \mathbb{R}^3) \colon (\tilde{\nabla} y)^T \tilde{\nabla} y = I_{2\times 2} \quad \text{a.e.}\}$$

has been extensively studied in the literature as this is the space of all bending deformations for isotropic sheets (as detailed rigorously by Friesecke et. al. [23]). For instance, Pakzad [37] showed that smooth isometric immersions are dense in $W^{2,2}_{iso}$ as long as the initially flat sheet ω is a convex regular domain. This was later generalized by Hornung [26] for flat sheets which belong to a much larger class of bounded and Lipschitz domains. For nematic elastomers, an appealing analogue to these results would be a similar density result for the space

$$W^{2,2}_{r}(\omega, \mathbb{R}^3) := \{y \in W^{2,2}(\omega, \mathbb{R}^3) \colon \tilde{\nabla} y \in \mathcal{D}_r \quad \text{a.e.}\}.$$

For instance, this space arises in compactness at the bending scale for the combined entropic and Frank energy studied in section 1.6. It does not appear that a result of this type has been considered so far. Our result for non-smooth midplane deformations satisfying $\tilde{\nabla} y \in \mathcal{D}_r$ a.e. is only stated for lifted surfaces, as these are the examples we can explicitly construct and approximate.

1.6 The metric constraint as a necessary condition for bending

We come to our last main result. So far, we exhibited constructions (nonisometric origami and lifted surfaces) which satisfy the metric constraint (1.6) and this guarantees that they have small entropic strain energy ($O(h^2)$ and $O(h^3)$ respectively).

The result we discuss now goes in the opposite direction. That is, we assume that the strain energy of a sequence of Y^h (indexed by h) is of order h^3 (i.e. is small) and we prove that then the midplane part of Y^h converges to a map $y : \omega \to \mathbb{R}^3$ satisfying the metric constraint.

This is a compactness result and as a technical ingredient it requires the Geometric Rigidity results of Friesecke, James and Müller [23]. To apply these in the present setting, we augment the original entropic elastic energy (1.1) with an additional term modeling (an approximation to) Frank elasticity. For this, we define the auxiliary map $N\colon \mathbb{R}^{3\times 3} \times \mathbb{S}^2 \to \mathbb{S}^2 \cup \{+\infty\}$ as

$$N(F, N_0) = \begin{cases} \frac{FN_0}{|FN_0|} & \text{if } |FN_0| \neq 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.27}$$

As the elastic energy augmented by a Frank elastic term, we take the function $\mathcal{I}^{h,\varepsilon}_{N_0^h}$ given by

$$\mathcal{I}^{h,\varepsilon}_{N_0^h}(Y^h) := \int_{\Omega_h} \left(\widehat{W}^e \left(\nabla Y^h, N(\nabla Y^h, N_0^h), N_0^h \right) + \varepsilon^2 |\nabla N(\nabla Y^h, N_0^h)|^2 \right) dx \tag{1.28}$$

for an appropriate space of admissible deformations (compare $\mathcal{A}^h(\Omega)$ below). Here we introduced $\widehat{W}^e := (\mu/2)^{-1} W^e$, the dimensionless entropic energy density, and the lengthscale $\varepsilon > 0$.

We come to the main result. For the statement, we find it convenient to *rescale the x_3 variable* so that we can work on the fixed domain $\Omega = \omega \times (-1/2, 1/2)$. This yields

$$W^h(z(x)) = Y^h(x), \qquad M_0^h(z(x)) = N_0^h(x), \qquad h^{-3}\mathcal{I}^{h,\varepsilon}_{N_0^h}(Y^h) = \mathcal{J}^{h,\varepsilon}_{M_0^h}(W^h) \tag{1.29}$$

$$\mathcal{J}^{h,\varepsilon}_{M_0^h}(W^h) := \int_\Omega \left(\frac{1}{h^2} \widehat{W}^e(\nabla_h W^h, N(\nabla_h W^h, M_0^h), M_0^h) + \frac{\varepsilon^2}{h^2} |\nabla_h N(\nabla_h W^h, M_0^h)|^2 \right) dz \tag{1.30}$$

where N is defined as in (1.27). We consider this energy functional on the space of admissible rescaled deformations

$$\mathcal{A}^h(\Omega) := \left\{ W^h \in W^{1,2}(\Omega, \mathbb{R}^3) \colon N(\nabla_h W^h, M_0^h) \in W^{1,2}(\Omega, \mathbb{S}^2) \right\}.$$

Here, for $f\colon \Omega \to \mathbb{R}^3$, we denote $\nabla_h f$ as $(\tilde{\nabla} f | \frac{1}{h} \partial_3 f)$, which reflects the rescaling of x_3 by $1/h$.

Given these rescalings, we have

Theorem 1.12 (Compactness)**.** *Let $r > 0$. Let $n_0 \in W^{1,2}(\omega, \mathbb{S}^2)$ and let*

$$c_l h \leq \varepsilon \equiv \varepsilon_h \leq c_u h, \tag{1.31}$$

for some constants $c_u \geq c_l > 0$. Moreover, let M_0^h satisfy

$$M_0^h(z) = n_0(\tilde{z}) + O(h), \quad \text{for a.e. } z \in \Omega, \quad \text{i.e., } \|M_0^h - n_0\|_{L^\infty(\Omega)} \leq \tau h. \tag{1.32}$$

For every sequence $\{W^h\} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ with $\mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h) \leq C < \infty$ as $h \to 0$, there exists a subsequence (not relabeled) and a $y \in W^{2,2}(\Omega, \mathbb{R}^3)$ independent of z_3 such that as $h \to 0$

$$\left(W^h - \frac{1}{|\Omega|} \int_\Omega W^h dz \right) \to y \quad \text{in } W^{1,2}(\Omega, \mathbb{R}^3) \quad \text{with} \quad (\tilde{\nabla} y)^T \tilde{\nabla} y = \tilde{\ell}_{n_0} \text{ a.e. on } \omega. \tag{1.33}$$

Remark 1.13. (i) Having control of the regularity $\nabla_h N(\nabla_h W^h, M_0^h)$ precisely with ε of order h is important to use Geometric Rigidity. In other words, we assume that Frank elasticity is comparable to entropic elasticity at the bending scale.

(ii) Let us discuss some background concerning Frank elasticity. Our augmented energy considers possible contributions due to Frank elasticity. Following de Gennes and Prost [20], Frank elasticity is a phenomenological continuum model for an energy penalizing distortions in the alignment of the current director n,

$$W_{Fr} = \frac{K_1}{2}(\text{div } n)^2 + \frac{K_2}{2}(n \cdot \text{curl } n)^2 + \frac{K_3}{2}|n \times \text{curl } n|^2.$$

Here, the three terms physically represent splay, twist and bend of the director field with respective moduli $K_1, K_2, K_3 > 0$. If the moduli are equal, i.e. $K_i = \kappa$ for $i = 1, 2, 3$, then W_{Fr} reduces to

$$W_{Fr} \equiv \frac{\kappa}{2}|\text{grad } n|^2. \tag{1.34}$$

More generally, since the moduli K_i are positive, we have the estimate

$$\frac{1}{2}\kappa_l|\text{grad } n|^2 \leq W_{Fr} \leq \frac{1}{2}\kappa_u|\text{grad } n|^2 \tag{1.35}$$

for $\kappa_l = \min\{K_1, K_2, K_3\}$ and $\kappa_u = \max\{K_1, K_2, K_3\}$.

We are interested foremost in how Frank energy may compete with the entropic energy at the bending scale. Thus, we consider only the simplified model (1.34) since the detailed model is sandwiched energetically by models of this type (1.35). We make a further assumption regarding how distortions in nematic alignment are accounted in the energetic framework. To elaborate, a model for Frank elasticity should ideally penalize spatial distortions in the alignment of the director field, i.e., the div, curl and grad operators should be with respect to the current frame. Unfortunately, this seems quite technical to capture in a variational setting, as notions of invertibility of Sobolev maps must be carefully considered. It is, however, an active topic of mathematical research. For instance, we refer the interested reader to the works of Barchiesi and DeSimone [3] and Barchiesi et al. [4] for Frank elasticity and nematic elastomers in this context. Nevertheless, for our purpose in understanding whether the metric constraint (1.6) is necessitated by a smallness in the energy, we find it sufficiently interesting to consider the simplified model

$$W_{Fr} \approx \frac{\kappa}{2}|\nabla N^h|^2, \quad N^h \colon \Omega_h \to \mathbb{S}^2 \tag{1.36}$$

where N^h refers to the current director field as a mapping from the initially flat sheet Ω_h and ∇ is the gradient with respect to this reference state. The term (1.36) is the one that appears in the energy (1.28).

(iii) The parameter $\varepsilon = \sqrt{\kappa/\mu}$ is likely quite small in nematic elastomers. Specifically, in liquid crystal fluids, the moduli K_i (which bound κ) have been measured in detail, and these moduli are likely similar for nematic elastomers (see, for instance, the discussion in Chapter 3 [43]). Further, the shear modulus μ of the rubbery network, which is distinct to elastomers, is much larger. Substituting the typical values for these parameters, we find $\varepsilon \sim 10 - 100nm$. Thus, entropic elasticity will often dominate Frank elasticity in these elastomers. However, a typical

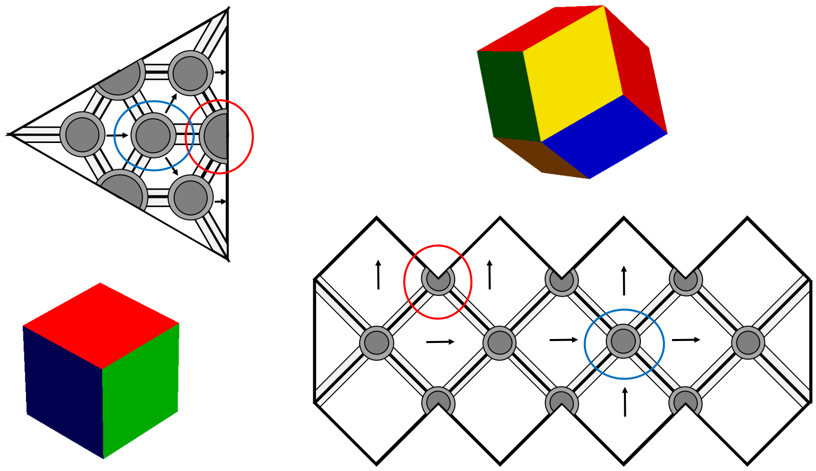


Figure 3: Sketch of smoothing procedure to approximate the Box and Rhombic Dodecahedron deformation. The interfaces are smoothed one dimensionally, while each junction is repaired in a disk or partial disk. For these symmetric constructions, there is a class of constructions suitable for all interior cases in blue and another class suitable for all the boundary cases in red.

thin sheet will have a thickness $h \sim 10 - 100\mu m$. So there are two small lengthscales to consider in this problem. For mechanical boundary conditions which induce stretch and stress in these sheets, the entropic energy does appear to dominate the Frank term. For instance, stripe domains of oscillating nematic orientation would be suppressed by a large Frank energy, and yet these have been observed by Kundler and Finkelmann [28] in the clamped stretch experiments on thin sheets. Mathematically, this dominance under stretch is made precise, for instance, by Cesana et al. [13] in studying an energy of type (1.28) for a general director N and the resulting membrane theory does not depend on Frank elasticity. These results notwithstanding, actuation of nematic sheets with controlled heterogeneity occurs at a much lower energy state. Therefore, it is possible that the actuated configuration emerges from a non-trivial competition between entropic and Frank elasticity at these small energy scales. Hence, we study this competition in an asymptotic sense by taking h and $\varepsilon \to 0$.

2 Approximating the two-dimensional deformations for idealized actuation

We now catalogue several results regarding the approximation of non-smooth two-dimensional deformations $y\colon \omega \to \mathbb{R}^3$ on a lengthscale $\delta > 0$ via smoothings $y^\delta\colon \omega \to \mathbb{R}^3$. Each of these constructions is amenable to a three dimensional incompressible extension for small h with $\delta = O(h)$. In Sections 2.1-2.3, we construct smooth approximations to junctions which form the unit cells of the nonisometric origami examples in Figure 13. These smoothings satisfy the approximation hypothesis (1.16), crucial to our energy argument. The idea is sketched in Figure 3. In section 2.4, we piece together these constructions to obtain global midplane deformations satisfying (1.16) for these examples of nonisometric origami. In addition, we construct an optimal vector associated to the out-of-plane deformation gradient at the midplane for any approximation to nonisometric origami in (1.16). Finally, in Section 2.5, we construct appropriate approximations to the lifted surface ansatz.

2.1 Smoothing with full-rank at each interface

Suppose ω is the union of K connected sectors about a junction O with each sector ω_α described by the angle θ_α. We assume $y\colon \omega \to \mathbb{R}^3$ is a continuous piecewise affine deformation satisfying

$$y(\tilde{x}) = \tilde{F}_\alpha \tilde{x} \quad \text{for} \quad \tilde{x} \in \omega_\alpha$$

and such that each $\tilde{F}_\alpha \in \mathbb{R}^{3\times 2}$ has $\operatorname{rank} \tilde{F}_\alpha = 2$. Let each interface be described by the outward tangent $g_\alpha \in \mathbb{S}^1$ as in Figure 4a with $g_\alpha^\perp$ its perpendicular as shown. We set $\tilde{x} := x_1 e_1 + x_2 e_2 = s_\alpha g_\alpha + t_\alpha g_\alpha^\perp$, and define $y_\alpha \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$y_\alpha(s_\alpha, t_\alpha) = \begin{cases} s_\alpha \tilde{F}_\alpha g_\alpha + t_\alpha \tilde{F}_\alpha g_\alpha^\perp & t_\alpha \leq 0 \\ s_\alpha \tilde{F}_{\alpha+1} g_\alpha + t_\alpha \tilde{F}_{\alpha+1} g_\alpha^\perp & t_\alpha > 0. \end{cases} \tag{2.1}$$

If $\alpha = K$, then $\alpha + 1 = 1$. Next, for the region $\omega_{g_\alpha} \subset \omega$ given by the sector of angle $(\theta_\alpha + \theta_{\alpha+1})/2$ as in the schematic in Figure 4a, we define $y_\alpha^\delta \colon \omega_{g_\alpha} \to \mathbb{R}^3$ given by

$$y_\alpha^\delta(s_\alpha, t_\alpha) = \int_{I_{\delta/2}(t_\alpha)} \eta_h(t_\alpha - t) y_\alpha(s_\alpha, t) dt \quad \text{on } \omega_{g_\alpha}. \tag{2.2}$$

Here $I_{\delta/2}(t_\alpha) = (-\delta/2 + t_\alpha, \delta/2 + t_\alpha)$, and η_δ is a standard symmetric mollifier with support on the interval $I_{\delta/2}(0)$. Finally, we set $y_0^\delta \colon \omega \to \mathbb{R}^3$ as

$$y_0^\delta = \begin{cases} y_\alpha^\delta & \text{on each } \omega_{g_\alpha} \\ y & \text{otherwise on } \omega. \end{cases} \tag{2.3}$$

Note that the exceptional set occurs if $\Sigma_{\alpha=1}^K \theta_\alpha < 2\pi$, in which case the junction has a boundary. We make the following observations regarding this construction:

Proposition 2.1. *Let $\theta_0 := \min\{\theta_\alpha \colon \alpha \in \{1, \ldots, K\}\}$, assume $\delta \ll 1$ and set $\rho_0^\delta := \delta/\sin(\theta_0/2)$. Let $\Gamma \subset \omega$ be the set containing all g_α interfaces and set $\Gamma_\delta := \{\tilde{x} \in \omega \colon \operatorname{dist}(\tilde{x}, \Gamma) < \delta/2\}$. If at each g_α interface*

$$\lambda \tilde{F}_\alpha g_\alpha \times \tilde{F}_\alpha g_\alpha^\perp + (1-\lambda) \tilde{F}_{\alpha+1} g_\alpha \times \tilde{F}_{\alpha+1} g_\alpha^\perp \neq 0 \quad \forall \ \lambda \in [0,1], \tag{2.4}$$

then the restriction of y_0^δ to $\omega \setminus B_{\rho_0^\delta}(O)$ is in $C^\infty(\overline{\omega \setminus B_{\rho_0^\delta}(O)}, \mathbb{R}^3)$ and satisfies

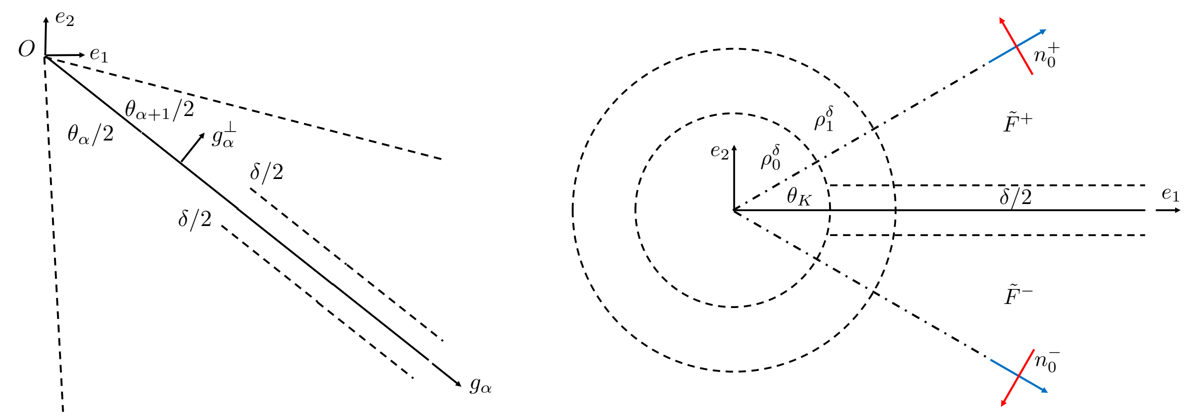
$$\|\partial_1 y_0^\delta \times \partial_2 y_0^h\|_{L^\infty} \geq c > 0, \quad \|\tilde{\nabla} y_0^\delta\|_{L^\infty} \leq C, \quad \|\tilde{\nabla}\tilde{\nabla} y_0^\delta\|_{L^\infty} \leq C h^{-1}, \quad \|\tilde{\nabla}^{(3)} y_0^\delta\|_{L^\infty} \leq C\delta^{-2} \tag{2.5}$$

on this set for some c, C depending only on y. Further, y_0^δ has the representation

$$y_0^\delta = y \quad \text{on } \omega \setminus (B_{\rho_0^\delta}(O) \cup \Gamma_\delta) \tag{2.6}$$
$$y_0^\delta(\tilde{x}) = s_\alpha \tilde{F}_\alpha g_\alpha + \gamma_\delta(t_\alpha) \tilde{F}_\alpha g_\alpha^\perp + (t_\alpha - \gamma_\delta(t_\alpha)) \tilde{F}_{\alpha+1} g_\alpha^\perp \quad \text{on each } \omega_{g_\alpha} \tag{2.7}$$

with $\gamma_\delta \in C^\infty(\mathbb{R}, \mathbb{R})$ given by

$$\gamma_\delta(t_\alpha) := \int_{I_{\delta/2}(t_\alpha) \cap \{t \leq 0\}} \eta_\delta(t_\alpha - t) t dt \tag{2.8}$$



(a) Local cooridnate system for α interface

(b) Symmetric Junctions construction

Figure 4: Building blocks to smoothing the piecewise affine constructions

and satisfying

$$\gamma_\delta(t_\alpha) = \begin{cases} t_\alpha & \text{for } t_\alpha < -\delta/2 \\ 0 & \text{for } t_\alpha > h/2 \end{cases} \tag{2.9}$$

$$\gamma_\delta'(t_\alpha) = \int_{I_{h/2}(t_\alpha)\cap\{t\leq 0\}} \eta_\delta(t_\alpha - t)dt =: \lambda_\delta(t_\alpha) \tag{2.10}$$

for $\lambda_\delta \in C^\infty(\mathbb{R}, [0,1])$.

Before we prove this result, we note that if one is interested in actuating three dimensional shape that is idealized as a piecewise affine deformation from a flat sheet, then shapes which fold into themselves are unphysical. This is what is meant by the assumption in (2.4). If this assumption holds, then the proposition provides a construction for smoothing the piecewise affine deformation away from the junction O, and this smoothing has nice properties (2.5). If δ is small, then this smoothing is exactly the piecewise affine deformation desired for actuation outside a small region of measure $O(\delta)$ as shown in (2.6). The remaining representation formulas in the proposition will be useful later for our construction to repair the region around the junction.

Proof. To begin, observe that since by hypothesis y is a continuous piecewise affine deformation, and each g_α interface is straight,

$$\tilde F_\alpha g_\alpha = \tilde F_{\alpha+1} g_\alpha =: f_\alpha \quad \text{for each } g_\alpha. \tag{2.11}$$

Thus given (2.1), $s_\alpha \mapsto y_\alpha(s_\alpha, t_\alpha)$ is smooth for t_α fixed. Hence, since y_α^δ is a δ-mollification of y_α in the direction for which y_α is not smooth but strictly Lipschitz continuous, we conclude $y_\alpha^\delta \in C^\infty(\mathbb{R}^2, \mathbb{R}^3)$ with

$$\|\tilde\nabla y_\alpha^\delta\|_{L^\infty} \leq C \quad \|\tilde\nabla\tilde\nabla y_\alpha^\delta\|_{L^\infty} \leq C\delta^{-1} \quad \|\tilde\nabla^{(3)} y_\alpha^\delta\|_{L^\infty} \leq C\delta^{-2} \tag{2.12}$$

where C depends only on y_α and so y.

Now, combining (2.1), (2.2), (2.3) and (2.11), we obtain the local characterization of y_0^δ in (2.7) using the fact that $\int_{\mathbb{R}} \eta_\delta(t)dt = 1$. Thus on interior of each ω_{g_α}, we find that the gradient operator in the global frame is given by

$$\tilde\nabla y_0^\delta(\tilde x) = (\partial_1 y_\alpha^\delta | \partial_2 y_\alpha^\delta)(\tilde x) = \Big(\lambda_\delta(t_\alpha)(\tilde F_\alpha g_\alpha | \tilde F_\alpha g_\alpha^\perp) + (1-\lambda_\delta(t_\alpha))(\tilde F_{\alpha+1} g_\alpha | \tilde F_{\alpha+1} g_\alpha^\perp)\Big)\, \tilde R_\alpha \tag{2.13}$$

where $\tilde{R}_\alpha := e_1 \otimes \tilde{\nabla} s_\alpha + e_2 \otimes \tilde{\nabla} t_\alpha$ is a constant on $SO(2)$. Note that we have assumed in this calculation that $\gamma_\delta'(t) = \lambda_\delta(t)$. This requires justification.

When $\delta/2 + t_\alpha \leq 0$, we find $I_{\delta/2}(t_\alpha) \cap \{t \leq 0\} = I_{\delta/2}(t_\alpha)$ and

$$\gamma_\delta(t_\alpha) = \int_{I_{\delta/2}(t_\alpha)} \eta_\delta(t_\alpha - t) t dt = \int_{-\delta/2}^{\delta/2} \eta_\delta(\tau)(t_\alpha - \tau) d\tau = t_\alpha$$

since η_δ is a symmetric mollifier and τ is antisymmetric, and so there product vanishes upon this integration. Alternatively, if $-\delta/2 + t_\alpha \geq 0$, then $|I_{\delta/2}(t_\alpha) \cap \{t \leq 0\}| = 0$ and thus $\gamma_\delta(t_\alpha) = 0$ in this case. These two results prove (2.9). For (2.10), we consider the case $-\delta/2 + t_\alpha < 0 < \delta/2 + t_\alpha$. In this case,

$$\begin{aligned}
\gamma_\delta'(t_\alpha) &= \int_{-\delta/2+t_\alpha}^{0} \eta_h'(t_\alpha - t) t dt = \int_{t_\alpha}^{\delta/2} \eta_\delta'(\tau)(t_\alpha - \tau) d\tau \\
&= -\eta_\delta(t_\alpha) t_\alpha - \int_{t_\alpha}^{h/2} (\eta_\delta(\tau)\tau)' d\tau - \int_{t_\alpha}^{\delta/2} \eta_\delta(\tau) d\tau \\
&= \int_{-\delta/2+t_\alpha}^{0} \eta_\delta(t_\alpha - t) dt = \lambda_\delta(t_\alpha)
\end{aligned}$$

using various properties of the mollifier. In addition, given that the derivatives of the result in (2.9) coincide with λ_δ on these exceptional sets, we conclude (2.10) Thus, the formula in (2.13) is justified.

Now, we note that for $\tilde{G} \in \mathbb{R}^{3\times 2}$ and $\tilde{R} \in SO(2)$, $|\tilde{G}\tilde{R}e_1 \times \tilde{G}\tilde{R}e_2| = |\tilde{G}e_1 \times \tilde{G}e_2|$. Hence, combining this observation with (2.13), we find that on the interior of ω_{g_α}

$$\begin{aligned}
|\partial_1 y_0^\delta \times \partial_2 y_0^\delta|(\tilde{x}) &= |f_\alpha \times (\lambda_\delta(t_\alpha)\tilde{F}_\alpha g_\alpha^\perp + (1 - \lambda_\delta(t_\alpha))\tilde{F}_{\alpha+1} g_\alpha^\perp)| \\
&= |\lambda_\delta(t_\alpha) f_\alpha \times \tilde{F}_\alpha g_\alpha^\perp + (1 - \lambda_\delta(t_\alpha)) f_\alpha \times \tilde{F}_{\alpha+1} g_\alpha^\perp| \\
&\geq \inf_{\lambda \in [0,1]} |\lambda f_\alpha \times \tilde{F}_\alpha g_\alpha^\perp + (1 - \lambda) f_\alpha \times \tilde{F}_{\alpha+1} g_\alpha^\perp| \geq c_\alpha > 0
\end{aligned} \tag{2.14}$$

where we have also used (2.11) and the fact that λ_δ maps to $[0,1]$. That this infimum is given by some $c_\alpha > 0$ is a consequence of (2.4). Indeed, since $\tilde{F}_\alpha$ and $\tilde{F}_{\alpha+1}$ are full rank, we need only consider $\lambda \in (0,1)$. Then by (2.4) and (2.11), $\lambda f_\alpha \times \tilde{F}_\alpha g_\alpha \neq (\lambda - 1) f_\alpha \times \tilde{F}_{\alpha+1} g_\alpha$ for any $\lambda \in (0,1)$, and this result implies a $c_\alpha > 0$.

We are now prepared to show that y_0^δ is smooth in $\omega \setminus B_{\rho_0^\delta}(O)$ and satisfies (2.5) and (2.6). For (2.6), note that by (2.7) and (2.9), we find $y_0^\delta = y$ on $\omega_{g_\alpha} \setminus \Gamma_\delta$. Thus by (2.3), we see that actually $y_0^\delta = y$ on $\omega \setminus \Gamma_\delta$. Now $(B_{\rho_0^\delta}(O) \cap \omega) \not\subset \Gamma_\delta$. This implies that on the radial boundaries of $\omega_{g_\alpha} \setminus B_{\rho_0^\delta}(O)$, $y_0^\delta = y$. Therefore, since the piecewise affine y has no jumps in its gradient on this set and since $y_0^\delta = y_\alpha^\delta$ on the interior of ω_{g_α} with y_α^δ smooth, we conclude that y_0^δ is smooth on $\omega \setminus B_{\rho_0^\delta}(O)$. Finally y_0^δ satisfies (2.6) since we have (2.12) and (2.14) and noting that the number of g_α interfaces is finite. □

2.2 Smoothing with full-rank in symmetric junctions: Interior Cases

We now consider the smoothing of each of the junctions given by the symmetric problem described in Figure 4b. The difficulty here is to smooth so that the midplane deformation after smoothing is full-rank everywhere. The simplest case to consider is the interior junctions where for K interfaces with $K \geq 3$, $\theta_K = \pi/K$ and the directors n_0^+ and n_0^- are planar and given in blue in the figure. We have implicitly assumed the cooling case $r > 1$ for this analysis. The exact analogue for heating can be obtained replacing the directors with their respective perpendiculars in the plane. In the cooling cases, we obtain a compatible pyramidal type deformation where in the global $\{e_1, e_2\}$ frame,

$$
\begin{aligned}
y_{e_1}(\tilde{x}) &:= \begin{cases} \tilde{F}^{-}\tilde{x} & \text{for } \tilde{x} \in \omega_{e_1} \cap \{x_2 \leq 0\} \\ \tilde{F}^{+}\tilde{x} & \text{for } \tilde{x} \in \omega_{e_1} \cap \{x_2 > 0\}, \end{cases} \\
\tilde{F}^{\pm} &= r^{-1/6} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{r-1}\cos(\pi/K) & \pm\sqrt{r-1}\sin(\pi/K) \end{pmatrix}
\end{aligned} \tag{2.15}
$$

with $\omega_{e_1} \subset \omega$ the sector described in Figure 4b. Then the complete pyramidal piecewise affine midplane deformation $\bar{y}\colon \omega \to \mathbb{R}^3$ corresponding to this design is given by

$$
y(\tilde{x}) = R_K^\alpha y_{e_1}((\tilde{R}_K^\alpha)^T \tilde{x}) \quad \text{for } \tilde{x} \in \omega_{\tilde{R}_K^\alpha e_1}, \quad \alpha \in \{0, \dots, K-1\}. \tag{2.16}
$$

Here $R_K^\alpha = R_{e_3}(2\pi\alpha/K) \in SO(3)$ and $\tilde{R}_K^\alpha = \tilde{R}_{e_3}(2\pi\alpha/K) \in SO(2)$ for

$$
R_{e_3}(\theta) := \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}_{e_3}(\theta) := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{2.17}
$$

and $\omega_{\tilde{R}_K^\alpha e_1}$ is the sector described by the $\tilde{R}_K^\alpha$ rotated ω_{e_1} depicted in Figure 4b.

This explicit construction characterizes the interior junctions circled in blue in Figure 3. Specifically, up to a rotation of the entire junction, the interior junctions for the box and rhombic dodecahedron are given explicitly by this result with $K = 3$ and 4 respectively. To smooth out the junction and maintain full rank in these cases is relatively straightforward. The intuition is that the projection of these deformations onto the $\{e_1, e_2\}$ plane is a homogeneous contraction $r^{-1/6} I_{2\times 2}$ as shown in Figure 5. This observation motivates a simple construction:

Proposition 2.2. *Let y as in (2.16) with $r > 1$ and assume $\delta \ll 1$. Let $\Gamma \subset \omega$ be the set containing all the $\tilde{R}_K^\alpha e_1$ interfaces, and $\Gamma_\delta := \{\tilde{x} \in \omega\colon \operatorname{dist}(\tilde{x}, \Gamma) < \delta/2\}$. Let $\rho_0^\delta := \delta/\sin(\pi/K)$ and $\rho_1^\delta := \delta/\sin(\pi/K) + \delta/2$ (see Figure 4b). There exists a $y^\delta \in C^\infty(\bar{\omega}, \mathbb{R}^3)$ such that*

$$
\|\partial_1 y^\delta \times \partial_2 y^\delta\|_{L^\infty} \geq c > 0, \quad \|\tilde{\nabla} y^\delta\|_{L^\infty} \leq C, \quad \|\tilde{\nabla}\tilde{\nabla} y^\delta\| \leq C\delta^{-1}, \quad \|\tilde{\nabla}^{(3)} y^\delta\|_{L^\infty} \leq C\delta^{-2} \tag{2.18}
$$

everywhere on ω for c, C depending only on y. Moreover,

$$
y^\delta = y \quad \text{on } \omega \setminus (B_{\rho_1^\delta}(O) \cup \Gamma_\delta). \tag{2.19}
$$

Proof. We apply Proposition 2.1 with $\theta_0/2 = \pi/K$ to obtain a $\bar{y}_0^\delta$ which has all the desired properties outside of the ball $B_{\rho_0^\delta}(O)$. We now modify this function in a superset of this bad set, the ball $B_{\rho_1^\delta}(O)$. Specifically, we define

$$
y^\delta = y_0^\delta + \psi_\delta(y_c - y_0^\delta)
$$

for $\psi_\delta \in C^\infty(\mathbb{R}, [0,1])$ a smooth radial cutoff function $\psi_\delta = \psi_\delta(\rho)$ equal to 1 in $B_{\rho_0^\delta}(O)$ and 0 outside of $B_{\rho_1^\delta}(O)$. Thus, since the restriction of y_0^δ to $\omega \setminus B_{\rho_0^\delta}(O)$ belongs to $C^\infty(\overline{\omega \setminus B_{\rho_0^\delta}(O)}, \mathbb{R}^3)$, we see that $y^\delta \in C^\infty(\bar{\omega}, \mathbb{R}^3)$ so long as $y_c \in C^\infty(\bar{\omega}, \mathbb{R}^3)$.

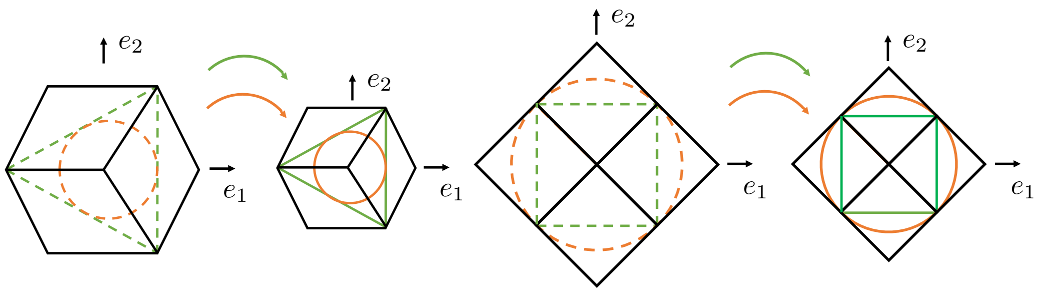


Figure 5: The interior junction for the box (left) and rhombic dodecahedron (right). The flat sheet becomes a pyramid, but in the projection shown, the dashed lines are uniformly contracted to the embolden lines.

Since the projection of y in (2.16) to the $\{e_1, e_2\}$ plane is a homogeneous contraction $r^{-1/6}(x_1e_1 + x_2e_2)$ (see also Figure 4b), we take this as y_c, i.e.

$$y_c = r^{-1/6}\rho e_\rho = r^{-1/6}(x_1e_1 + x_2e_2). \tag{2.20}$$

Hence, $y^\delta \in C^\infty(\bar{\omega}, \mathbb{R}^3)$ and we have the equality in (2.19) by (2.6) in Proposition 2.1.

It remains to show the estimates in (2.18). These hold trivially in $B_{\rho_0^\delta}(O)$ since $y^\delta = y_c$ in this case. In fact, the bound on the gradients in this regime is independent of $h \ll 1$. Outside of $B_{\rho_1^\delta}(O)$, $y^\delta = y_0^\delta$ and these estimates follow from (2.5) in Proposition 2.1. Finally, on the punctured disk $B_{\rho_1^\delta}(O) \setminus B_{\rho_0^\delta}(O)$ (see Figure 4b), we compute

$$\tilde{\nabla} y^\delta = (1 - \psi_\delta)\tilde{\nabla} y_0^\delta + \psi_\delta \tilde{\nabla} y_c + \psi_\delta'(y_c - y_0^\delta) \otimes e_\rho.$$

We note that $|\psi_\delta'| \leq C\delta^{-1}$ while $|y_c - y_0^\delta| \leq C\delta$, hence the third term above is at most $O(1)$. Thus, we conclude $|\tilde{\nabla} y^\delta| \leq C$ for C depending only on y since $\tilde{\nabla} y_0^\delta$ has this property (Proposition 2.1). Likewise, we find $|\tilde{\nabla}\tilde{\nabla} y^\delta| \leq C\delta^{-1}$ and $|\tilde{\nabla}^{(3)} y^\delta| \leq C\delta^{-2}$ by the chain rule since y_0^δ has this property and we may choose a cutoff function satisfying $|\psi_\delta''| \leq C\delta^{-2}$ and $|\psi_\delta^{(3)}| \leq C\delta^{-3}$ where each C above is uniform or depends only on y.

For the cross-product bound, we denote $\tilde{y}^\delta : \omega \to \mathbb{R}^2$ as the projection of y^δ to the $\{e_1, e_2\}$ plane (and similarly for the other functions involved). Given this, we have $|\partial_1 y^\delta \times \partial_2 y^\delta| \geq |\det(\tilde{\nabla}\tilde{y}^\delta)|$, so we simply compute this determinant explicitly and show it does not vanish to prove the result. We note that in this calculation it suffices to consider $\tilde{y}^\delta$ restricted to ω_{e_1}, i.e. $\tilde{y}^\delta_{e_1}$, as in Figure 4b since it can be verified (akin to (2.16)) that

$$\tilde{y}^\delta(\tilde{x}) = \tilde{R}_K^\alpha \tilde{y}^\delta_{e_1}((\tilde{R}_K^\alpha)^T\tilde{x}), \quad \tilde{x} \in \omega_{\tilde{R}_K^\alpha e_1}, \quad \alpha = \{0, \dots, K-1\}.$$

This leads to the identity $\det(\tilde{\nabla}\tilde{y}^\delta(\tilde{x})) = \det(\tilde{\nabla}\tilde{y}^\delta_{e_1}((\tilde{R}_K^\alpha)^T\tilde{x}))$, hence the sufficiency.

For this computation, we observe that $\tilde{F}^+_{2\times 2} = \tilde{F}^-_{2\times 2} = \bar{r}^{-1/6} I_{2\times 2}$ where $\tilde{F}^\pm_{2\times 2} \in \mathbb{R}^{2\times 2}$ is the 2×2 submatrix of $\tilde{F}^\pm$ in (2.15) associated to the projection $\tilde{y}$. Thus, following (2.7), we observe

$$\tilde{y}_0^\delta(\tilde{x}) = x_1\tilde{F}^-_{2\times 2}e_1 + \gamma_\delta(x_2)\tilde{F}^-_{2\times 2}e_2 + (x_2 - \gamma_\delta(x_2))\tilde{F}^+_{2\times 2}e_2 = r^{-1/6}I_{2\times 2}\tilde{x} \quad \text{ on } \omega_{e_1},$$

and following (2.20), we also have $\tilde{y}_c(\tilde{x}) = r^{-1/6}I_{2\times 2}\tilde{x}$ on ω. Thus, we conclude

$$\tilde{\nabla}\tilde{y}^\delta = (1 - \psi_\delta)\tilde{\nabla}\tilde{y}_0^\delta + \psi_\delta\tilde{\nabla}\tilde{y}_c + \psi_\delta'(\tilde{y}_c - \tilde{y}_0^\delta) \otimes e_\rho = r^{-1/6}I_{2\times 2}\tilde{x} \quad \text{ on } \omega_{e_1}.$$

Evidently then, $\det \tilde{\nabla}\tilde{y}^\delta = r^{-1/3}$ everywhere on ω, so the proof is complete. $\square$

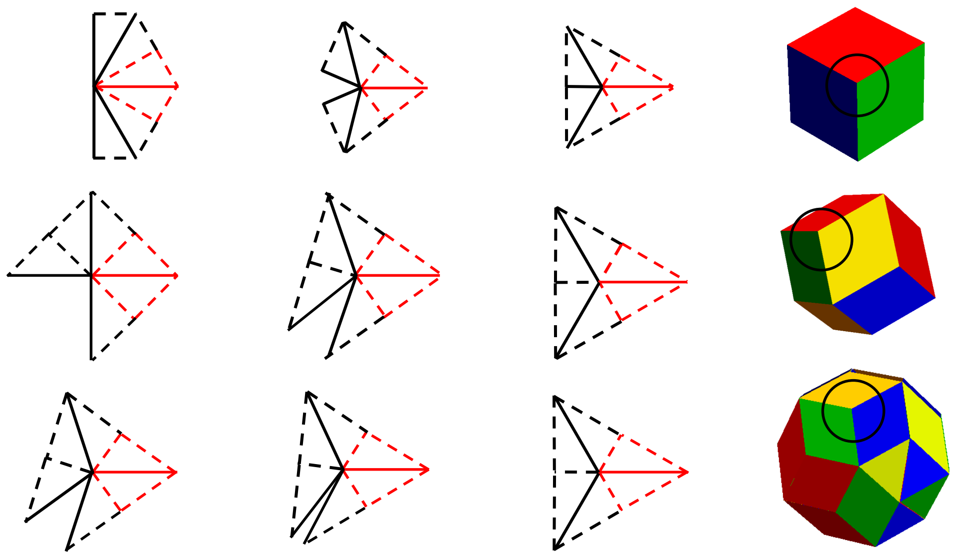


Figure 6: The unit cell in Figure 4b is emphasized in red. (Far Left) The initially undeformed flat boundary case. (Left) The projection of an intermediate deformation onto the $\{e_1, e_2\}$ plane. (Right) When the regions finally merge to form a complete corner at $r = 3$ (box), $= 2$ (rhombic dodecahedron) ≈ 1.38 (rhombic triacontahedron).

2.3 Smoothing with full-rank in symmetric junctions: Boundary Cases

Next, we turn to the boundary cases characterized by the circled red regions in Figure 3. We can capture deformations where these boundaries eventually merge to form a regular polyhedra by careful consideration of the unit cell in Figure 4b with the director now given in red. In these cases, we have the following description of the unit cell in the global $\{e_1, e_2\}$ frame:

$$y_{e_1}(\tilde{x}) := \begin{cases} \tilde{F}^- \tilde{x} & \text{for } \tilde{x} \in \omega_{e_1} \cap \{x_2 \leq 0\} \\ \tilde{F}^+ \tilde{x} & \text{for } \tilde{x} \in \omega_{e_1} \cap \{x_2 > 0\}, \end{cases} \tag{2.21}$$

$$\tilde{F}^{\pm} = \frac{r^{5/6}}{c_K^r} \begin{pmatrix} 1 & \mp \frac{(r-1)}{r \tan(\theta_K)} \\ 0 & \frac{c_K^r}{r} \\ \frac{\sqrt{r-1}\cos(\theta_K)}{r \tan(\theta_K)} & \pm \frac{\sqrt{r-1}\cos(\theta_K)}{r} \end{pmatrix} \tag{2.22}$$

$$c_K^r := \frac{\sqrt{r - \cos^2(\theta_K)}}{\sin(\theta_K)}$$

with ω_{e_1} the sector described in Figure 4b.

This unit cell captures the deformation at boundaries where a flat sheet merges to form a corner of the box, rhombic dodecahedron or rhombic triacontahedron. In particular, for these corners, we have

$$\theta_K \equiv \theta_3 = \pi/6: \quad y_{box}(\tilde{x}) = Q_3^{\alpha} y_{e_1}((\tilde{R}_3^{\alpha})^T \tilde{x}), \quad \tilde{x} \in \omega_{\tilde{R}_3^{\alpha} e_1} \quad \alpha \in \{0, \pm 1\};$$

$$\theta_K \equiv \theta_4 = \pi/4: \quad y_{rd}(\tilde{x}) = \begin{cases} Q_4^{\alpha} y_{e_1}((\tilde{R}_4^{\alpha})^T \tilde{x}), & \tilde{x} \in \omega_{\tilde{R}_4^{\alpha} e_1} \quad \alpha \in \{0, 1\} \\ Q_4^{-1} y_{e_1}((\tilde{R}_4^{-1})^T \tilde{x}), & \tilde{x} \in \omega^{+}_{\tilde{R}_4^{-1} e_1} \\ Q_4^{2} y_{e_1}((\tilde{R}_4^{2})^T \tilde{x}), & \tilde{x} \in \omega^{-}_{\tilde{R}_4^{2} e_1}; \end{cases} \tag{2.23}$$

$$\theta_K \equiv \theta_5 = 3\pi/10: \quad y_{rt}(\tilde{x}) = \begin{cases} Q_5^{\alpha} y_{e_1}((\tilde{R}_5^{\alpha})^T \tilde{x}), & \tilde{x} \in \omega_{\tilde{R}_5^{\alpha} e_1} \quad \alpha \in \{0,1\} \\ Q_5^{-1} y_{e_1}((\tilde{R}_5^{-1})^T \tilde{x}), & \tilde{x} \in \omega^{+}_{\tilde{R}_5^{-1} e_1} \\ Q_5^{2} y_{e_1}((\tilde{R}_5^{2})^T \tilde{x}), & \tilde{x} \in \omega^{-}_{\tilde{R}_5^{2} e_1} \end{cases}$$

for y_{e_1} as in (2.21) and $\tilde{F}^{\pm}$ as in (2.22), $\tilde{R}_K^{\alpha} = \tilde{R}_{e_1}(2\alpha\theta_K)$ for $\tilde{R}_{e_1}$ in (2.17), and $Q_K^{\alpha} = R_{e_3}(2\alpha\tilde{\theta}_K)$ with

$$\tilde{\theta}_K := \arctan\left(c_K^r \tan(\theta_K)\right) \tag{2.24}$$

and for R_{e_3} in (2.17). In the diagram in Figure 6, we have plotted these deformations projected onto the $\{e_1, e_2\}$ plane. The key observations for these boundary cases is periodicity, and the fact that all these deformations have consistent behavior and this holds across all relevant values of $r \geq 1$. These observations allow us to focus exclusively on the unit cell in Figure 4b via a modification consistent across all cases to obtain a full rank smooth ($C^3(\bar{\omega})$) junction, Figure 7.

Proposition 2.3. *Let $y \in \{y_{box}, y_{rd}, y_{rt}\}$, and let $r \in [1, r_c]$ where r_c defines the deformation when the boundaries merge to form a corner ($r_c = 3$ (box), $= 2$ (rhombic dodecahedron), ≈ 1.38 (rhombic triacontahedron)). Assume $\delta \ll 1$, and let $\Gamma \subset \omega$ be the set containing all the $\tilde{R}_K^{\alpha} e_1$ interfaces and $\Gamma_\delta := \{\tilde{x} \in \omega \colon \operatorname{dist}(\tilde{x}, \Gamma) < \delta/2\}$. There exists a $\rho_1^\delta = M\delta$ and a $y^\delta \in C^3(\bar{\omega}, \mathbb{R}^3)$ such that*

$$y^\delta = y \quad \text{on } \omega \setminus (B_{\rho_1^\delta}(O) \cup \Gamma_\delta). \tag{2.25}$$

for some $M \geq 1/\sin(\theta_K)$ depending only on y. Moreover, y^δ satisfies the estimates

$$\|\partial_1 y^\delta \times \partial_2 y^\delta\|_{L^\infty} \geq c > 0, \quad \|\tilde{\nabla} y^\delta\| \leq C, \quad \|\tilde{\nabla}\tilde{\nabla} y^\delta\|_{L^\infty} \leq C\delta^{-1}, \quad \|\tilde{\nabla}^{(3)} y^\delta\|_{L^\infty} \leq C\delta^{-2} \tag{2.26}$$

for c, C depending only on y.

We prove this proposition in three steps. In the first step, we repair y on the ball $B_{\rho_1^\delta}(O)$ via the deformation described in Figure 7. This construction is good (i.e. satisfies (2.25) and (2.26)) outside of a neighborhood of the origin O. However, in this neighborhood its second and third derivative become unbounded. Thus in the second step, we modify the construction of the first step in the ball $B_{\delta/2}(O)$ to obtain a construction in $C^3(\bar{\omega}, \mathbb{R}^3)$ which satisfies the upper bound estimates in (2.26). However, the rank of the gradient diminishes at the origin for this modification. Thus, in the final step, we make a small perturbation in the ball $B_{\rho_1^\delta}(O)$ which gives a full-rank gradient on all of ω and does not modify the gradient estimates established previously. The calculation is tedious, so we present the proof of this proposition in Appendix D.

2.4 Smoothing of nonisometric origami

Let $y \colon \omega \to \mathbb{R}^3$ such that $y \in \{y_{box}, y_{rd}, y_{rt}\}$ where the deformation is now a mapping from the entire domain ω as depicted with the box to the far left in Figure 8. We break up the domain ω into disjoint polygonal regions ω_β each containing a single junction and such that $\cup_\beta \omega_\beta = \omega$ as depicted with the dashed lines in the figure. We then translate and rotate our coordinate frame as necessary so that the deformation y restricted to the region regions ω_β is captured by a deformation y_β characterized by (2.15) and (2.16) or (2.22) and (2.23) up to a restriction of the reference domain in Figure 4b. This is depicted for the box in the middle/left part of Figure 8 where the red region highlights the sector captured in Figure 4b. We perform the appropriate smoothing of y_β in this local frame, either with Proposition 2.2 or Proposition 2.3, to obtain a y_β^δ on ω_β. This is depicted

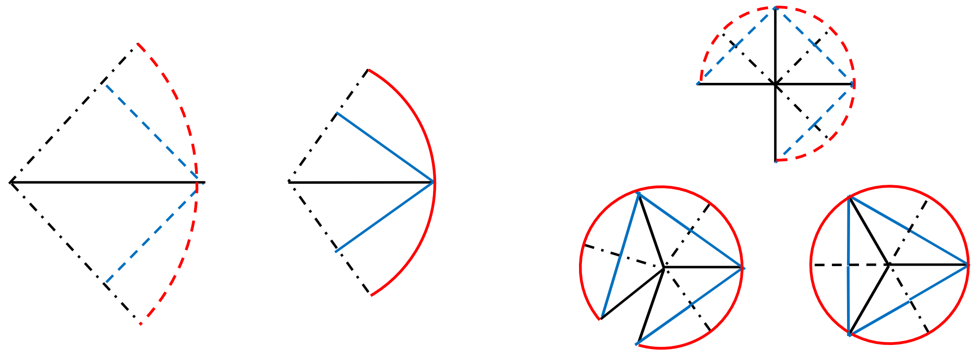


Figure 7: Unit cell (left) and examples of the global boundary deformation for rhombic dodecahedron (right). The dashed blue lines are deformed by the piecewise affine deformation to the blue lines. We correct the piecewise affine deformation employing a deformation (D.1) which takes the dashed red circular arcs to the red circular arcs. The affine deformation and correction are identical at each interface.

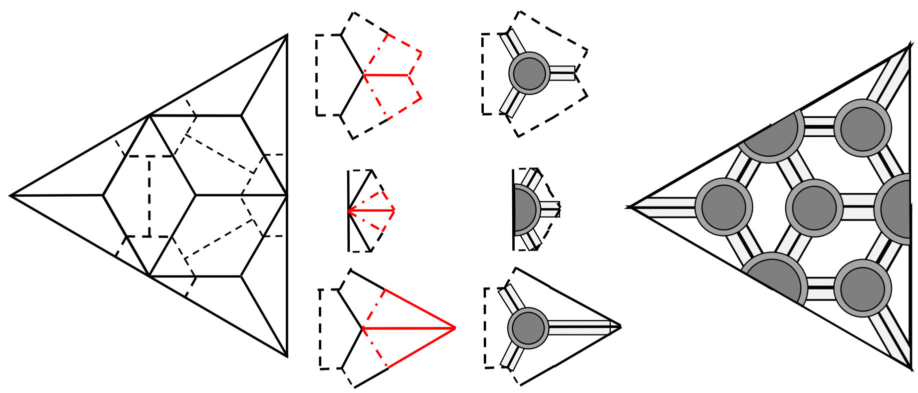


Figure 8: Smoothing of the box construction via local modifiications

for the box in the middle/right part of Figure 8 where the shaded regions highlight our δ-dependent smoothing. Finally, we translate and/or rotate our coordinate system as necessary back to the original frame. From this, we obtain a global deformation $y^\delta : \omega \to \mathbb{R}^3$ which is smooth across each $\partial\omega_\beta \cap \operatorname{int}(\omega)$ due to the compatibility of our constructions at these boundaries. This is depicted for the box on the far right in Figure 8.

Now, since each y^δ is obtained by the smoothing of junctions via Proposition 2.2 and 2.3, these globally smoothed midplane deformations actually satisfy the approximation hypothesis (1.16) for the appropriate $y \in \{y_{box}, y_{rd}, y_{rt}\}$. In addition, these are not the only nonisometric origami examples for which the explicit constructions in sections 2.1-2.3 are amenable. For instance, the examples in Figure 13c can be smoothed to obtain a global approximation y^δ satisfying (1.16) via application of Proposition 2.2 at each junction. Clearly then, there is not a dearth of examples privy to the analysis of the previous sections which leads to the approximation properties in (1.16). Hence, we assume (1.16) for our study of nonisometric origami in sequel.

We now construct a vector field $b^\delta : \omega \to \mathbb{R}^3$ that serves as the out-of-plane deformation gradient at the midplane ω which is consistent with a nearly zero energy midplane for any nonisometric origami in which the approximation property (1.16) holds:

Proposition 2.4. *Let $r > 0$ and $\delta \ll 1$. Let ω satisfy (1.9), n_0 satisfy (1.10), $y \in W^{1,\infty}(\omega, \mathbb{R}^3)$*

satisfy (1.12) and each $\tilde{F}_\alpha$ and $n_{0\alpha}$ in these relations satisfy (1.13). If there exists a y^δ satisfying (1.16), then for $\delta > 0$ sufficiently small, there exists a $b^\delta \in C^2(\bar{\omega}, \mathbb{R}^3)$ such that

$$(\tilde{\nabla} y^\delta | b^\delta)^T (\tilde{\nabla} y^\delta | b^\delta) = \ell_{n_0} \quad \text{and} \quad \tilde{\nabla} b^\delta = 0 \quad \text{on } \omega \setminus \tilde{\omega}_\delta \quad \text{with } |\tilde{\omega}_\delta| \leq O(\delta), \tag{2.27}$$

$$\det(\tilde{\nabla} y^\delta | b^\delta) = 1 \quad \text{everywhere on } \omega. \tag{2.28}$$

Moreover, b^δ satisfies

$$\|b^\delta\|_{L^\infty} \leq C, \quad \|\tilde{\nabla} b^\delta\|_{L^\infty} \leq C\delta^{-1}, \quad \|\tilde{\nabla}\tilde{\nabla} b^\delta\|_{L^\infty} \leq C\delta^{-2} \tag{2.29}$$

everywhere on ω for some $C > 0$ which can depend on y but is independent of δ.

Proof. We define the vector valued function $b_0^\delta \colon \omega \to \mathbb{R}^3$ such that

$$b_0^\delta := \frac{\partial_1 y^\delta \times \partial_2 y^\delta}{|\partial_1 y^\delta \times \partial_2 y^\delta|^2}. \tag{2.30}$$

Given this definition and the properties of y^h in (1.16), we see that $b_0^\delta \in C^2(\bar{\omega}, \mathbb{R}^3)$ such that

$$|b_0^\delta| \leq C, \quad |\tilde{\nabla} b_0^\delta| \leq C\delta^{-1}, \quad |\tilde{\nabla}\tilde{\nabla} b_0^\delta| \leq C\delta^{-2} \tag{2.31}$$

$$\det(\tilde{\nabla} y^\delta | b_0^\delta) = (\partial_1 y^\delta \times \partial_2 y^\delta) \cdot b_0^\delta = 1 \tag{2.32}$$

everywhere on ω and for some $C > 0$ which depends on y but is independent of δ.

We will define b^δ as a modification of b_0^δ. For this we note that the domain ω is composed of N connected, disjoint polygonal regions ω_α such the $\cup_{\alpha \in \{1,\dots,N\}} \omega_\alpha = \omega$, e.g. the box example in Figure 8. We now focus on one particular ω_α for the modification. Note that on this set, y^δ is modified from y in a $O(\delta)$ neighborhood of $\partial\omega_\alpha$ only. Thus, for $\delta > 0$ sufficiently small, we define $\tilde{\omega}_\alpha := \bar{\gamma}\omega_\alpha + \bar{b} \subset \omega_\alpha \setminus \omega_\delta$ for some $\bar{\gamma} \in (0,1)$ and $\bar{b} \in \mathbb{R}^3$ with the following properties: Their exists a cutoff function $\psi_\alpha^\delta \in C_0^\infty(\omega_\alpha, [0,1])$ such that $\psi_\alpha^\delta = 1$ on $\tilde{\omega}_\alpha$, $\psi_\alpha^\delta = 0$ on $\omega_\alpha \cap \omega_\delta$, and

$$|\tilde{\nabla}\psi_\alpha^\delta| \leq C\delta^{-1}, \quad |\tilde{\nabla}\tilde{\nabla}\psi_\alpha^\delta| \leq Ch^{-2} \tag{2.33}$$

on all of ω_α. Further, $\tilde{\omega}_\alpha$ is such that $|\omega_\alpha \setminus \tilde{\omega}_\alpha| = M_\alpha \delta$ for some constant M_α which may depend on ω_α but is independent of $\delta \ll 1$.

For $\delta > 0$ sufficiently small, we now fix each $\tilde{\omega}_\alpha$ and ψ_α^δ for $\alpha = 1, \dots, N$ such that they have the properties described above. We observe that

$$\tilde{\nabla} y^\delta = \tilde{\nabla} y = \tilde{F}_\alpha \quad \text{on each } \tilde{\omega}_\alpha$$

where $\tilde{F}_\alpha \in \mathbb{R}^{3\times 2}$ such that $\tilde{F}_\alpha^T \tilde{F}_\alpha = \tilde{\ell}_{n_{0\alpha}}$ for $n_{0\alpha} \in \mathbb{S}^2$ the uniform director associated to the director field $n_0 \colon \omega \to \mathbb{S}^2$ in region ω_α. This follows directly from (1.12) and (1.16). Hence, by Proposition A.5, there exists a $b_\alpha \in \mathbb{R}^3$ such that

$$(\tilde{F}_\alpha | b_\alpha)^T (\tilde{F}_\alpha | b_\alpha) = \ell_{n_{0\alpha}}, \quad \det(\tilde{F}_\alpha | b_\alpha) = 1.$$

Finally, we define $b^\delta \colon \omega \to \mathbb{R}^3$ such that

$$b^\delta := b_0^\delta + \psi_\alpha^\delta (b_\alpha - b_0^\delta) \quad \text{on each } \omega_\alpha.$$

It remains to verify the desired identities (2.27), (2.28) and (2.29). For the first identity, we observe that $b^\delta = b_\alpha$ on each $\tilde{\omega}_\alpha$ by the definition of ψ_α^h. Thus, it follows that $(\tilde{\nabla} y^\delta | b^\delta)^T (\tilde{\nabla} y^\delta | b^\delta) =$

$(\tilde{F}_\alpha|b_\alpha)^T(\tilde{F}_\alpha|b_\alpha) = \ell_{n_{0\alpha}}$ and $\tilde{\nabla} b^\delta = 0$ on each $\tilde{\omega}_\alpha$. Hence, $(\tilde{\nabla} y^\delta|b^\delta)^T(\tilde{\nabla} y^\delta|b^\delta) = \ell_{n_0}$ and $\tilde{\nabla} b^\delta = 0$ on $\cup_{\alpha\in\{1,\dots,N\}}\tilde{\omega}_\alpha$. Given this observation, we define $\tilde{\omega}_\delta := \omega \setminus \cup_{\alpha\in\{1,\dots,N\}}\tilde{\omega}_\alpha$. Further, we recall that $\tilde{\omega}_\alpha$ was defined such that $|\omega_\alpha \setminus \tilde{\omega}_\alpha| = O(\delta)$. Therefore, we conclude that $|\tilde{\omega}_\delta| \leq O(\delta)$ as desired, and the identity in (2.27) follows. For (2.28), we find

$$\begin{aligned}\det(\tilde{\nabla} y^\delta|b^\delta) &= (\partial_1 y^\delta \times \partial_2 y^\delta)\cdot b^\delta \\ &= (1-\psi^\delta_\alpha)(\partial_1 y^\delta \times \partial_2 y^\delta)\cdot b^\delta_0 + \psi^\delta_\alpha \det(\tilde{F}_\alpha|b_\alpha) = 1\end{aligned}$$

due to (2.31) and since $\tilde{\nabla} y^\delta = \tilde{F}_\alpha$ when $\psi^\delta_\alpha > 0$. Finally, (2.29) follows from the chain rule given (2.32) and (2.33). ☐

2.5 Smoothing of lifted surfaces

Let $r > 1$. Let n_0 as in (1.22) and y as in (1.20) for a graph φ as in (1.21). Here n_0 can be a $W^{1,\infty}(\omega,\mathbb{S}^2)$ design and y can be a $W^{2,\infty}(\omega,\mathbb{R}^3)$ depending on the regularity of φ, and our analysis requires smooth fields to extend the deformation y to a three dimensional deformation. Thus, we approximate the fields by replacing φ with a smooth $\delta-$dependent approximation φ_δ. Specifically, we extended φ to all of $\mathbb{R}^2$ yielding $\varphi \in W^{2,\infty}(\mathbb{R}^2,\mathbb{R}^3)$ (the extension is not relabeled), and we set

$$\varphi_\delta := \eta_\delta * \varphi \quad \text{ on } r^{-1/6}\omega \tag{2.34}$$

for a standard mollifier η_h supported on a ball of radius $h/2$. Thus:

Proposition 2.5. *For $\delta > 0$ sufficiently small, φ_δ in (2.34) belongs to $C^\infty_0(r^{-1/6}\omega,\mathbb{R})$ and satisfies the estimates*

$$\begin{aligned}&\|\varphi - \varphi_\delta\|_{W^{1,\infty}} \leq O(\delta), \quad \|\tilde{\nabla}\varphi_\delta\|_{L^\infty} < \lambda_r, \\ &\|\tilde{\nabla}^{(n)}\varphi_\delta\|_{L^\infty} \leq O(\delta^{2-n}), \quad \textit{for any integer } n \geq 2.\end{aligned}$$

Proof. φ_δ is smooth by mollification. It vanishes on the boundary of $r^{-1/6}\omega$ for $\delta > 0$ sufficiently small since by (1.21), $\operatorname{spt}\varphi \subset r^{-1/6}\omega_m := r^{-1/6}\{\tilde{x} \in \omega\colon \operatorname{dist}(\tilde{x},\omega) > m\}$ and since η_δ is supported on a ball of radius $\delta/2$. From standard manipulation of the mollification (2.34), the estimate on the $W^{1,\infty}$ norm follows from the Lipschitz continuity of φ and $\tilde{\nabla}\varphi$, the estimate on $\tilde{\nabla}\varphi_h$ follows from that fact that $\|\tilde{\nabla}\varphi\|_{L^\infty} < \lambda_r$ and the estimates on the higher derivatives follow from the fact that $\tilde{\nabla}\tilde{\nabla}\varphi \in L^\infty$. ☐

Now, we define n^δ_0 as n_0 in (1.22) and y^δ as y in (1.20) each with the graph φ replace by φ_δ in (2.34). We make the following observations:

Proposition 2.6. *Let $\delta > 0$ sufficiently small. Let n^δ_0 and y^δ as defined above for φ_δ as in (2.34), φ as in (1.21), n_0 as in (1.22) and y as in (1.20). Then $n^\delta_0 \in C^\infty(\bar{\omega},\mathbb{S}^2)$ and $y^\delta \in C^\infty(\bar{\omega},\mathbb{R}^3)$ and they satisfy*

$$\begin{aligned}&(\tilde{\nabla} y^\delta)^T(\tilde{\nabla} y^\delta) = \tilde{\ell}_{n^\delta_0} \quad \textit{on } \omega, \quad \|n^\delta_0 - n_0\|_{L^\infty} \leq O(\delta), \quad \|y^\delta - y\|_{W^{1,\infty}} \leq O(\delta), \\ &\|\tilde{\nabla} y^\delta\|_{L^\infty} + \|\tilde{\nabla}\tilde{\nabla} y^\delta\|_{L^\infty} + \|\tilde{\nabla} n^\delta_0\|_{L^\infty} \leq C, \quad \|\tilde{\nabla}^{(3)} y^\delta\|_{L^\infty} + \|\tilde{\nabla}\tilde{\nabla} n^\delta_0\|_{L^\infty} \leq C\delta^{-1}\end{aligned}$$

for C independent of δ.

Proof. These properties are a consequence of the properties on φ_δ established in Proposition 2.5. In particular, smoothness follows since φ_δ is a mollification; the metric constraint holds by the equivalence (1.23) since $\|\tilde{\nabla}\varphi_\delta\|_{L^\infty} < \lambda_r$; the estimates on the approximations $n_0^\delta - n_0$ and $y^\delta - y$ follow from the $W^{1,\infty}$ estimate of $\varphi_\delta - \varphi$ using the explicit definition of each field; and the δ-dependent derivative estimates follow from the δ-dependent derivative estimates of φ_δ again using the explicit definition of each field. ☐

Now, for the midplane fields approximating the lifted surface ansatz, it remains to construct the out-of-plane vector $b^\delta \colon \omega \to \mathbb{R}^3$ approximating the out-of-plane deformation gradient at the midplane. To this end, we have:

Proposition 2.7. *Let $\delta > 0$ sufficiently small. Let n_0^δ and y^δ as in Proposition 2.6. There exists a $b^\delta \in C^\infty(\bar{\omega}, \mathbb{R}^3)$ such that*

$$\begin{aligned} &(\tilde{\nabla} y^\delta | b^\delta)^T (\tilde{\nabla} y^\delta | b^\delta) = \ell_{n_0^\delta}, \quad \det(\tilde{\nabla} y^h | b^h) = 1, \\ &\|b^\delta\|_{L^\infty} + \|\tilde{\nabla} b^\delta\|_{L^\infty} \leq C, \quad \|\tilde{\nabla}\tilde{\nabla} b^\delta\|_{L^\infty} \leq C\delta^{-1} \end{aligned} \tag{2.35}$$

for C independent of δ.

Proof. Since by Proposition 2.6, we have $(\tilde{\nabla} y^\delta)^T \tilde{\nabla} y^\delta = \tilde{\ell}_{n_0^\delta}$ everywhere on ω, we apply Proposition A.5 pointwise everywhere on ω. Thus, we define the vector $b^\delta \colon \omega \to \mathbb{R}^3$ as in (A.5) with $\tilde{\nabla} y^\delta$ replacing $\tilde{F}$ and n_0^δ replacing n_0 in these relations. Hence, (2.35) holds on ω. Smoothness follows since n_0^δ, y^δ and the parameterization (A.5) are each themselves smooth. The estimates on the derivatives of b^δ follow from the estimates on the derivative of y^δ and n_0^δ in Proposition 2.6 by explicit differentiation of the parameterization in (A.5). ☐

3 Incompressibility and the energy of three dimensional constructions

In this section, for $h > 0$ sufficiently small we construct deformations $Y^h \colon \Omega_h \to \mathbb{R}^3$ which approximate nonisometric origami, lifted surfaces and sufficiently smooth surfaces. We show that each of these constructions has a nontrivial $h-$dependence entropic energy scaling. In Section 3.1, we take the smooth approximations in Section 2, and extend them through the thickness to an incompressible global deformation. Next in Section 3.2, we define each of the constructions approximating the desired actuation. Finally, we prove the thickness scaling of the entropic energy: Theorem 1.4 for nonisometric origami is proved in Section 3.3, and Theorem 1.8 for lifted surfaces and Corollary 1.9 for sufficiently smooth surfaces are proved in Section 3.4.

3.1 Incompressibility of thin elastomers

We begin by using the approximations developed in Section 2 to construct incompressible three dimensional deformations for $h > 0$ sufficiently small. The lemma below contains, by hypothesis, the essential features to these approximations which allow for incompressible extensions. This is parameterized by $\alpha \in \{-1, 0, 1\}$. $\alpha = 1$ corresponds to nonisometric origami, $\alpha = 0$ corresponds to lifted surfaces, and $\alpha = -1$ corresponds to smooth surfaces.

Lemma 3.1. *Let $\alpha \in \{-1, 0, 1\}$. Suppose for any $0 < \delta \ll 1$ we have $y_\alpha^\delta \in C^3(\bar{\omega}, \mathbb{R}^3)$ and $b_\alpha^\delta \in C^2(\bar{\omega}, \mathbb{R}^3)$ satisfying*

$$\det(\tilde{\nabla} y_\alpha^\delta | b_\alpha^\delta) = 1 \quad \textit{on } \omega,$$

$$
\begin{aligned}
&\|\tilde{\nabla} y_\alpha^\delta\|_{L^\infty(\omega)} + \|b_\alpha^\delta\|_{L^\infty(\omega)} \leq M,\\
&\|\tilde{\nabla}\tilde{\nabla} y_\alpha^\delta\|_{L^\infty(\omega)} + \|\tilde{\nabla} b_\alpha^\delta\|_{L^\infty(\omega)} \leq M\delta^{\min\{-\alpha,0\}},\\
&\|\tilde{\nabla}^{(3)} y_\alpha^\delta\|_{L^\infty(\omega)} + \|\tilde{\nabla}\tilde{\nabla} b_\alpha^\delta\|_{L^\infty(\omega)} \leq M\delta^{-\alpha-1}
\end{aligned}
$$

for some uniform constant $M > 0$. Then there exists an $m \equiv m(M,\alpha) \geq 1$ such that for any $h > 0$ sufficiently small, there exists a $\xi_\alpha^h \in C^1(\overline{\Omega}_h, \mathbb{R})$ and an extension $Y_\alpha^h \in C^1(\overline{\Omega}_h, \mathbb{R}^3)$ satisfying

$$
Y_\alpha^h = y_\alpha^{\delta_h} + \xi_\alpha^h b_\alpha^{\delta_h}, \quad \textit{with} \quad \delta_h = mh \quad \textit{and} \quad \det \nabla Y_\alpha^h = 1 \quad \textit{on } \Omega_h. \tag{3.1}
$$

In additions, ξ_α^h satisfies the pointwise estimates

$$
|\xi_\alpha^h - x_3| \leq Ch^{\min\{-\alpha,0\}}|x_3|^2, \;\; |\partial_3\xi_\alpha^h - 1| \leq Ch^{\min\{-\alpha,0\}}|x_3|, \;\; |\tilde{\nabla}\xi_\alpha^h| \leq Ch^{-\alpha-1}|x_3|^2. \tag{3.2}
$$

everywhere on Ω_h. Here, each $C \equiv C(M)$ and does not depend on h.

Proof. We set $\delta_h = mh$ for $m \geq 1$ to be determined in Proposition 3.2. We consider the function

$$
V_\alpha^h(\tilde{x}, x_3) := y_\alpha^{\delta_h}(\tilde{x}) + x_3 b_\alpha^{\delta_h}(\tilde{x}) \tag{3.3}
$$

and assume $x_3 \in (-h/2, h/2)$. Since $\nabla V_\alpha^h = (\tilde{\nabla} y_\alpha^{\delta_h}|b_\alpha^{\delta_h}) + x_3(\tilde{\nabla} b_\alpha^{\delta_h}|0)$ and $\det(\tilde{\nabla} y_\alpha^{\delta_h}|b_\alpha^{\delta_h}) = 1$, we let $S_\alpha^h := (\tilde{\nabla} y_\alpha^{\delta_h}|b_\alpha^{\delta_h})^{-1}(\tilde{\nabla} b_\alpha^{\delta_h}|0)$ and find

$$
\begin{aligned}
\det \nabla V_\alpha^h &= \det((\tilde{\nabla} y_\alpha^{\delta_h}|b_\alpha^{\delta_h})^{-1}\nabla V_\alpha^h) = \det(I + x_3 S_\alpha^h)\\
&= 1 + x_3 \operatorname{Tr}(S_\alpha^h) + x_3^2 \operatorname{Tr}(\operatorname{cof} S_\alpha^h) + x_3^3 \det(S_\alpha^h).
\end{aligned} \tag{3.4}
$$

For the estimates below, $C \equiv C(M)$. We note that $\|(\tilde{\nabla} y^{\delta_h}|b^{\delta_h})^{-1}\|_{L^\infty(\omega)} \leq C$ since the determinant is unity, and therefore $|S_\alpha^h| \leq C\delta_h^{\min\{-\alpha,0\}}$ by hypothesis and

$$
\begin{aligned}
|\det \nabla V_\alpha^h - 1| &\leq C\left(|x_3|\delta_h^{\min\{-\alpha,0\}} + |x_3|^2\delta_h^{2\min\{-\alpha,0\}} + |x_3|^3\delta_h^{3\min\{-\alpha,0\}}\right)\\
&\leq C|x_3|\delta_h^{\min\{-\alpha,0\}} \quad \text{for} \quad \alpha \in \{-1,0,1\}, \;\; m \geq 1.
\end{aligned} \tag{3.5}
$$

In addition for $\beta = 1,2$, since $\|\partial_\beta S_\alpha^h\|_{L^\infty(\omega)} \leq C(\delta_h^{2\min\{-\alpha,0\}} + \delta_h^{-\alpha-1}) \leq C\delta_h^{-\alpha-1}$ for $\alpha \in \{-1,0,1\}$ and since $|\partial_\beta \operatorname{Tr}(S_\alpha^h)| \leq |\partial_\beta S_\alpha^h|$, $|\partial_\beta \operatorname{Tr}(\operatorname{cof} S_\alpha^h)| \leq 2|S_\alpha^h||\partial_\beta S_\alpha^h|$ and $|\partial_\beta \det(S_\alpha^h)| \leq |S_\alpha^h|^2|\partial_\beta S_\alpha^h|$, we conclude that

$$
\begin{aligned}
|\partial_\beta \det \nabla V_\alpha^h| &\leq C(|x_3|\delta_h^{-\alpha-1} + |x_3|^2\delta_h^{\min\{-\alpha,0\}}\delta_h^{-\alpha-1} + |x_3|^3\delta_h^{2\min\{-\alpha,0\}}\delta_h^{-\alpha-1})\\
&\leq C|x_3|\delta_h^{-\alpha-1} \quad \text{for} \;\; \beta \in \{1,2\}, \;\; \alpha \in \{-1,0,1\}, \;\; m \geq 1.
\end{aligned} \tag{3.6}
$$

Now since V_α^h is not incompressible, we modify it through a non-linear change in coordinates. We let $\Xi^h(\tilde{x}, x_3) = (\tilde{x}, \xi^h(\tilde{x}, x_3))$ for $\xi^h \in C^1(\overline{\Omega}_h, \mathbb{R})$ to be determined, and we define $Y_\alpha^h := V_\alpha^h \circ \Xi^h$. Hence, by the column linearity of the determinant, we find that

$$
\det \nabla Y_\alpha^h = \det(\nabla V_\alpha^h \circ \Xi^h)\partial_3\xi^h.
$$

Thus, satisfying the determinant constraint on ∇Y^h amounts to satisfying the ordinary differential equation

$$
\partial_3\xi^h = \frac{1}{\det(\nabla V_\alpha^h \circ \Xi^h)} \quad \text{on} \quad \Omega_h \tag{3.7}
$$

for some ξ^h. There is an $m = m(\alpha, M) \geq 1$ such that for $h > 0$ sufficiently small, there is a solution to (3.7), i.e. $\xi^h \equiv \xi^h_\alpha$ for a $\xi^h_\alpha \in C^1(\overline{\Omega}_h, \mathbb{R})$ with the initial condition $\xi^h_\alpha(\tilde{x}, 0) = 0$, see Proposition 3.2.

It remains to prove the estimates in (3.2). By Proposition 3.2, the map ξ^h_α satisfies pointwise

$$|\xi^h_\alpha| \leq 2|x_3|, \quad |\partial_3 \xi^h_\alpha| \leq 2 \tag{3.8}$$

everywhere on Ω_h. Thus, given (3.7),(3.5) and the estimates above, we deduce

$$|\partial_3 \xi^h_\alpha - 1| \leq |\partial_3 \xi^h_\alpha||\det(\nabla V^h \circ \Xi^h_\alpha) - 1| \leq Ch^{\min\{-\alpha,0\}}|\xi^h_\alpha| \leq Ch^{\min\{-\alpha,0\}}|x_3|$$

everywhere on Ω_h. Similarly,

$$|\xi^h_\alpha - x_3| \leq |\int_0^{x_3} (\partial_3 \xi^h_\alpha - 1) d\bar{x}_3| \leq \int_0^{|x_3|} |\partial_3 \xi^h_\alpha - 1| d\bar{x}_3 \leq Ch^{\min\{-\alpha,0\}}|x_3|^2$$

everywhere on Ω_h. Finally, to estimate the first and second derivative of ξ_h, we define $F_h(\tilde{x}, t) := \int_0^s \det(\nabla V^h(\tilde{x}, s))ds$, and notice that the ordinary differential equation in (3.7) is equivalent to the implicit equation $F_h(\tilde{x}, \xi_h(x)) = x_3$. Differentiating this equation with respect to x_β, $\beta = 1$, or 2, we find

$$\int_0^{\xi^h_\alpha} \partial_\beta \det(\nabla V^h) ds + \det(\nabla V^h \circ \Xi^h_\alpha)\partial_\beta \xi^h_\alpha = 0.$$

Hence using (3.7), (3.6) and (3.8),

$$|\partial_\beta \xi^h_\alpha| \leq |\partial_3 \xi^h_\alpha| \int_0^{|\xi_h|} |\partial_\beta \det \nabla V^h| ds \leq Ch^{-\alpha-1} \int_0^{|\xi^h_\alpha|} |s| ds \leq Ch^{-\alpha-1}|x_3|^2$$

everywhere on Ω_h for $\beta = 1, 2$. These are the desired estimates. $\square$

Proposition 3.2. *Let $\alpha \in \{-1, 0, 1\}$. Let V^h_α defined in (3.3) with $y^{\delta_h}_\alpha$ and $b^{\delta_h}_\alpha$ as in Lemma 3.1 with $\delta_h = mh$. There is an $m = m(\alpha, M) \geq 1$ such that for any $h > 0$ sufficiently small, there exists a $\xi^h_\alpha \in C^\infty(\overline{\Omega}_h, \mathbb{R})$ such that*

$$\partial_3 \xi^h_\alpha = \frac{1}{\det(\nabla V^h \circ \Xi^h)} \quad \text{on } \Omega_{h_0}, \quad \text{with} \quad \xi^h_\alpha(\tilde{x}, 0) = 0. \tag{3.9}$$

Moreover ξ^h_α satisfies pointwise the estimate

$$|\xi^h_\alpha| \leq 2|x_3|, \quad |\partial_3 \xi^h_\alpha| \leq 2 \quad \text{on } \Omega_h. \tag{3.10}$$

Remark 3.3. We can choose $m = 1$ for $\alpha = \{-1, 0\}$. For $\alpha = 1$, we generally have to choose m such that $m \geq \max\{C(M), 1\}$ where $C(M)$ is a constant that depends on M but is independent of h.

Proof. For $\alpha \in \{-1, 0, 1\}$ and $h > 0$, we consider the mapping $T^h_\alpha \colon \mathcal{M}^h_\alpha \to C(\bar{\Omega}_h)$ given by

$$T^h_\alpha(\phi)(\tilde{x}, x_3) = \int_0^{x_3} \frac{1}{\det(\nabla V^h_\alpha(\tilde{x}, \phi(\tilde{x}, s)))} ds \quad \text{for each} \quad (\tilde{x}, x_3) \in \Omega_h,$$

where $\mathcal{M}^h_\alpha$ is given by

$$\mathcal{M}^h_\alpha := \big\{\phi \in C(\overline{\Omega}_h) \colon \phi(\tilde{x}, 0) = 0, \ \ |\phi(\tilde{x}, x_3)| \leq 2|x_3|,$$

$$\det(\nabla V_\alpha^h(\tilde{x}, \phi(\tilde{x}, x_3))) \geq 1/2 \text{ for each } (\tilde{x}, x_3) \in \Omega_h \Big\}.$$

This is a (non-empty) complete space under the infinity norm. Thus, we aim to show that there is an appropriate choice of $m = m(\alpha, M)$ in δ_h such that for $h > 0$ sufficiently small, the mapping T_α^h is, in fact, a contraction map in the space $\mathcal{M}_\alpha^h$ under the infinity norm. The proposition will follow by the equivalence of the integral representation of (3.9).

We first prove that T_α^h is an operator (i.e. $T_\alpha^h \colon \mathcal{M}_\alpha^h \to \mathcal{M}_\alpha^h$) for an appropriate choice of $m = m(\alpha, M)$ and small enough h. For the estimates below, $C \equiv C(M)$. Since $\phi \in \mathcal{M}_\alpha^h$, we have

$$|T_\alpha^h(\phi)(\tilde{x}, x_3)| \leq 2|x_3|, \quad \text{for each} \quad (\tilde{x}, x_3) \in \Omega_h.$$

In addition, using a similar estimate to (3.5), we obtain

$$|\det \nabla V_\alpha^h(\tilde{x}, T_\alpha^h(\tilde{x}, x_3)) - 1| \leq C|T_\alpha^h(\tilde{x}, x_3)|\delta_h^{\min\{-\alpha,0\}} \leq C|x_3|h^{\min\{-\alpha,0\}}m^{\min\{-\alpha,0\}}$$

for $\alpha \in \{-1, 0, 1\}$. Thus, for $\alpha \in \{-1, 0\}$, we need only enforce $m \geq 1$ and for $\alpha = 1$ we enforce $m = m(\alpha, M) \geq \max\{2C, 1\}$ to ensure T_α^h is an operator for small h.

It remains to prove that T_α^h is a contraction under the L^∞ norm. Observe for $\phi, \psi \in \mathcal{M}_\alpha^h$,

$$\begin{aligned}
|T_\alpha^h(\phi)(\tilde{x}, x_3) - T_\alpha^h(\psi)(\tilde{x}, x_3)| &\leq 4 \int_0^{|x_3|} |\det(\nabla V_\alpha^h(\tilde{x}, \psi(\tilde{x}, s)) - \det(\nabla V_\alpha^h(\tilde{x}, \phi(\tilde{x}, s))| ds \\
&\leq C\delta_h^{\min\{-\alpha,0\}} \int_0^{|x_3|} |\psi(\tilde{x}, s) - \phi(\tilde{x}, s)| ds \\
&\leq C m^{\min\{-\alpha,0\}} h^{\min\{-\alpha,0\}} h \|\psi - \phi\|_{L^\infty(\Omega_h)}
\end{aligned}$$

for any $(\tilde{x}, x_3) \in \Omega_h$. Here the first inequality uses the determinant constraint on $\mathcal{M}_\alpha^h$, the second uses the equation (3.4), and the third uses that $\delta_h = mh$. Finally, from this estimate, it is clear that we can choose $m = m(\alpha, M) \geq 1$ independent of h (in fact $m = 1$ suffices for $\alpha = -1, 0$ as in the remark), such that for h sufficiently small

$$\|T_\alpha^h(\phi) - T_\alpha^h(\psi)\|_{L^\infty(\Omega_h)} < \|\psi - \phi\|_{L^\infty(\Omega_h)},$$

i.e. it is a contraction map.

We now fix this $m = m(\alpha, M)$ and an $h > 0$ sufficiently small. Since T_α^h is a contraction map, there exists a ξ_α^h such that

$$\xi_\alpha^h = T_\alpha^h(\xi_\alpha^h) = \int_0^{x_3} \frac{1}{\det(\nabla V_\alpha^h(\tilde{x}, \xi_\alpha^h(\tilde{x}, s)))} ds \quad \text{for each} \quad (\tilde{x}, x_3) \in \Omega_h.$$

This is equivalent to the ordinary differential equation (3.9). The regularity $\xi_\alpha^h \in C^1(\bar{\Omega}_h, \mathbb{R})$ follows from the regularity of $y_\alpha^{\delta_h}$ and $b_\alpha^{\delta_h}$. The estimates (3.10) follow from the fact that $\xi_h^\alpha \in \mathcal{M}_\alpha^h$. This completes the proof. $\square$

3.2 Definition of three dimensional deformations for low energy constructions

We now defined the three dimensional fields used in our subsequent energy arguments for actuation.

Definition of three dimensional deformation for nonisometric orgiami: Let $r > 0$. We suppose $\omega \subset \mathbb{R}^2$ satisfies (1.9), N_0^h satisfies (1.7) with the design n_0 as in (1.10) and (1.11), $y \in W^{1,\infty}(\omega, \mathbb{R}^3)$ is an origami deformation as in (1.12), and each $\tilde{F}_\alpha$ and $n_{0\alpha}$ in these relations satisfies (1.13). In addition, we assume there exists a $y^\delta \in C^3(\bar{\omega}, \mathbb{R}^3)$ satisfying the approximations property (1.16).

Hence, by Proposition 2.4, we obtain a vector field $b^\delta \in C^2(\bar{\omega}, \mathbb{R}^3)$ with the identities (2.27) and (2.28) and the estimate (2.29). Thus by Lemma 3.1 with $\alpha = 1$, there exists a $m = m(\tilde{\nabla} y) \geq 1$ such that for $h > 0$ sufficiently small there exists a $\xi^h \in C^1(\overline{\Omega}_h, \mathbb{R})$ and an extension $Y^h \in C^1(\overline{\Omega}_h, \mathbb{R}^3)$ with the properties:

$$\begin{cases} Y^h := y^{\delta_h} + \xi^h b^{\delta_h} \quad \text{with} \quad \delta_h = mh, \quad \det \nabla Y^h = 1 \quad \text{on } \Omega_h, \\ |\xi^h - x_3| \leq Ch^{-1}|x_3|^2, \ \ |\partial_3 \xi^h - 1| \leq Ch^{-1}|x_3|, \ \ |\tilde{\nabla}\xi^h| \leq Ch^{-2}|x_3|^2 \quad \text{on } \Omega_h \end{cases} \tag{3.11}$$

for $C = C(\tilde{\nabla} y) > 0$ independent of h. With this construction, we prove Theorem 1.4 in Section 3.3.

Definition of three dimensional deformation for lifted surfaces: Let $r > 1$. We suppose $\{\varphi, y, n_0\}$ are as in the lifted surface ansatz (i.e. y satsifying (1.20) and n_0 satisfying (1.22) for φ as in (1.21) for some $m > 0$) and N_0^h is as in (1.7). For $\delta > 0$ sufficiently small, there exists $\delta-$dependent functions $\{\varphi_\delta, y^\delta, n_0^\delta, b^\delta\}$ approximating this ansatz with the properties as detailed in Section 2.5. Hence, by Lemma 3.1 with $\alpha = 0$, for $h > 0$ sufficiently small, we set $\delta = h$ (Remark 3.3) and there exists a $\xi^h \in C^1(\overline{\Omega}_h, \mathbb{R})$ and an extension $Y^h \in C^1(\overline{\Omega}_h, \mathbb{R}^3)$ with the properties:

$$\begin{cases} Y^h := y^h + \xi^h b^h, \quad \det \nabla Y^h = 1 \quad \text{on } \Omega_h, \\ |\xi^h - x_3| \leq C|x_3|^2, \ \ |\partial_3 \xi^h - 1| \leq C|x_3|, \ \ |\tilde{\nabla}\xi^h| \leq Ch^{-1}|x_3|^2 \quad \text{on } \Omega_h \end{cases} \tag{3.12}$$

for $C = C(\tilde{\nabla} y) > 0$ independent of h. With this construction, we prove Theorem 1.8 in Section 3.4.

Definition of three dimensional deformation for smooth surfaces: Let $r > 0$. We suppose that $n_0 \in C^2(\bar{\omega}, \mathbb{S}^2)$ and $y \in C^3(\bar{\omega}, \mathbb{R}^3)$ such that $(\tilde{\nabla} y)^T \tilde{\nabla} y = \tilde{\ell}_{n_0}$ on ω. Following Proposition A.5, there exists a $b \in C^2(\bar{\omega}, \mathbb{R}^3)$ such that $(\tilde{\nabla} y|b)^T(\tilde{\nabla} y|b) = \ell_{n_0}$ and $\det(\tilde{\nabla} y|b) = 1$. The smoothness is due to the regularity of n_0 and y by explicit differentiation of the parameterization in (A.5). Now y and b satisfy the hypotheses of Lemma 3.1 with $\alpha = -1$ since these fields are h-independent. Hence, for $h > 0$ sufficiently small there exists a $\xi^h \in C^1(\overline{\Omega}_h, \mathbb{R})$ and an extension $Y^h \in C^1(\overline{\Omega}_h, \mathbb{R}^3)$ with the properties:

$$\begin{cases} Y^h := y + \xi^h b, \quad \det \nabla Y^h = 1 \quad \text{on } \Omega_h, \\ |\xi^h - x_3| \leq C|x_3|^2, \ \ |\partial_3 \xi^h - 1| \leq C|x_3|, \ \ |\tilde{\nabla}\xi^h| \leq C|x_3|^2 \quad \text{on } \Omega_h \end{cases}$$

for $C = C(\tilde{\nabla} y) > 0$ independent of h. With this construction, we prove Corollary1.9 also in Section 3.4.

3.3 The $O(h^2)$ energy argument for nonisometric origami

We turn now to the proof of Theorem 1.4. Here and in the following, we find it useful to introduce the notation

$$W^e(F, n, n_0) = W_{nH}((\ell_n^f)^{-1/2} F (\ell_{n_0}^0)^{1/2}) \tag{3.13}$$

where

$$W_{nH}(F) := \frac{\mu}{2} \begin{cases} |F|^2 - 3 & \text{if } \det F = 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{3.14}$$

We consider any three dimensional deformation as in the definition for nonisometric origami above:

Proof of Theorem 1.4. We first remark that $Y^h(\tilde{x},0) = y^{\delta_h}(\tilde{x})$ for every $\tilde{x} \in \omega$ since $\xi^h(\tilde{x},0) = 0$ following the first estimate for ξ^h in (3.11). Thus, it remains only to show that the energy scales as $O(h^2)$ for this deformation.

To this end, we first compute ∇Y^h explicitly. We find that

$$\nabla Y^h = (\tilde{\nabla} y^{\delta_h}|0) + \xi^h(\tilde{\nabla} b^{\delta_h}|0) + (\partial_1 \xi^h b^{\delta_h}|\partial_2 \xi^h b^{\delta_h}|\partial_3 \xi^h b^{\delta_h}),$$

and note that from Proposition 2.4, $\tilde{\nabla} b^{\delta_h} = 0$ on the set $\omega \setminus \tilde{\omega}_{\delta_h}$. It follows that $\xi^h = x_3$ on this set. Indeed, since $\det \nabla Y^h = 1$, we find that on $\omega \setminus \tilde{\omega}_{\delta_h}$,

$$1 = \det((\tilde{\nabla} y^{\delta_h}|0) + (\partial_1 \xi^h b^{\delta_h}|\partial_2 \xi^h b^{\delta_h}|\partial_3 \xi^h b^{\delta_h})) = \partial_3 \xi^h \det(\tilde{\nabla} y^{\delta_h}|b^{\delta_h}).$$

Also from Proposition 2.4, $\det(\tilde{\nabla} y^{\delta_h}|b^{\delta_h}) = 1$. Thus, $\partial_3 \xi^h = 1$ on $\omega \setminus \tilde{\omega}_{\delta_h}$. Consequently, $\xi^h = x_3$ on this set since we have the condition $\xi^h(\tilde{x},0) = 0$.

To recap, we find that

$$\nabla Y^h = (\tilde{\nabla} y^{\delta_h}|b^{\delta_h}) \quad \text{on } \omega \setminus \tilde{\omega}_{\delta_h}. \tag{3.15}$$

On the exceptional set $\tilde{\omega}_h$, we find that

$$\begin{aligned} |\nabla Y^h| &= |(\tilde{\nabla} y^{\delta_h}|b^{\delta_h}) + (\partial_3 \xi^h - 1) b^{\delta_h} \otimes e_3 + x_3(\nabla' b^{\delta_h}|0) + (\xi^h - x_3)(\tilde{\nabla} b^{\delta_h}|0) + b^{\delta_h} \otimes \tilde{\nabla} \xi^h| \\ &\leq |(\tilde{\nabla} y^{\delta_h}|b^{\delta_h})| + |\partial_3 \xi^h - 1||b^{\delta_h}| + |x_3||\tilde{\nabla} b^{\delta_h}| + |\xi^h - x_3||\tilde{\nabla} b^{\delta_h}| + |b^{\delta_h}||\tilde{\nabla} \xi^h| \\ &\leq C\left(1 + h^{-1}|x_3| + h^{-2}|x_3|^2\right) \leq C \end{aligned} \tag{3.16}$$

where each $C = C(\tilde{\nabla} y, m(\tilde{\nabla} y)) > 0$ is independent of h. These estimates follow from the estimates (1.16), (2.29) in Proposition 2.4, and (3.11).

Now, we recall from Proposition 2.4 that $(\tilde{\nabla} y^{\delta_h}|b^{\delta_h})^T(\tilde{\nabla} y^{\delta_h}|b^{\delta_h}) = \ell_{n_0}$ on $\omega \setminus \tilde{\omega}_{\delta_h}$. Thus,

$$\begin{aligned} &W^e((\tilde{\nabla} y^{\delta_h}|b^{\delta_h}), n^h, n_0) \\ &\qquad = W_{nH}((\ell^f_{n^h})^{-1/2}(\tilde{\nabla} y^h|b^h)(\ell^0_{n_0})^{1/2}) = 0 \quad \text{on } \omega \setminus \tilde{\omega}_{\delta_h} \end{aligned} \tag{3.17}$$

following Proposition A.4 and the identity (3.13). Here, we have set

$$n^h := \frac{(\tilde{\nabla} y^{\delta_h}|b^{\delta_h}) n_0}{|(\tilde{\nabla} y^{\delta_h}|b^{\delta_h}) n_0|} \quad \text{on } \omega.$$

Since the energy density (3.17) vanishes, we deduce from Proposition A.3 that

$$(\ell^f_{n^h})^{-1/2}(\tilde{\nabla} y^h|b^h)(\ell^0_{n_0})^{1/2} =: R^h \in SO(3) \quad \text{on } \omega \setminus \tilde{\omega}_{\delta_h}. \tag{3.18}$$

We have yet to account for the non-ideal terms on this set as N_0^h in (1.7) is the appropriate argument for the energy density, not n_0. To do this, we exploit the observation in (3.18). Indeed, we set

$$N^h := \frac{(\nabla Y^h) N_0^h}{|(\nabla Y^h) N_0^h|} \quad \text{on } \Omega_h$$

and observe

$$(\ell^0_{N_0^h})^{1/2} = (\ell^0_{n_0})^{1/2} + O(x_3) + O(h), \quad (\ell^f_{N^h})^{-1/2} = (\ell^f_{n^h})^{-1/2} + O(x_3) + O(h) \tag{3.19}$$

following the scaling of the non-ideal terms in (1.7). Hence on $\omega \setminus \tilde{\omega}_{\delta_h}$, we find

$$\begin{aligned} W^e(\nabla Y^h, N^h, N_0^h) &= W_{nH}((\ell^f_{N^h})^{-1/2}(\tilde{\nabla} y^{\delta_h}|b^{\delta_h})(\ell^0_{N_0^h})^{1/2}) \\ &= W_{nH}((\ell^f_{n^h})^{-1/2}(\tilde{\nabla} y^{\delta_h}|b^{\delta_h})(\ell^0_{n_0})^{1/2} + O(x_3) + O(h)) \\ &= W_{nH}((R^h)^T((\ell^f_{n^h})^{-1/2}(\tilde{\nabla} y^{\delta_h}|b^{\delta_h})(\ell^0_{n_0})^{1/2} + O(x_3) + O(h))) \\ &= W_{nH}(I_{3\times 3} + O(x_3) + O(h)) \leq O(h^2). \end{aligned} \tag{3.20}$$

For the equalities, we used (3.15), (3.19), the frame invariance of W_{nH}, and (3.18). For the inequality, we used the estimate in Proposition A.2. Thus, on the set $\omega \setminus \tilde{\omega}_{\delta_h}$ the energy is at most bending for nonisometric origami.

Now, on the exceptional set $\tilde{\omega}_{\delta_h}$, we have

$$W^e(\nabla Y^h, N^h, N_0^h) \leq c(|\nabla Y^h|^2 + 1) \leq C \tag{3.21}$$

given the estimate in Proposition A.1 and (3.16). Thus, on the set $\tilde{\omega}_h$ characterizing the creases and junctions, the energy is stretching energy, but this set is small for nonisometric origami, i.e. $|\tilde{\omega}_{\delta_h}| = O(\delta_h) = O(h)$ from Proposition 2.4 since $\delta_h = mh$.

Combining the estimates (3.20) and (3.21), we conclude

$$\begin{aligned} \mathcal{I}^h_{N_0^h}(Y^h) &= \int_{-h/2}^{h/2}\int_{\tilde{\omega}_{\delta_h}} W^e(\nabla Y^h, N^h, N_0^h)dx + \int_{-h/2}^{h/2}\int_{\omega\setminus\tilde{\omega}_{\delta_h}} W^e(\nabla Y^h, N^h, N_0^h)dx \\ &\leq Ch|\tilde{\omega}_{\delta_h}| + O(h^3) \leq O(h^2). \end{aligned}$$

This completes the proof. □

3.4 The $O(h^3)$ energy argument for lifted surfaces and smooth surfaces

We now prove Theorem 1.8 and Corollary 1.9. First for Theorem 1.8, we consider any three dimensional deformation as in the definition for lifted surfaces in Section 3.2.

Proof of Theorem 1.8. We first note the $Y^h(\tilde{x}, 0) = y^h(\tilde{x})$ for $\tilde{x} \in \omega$ since $\xi^h(\tilde{x}, 0) = 0$ by the first estimate for ξ^h in (3.12). Moreover, $\|y^h - y\|_{W^{1,\infty}} \leq O(h)$ was shown in Proposition 2.6. So it remains to prove only the $O(h^3)$ scaling of the energy $\mathcal{I}_{N_0^h}(Y^h)$.

We compute explicitly

$$\begin{aligned} \nabla Y^h = (\tilde{\nabla} y^h|b^h) + x_3(\tilde{\nabla} b^h|0) + (\xi^h - x_3)(\tilde{\nabla} b^h|0) \\ + (\partial_3 \xi^h - 1)b^h \otimes e_3 + b^h \otimes \tilde{\nabla}\xi^h. \end{aligned}$$

Hence, by the estimates on ξ^h in (3.12) and the estimates on b^h in Proposition 2.7, we conclude

$$\nabla Y^h = (\tilde{\nabla} y^h|b^h) + O(x_3). \tag{3.22}$$

Now since Proposition 2.7 gives $(\tilde{\nabla} y^h|b^h)^T(\tilde{\nabla} y^h|b^h) = \ell_{n_0^h}$, we find by an identical argument as that which lead to (3.18) that

$$(\ell^f_{n^h})^{-1/2}(\tilde{\nabla} y^h|b^h)(\ell^0_{n_0^h})^{1/2} =: R^h \in SO(3) \quad \text{on } \omega \quad \text{for} \quad n^h := \frac{(\tilde{\nabla} y^h|b^h)n_0^h}{|(\tilde{\nabla} y^h|b^h)n_0^h|} \quad \text{on } \omega. \tag{3.23}$$

Note n^h here is defined with the approximation n_0^h and not by the design field n_0 as in (3.18). Now, we let $N^h := (\nabla Y^h)N_0^h/|(\nabla Y^h)N_0^h|$ on Ω_h, and observe that

$$\begin{aligned}(\ell^0_{N_0^h})^{1/2} &= (\ell^0_{n_0})^{1/2} + O(h)\\ &= (\ell^0_{n_0^h})^{1/2} + O(h),\end{aligned}$$

where the first equality follows from the non-ideal terms in (1.7) and the second follows since $\|n_0 - n_0^h\|_{L^\infty} \leq O(h)$ by Proposition 2.6. Additionally given (3.22), we conclude

$$(\ell^f_{N^h})^{-1/2} = (\ell^f_{n^h})^{-1/2} + O(x_3) + O(h)$$

for n^h as in (3.23). Hence, analogous to the reasoning of (3.20),

$$\begin{aligned}W^e(\nabla Y^h, N^h, N_0^h) &= W_{nH}((\ell^f_{n^h})^{-1/2}(\tilde{\nabla} y^h|b^h)(\ell^0_{n_0^h})^{1/2} + O(x_3) + O(h))\\ &= W_{nH}((R^h)^T(\ell^f_{n^h})^{-1/2}(\tilde{\nabla} y^h|b^h)(\ell^0_{n_0^h})^{1/2} + O(x_3) + O(h))\\ &= W_{nH}(I_{3\times 3} + O(x_3) + O(h)) \leq O(h^2)\end{aligned}$$

for R^h in (3.23). Since this inequality holds on all of ω,

$$\mathcal{I}^h_{N_0^h}(Y^h) = \int_{-h/2}^{h/2}\int_\omega W^e(\nabla Y^h, N^h, N_0^h)dx \leq O(h^3).$$

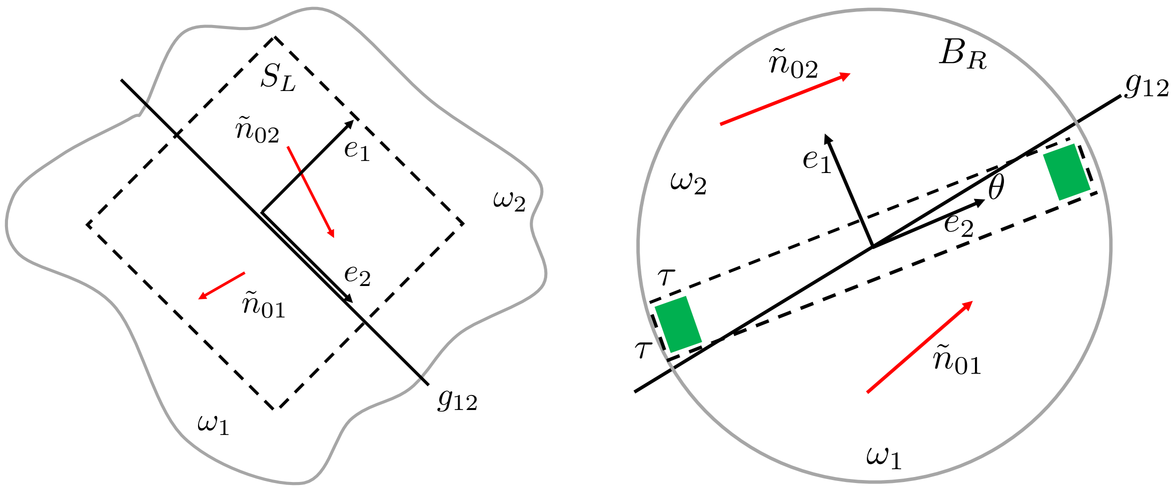
This completes the proof. $\square$

Now for Corollary 1.9, we consider any three dimensional deformation as in the definition for smooth surfaces in Section 3.2.

Proof of Corollary 1.9. We repeat the proof of Theorem 1.8 replacing $\{y^h, b^h, n_0^h\}$ with $\{y, b, n_0\}$ for these smooth surfaces and programs. $\square$

4 Nonisometric origami and an opitimal scaling law

In this section, we prove Theorem 1.6. Specifically, we show that for the two-dimensional analog to the entropic energy given by $\tilde{\mathcal{I}}^h_{n_0}$ in (1.18), a piecewise constant director design (with ω as in (1.9) and n_0 as in (1.10) and (1.11)) necessarily implies an energy of at least $O(h^2)$ upon actuation. In section 4.1, we show that this estimate can be reduced to a unit cell problem localized at a single interface. Further, we show that a lowerbound for this unit cell problem is described by a one-dimensional Modica-Mortola type functional. In their celebrated result, Modica and Mortola [35] (see also Modica [34]) prove that such functionals (under suitable hypothesis) Γ-converge to functionals which are proportional to the number of jumps of their argument. In our setting, these jumps correspond to the jump in the preferred metric over the interface. In Section 4.2, we present a self-contained argument which shows that the minimum of our Modica-Mortola type functional is necessarily bounded away from zero for $h > 0$ sufficiently small. This is the key result we use to prove Theorem 1.6.



(a) Case 1 (b) Case 2

Figure 9: Schematic for unit cell problem of Theorem 1.6

4.1 The unit cell problem

We assume ω and $n_0 \colon \omega \to \mathbb{S}^2$ satisfy (1.9), (1.10) and (1.11). Then there exists a straight interface $g_{\alpha\beta} \in \mathbb{S}^1$ adjoining two regions ω_α and ω_β such that $\tilde{n}_{0\alpha} \neq \pm\tilde{n}_{0\beta}$. Focusing on this single interface, we have two cases to consider:

1. *Case 1.* $(\tilde{n}_{0\alpha} \cdot e_i)^2 \neq (\tilde{n}_{0\beta} \cdot e_i)^2$ for at least one $i \in \{1, 2\}$.
2. *Case 2.* $(\tilde{n}_{0\alpha} \cdot e_i)^2 = (\tilde{n}_{0\beta} \cdot e_i)^2$ for both $i \in \{1, 2\}$

Definition for Case 1: In this case, we relabel so that $\alpha = 1$ and $\beta = 2$. We fix a global frame so that e_2 lies on the g_{12} interface and e_1 points in the direction of ω_2. We let the origin of this frame lie on the g_{12} interface such that for some $L > 0$ there exists a $S_L := (-L, L)^2 \subset \omega_1 \cup \omega_2$. A schematic of this description is provide in Figure 9a. We make the following observation in this case:

Proposition 4.1. *If ω and n_0 have an interface as in the definition of Case 1 (see Figure 9a), then for any $y \in W^{2,2}(\omega, \mathbb{R}^3)$,*

$$\tilde{\mathcal{I}}^h_{n_0}(y) \geq 2L^2 h M^h_1$$

where

$$M^h_1 := \left\{ \int_{-1}^{1} \left((u^2 - \sigma(t))^2 + \frac{h^2}{L^2}(u')^2 \right) dt \colon u \in W^{1,2}((-1,1), \mathbb{R}) \quad \text{with} \;\; u \geq 0 \;\; a.e \right\}$$

$$\alpha(t) = \begin{cases} \alpha_1 & \text{if } t < 0 \\ \alpha_2 & \text{if } t > 0. \end{cases}$$

Here $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 \neq \alpha_2$.

Proof. Let $y \in W^{2,2}(\omega, \mathbb{R}^3)$. Since $S_L \subset \omega_1 \cup \omega_2 \subset \omega$ and the integrand in (1.18) is non-negative, we have

$$\tilde{\mathcal{I}}^h_{n_0}(y) \geq h \int_{S_L} \left(|(\tilde{\nabla} y)^T \tilde{\nabla} y - \tilde{\ell}_{n_0}|^2 + h^2 |\tilde{\nabla}\tilde{\nabla} y|^2 \right) d\tilde{x}$$

$$
\begin{aligned}
&\geq h \int_{S_L} \left(|e_i \cdot ((\tilde{\nabla} y)^T \tilde{\nabla} y - \tilde{\ell}_{n_0}) e_i|^2 + h^2 |\partial_1 \partial_i y|^2 \right) d\tilde{x} \\
&= h \int_{S_L} \left(||\partial_i y|^2 + \sigma(x_1)|^2 + h^2 |\partial_1 \partial_i y|^2 \right) d\tilde{x} \qquad (4.1)
\end{aligned}
$$

where $i \in \{1, 2\}$ is chosen such that $(\tilde{n}_{01} \cdot e_i)^2 \neq (\tilde{n}_{02} \cdot e_i)^2$. We see then that σ is given by

$$
\sigma(t) = \begin{cases} r^{-1/3}(1 + (r-1)(\tilde{n}_{01} \cdot e_i)^2) & \text{if } t < 0 \\ r^{-1/3}(1 + (r-1)(\tilde{n}_{02} \cdot e_i)^2) & \text{if } t > 0. \end{cases}
$$

Thus, we set $\sigma_1 := r^{-1/3}(1 + (r-1)(\tilde{n}_{01} \cdot e_i)^2)$ and $\sigma_2 := r^{-1/3}(1 + (r-1)(\tilde{n}_{02} \cdot e_i)^2)$, and note that $\sigma_1 \neq \sigma_2$ by definition of this case, and $\sigma_1, \sigma_2 > 0$ since $r > 0$.

Given the chain of inequalities (4.1), we deduce that

$$
\begin{aligned}
\tilde{\mathcal{I}}^h_{n_0}(y) &\geq 2Lh \inf \left\{ \int_{-L}^{L} \left((|w|^2 - \sigma(t))^2 + h^2 |w'|^2 \right) dt \colon w \in W^{1,2}((-L, L), \mathbb{R}^3) \right\} \\
&\geq 2Lh \inf \left\{ \int_{-L}^{L} \left((|w|^2 - \sigma(t))^2 + h^2 (|w|')^2 \right) dt \colon w \in W^{1,2}((-L, L), \mathbb{R}^3) \right\} \\
&= 2Lh \inf \left\{ \int_{-L}^{L} \left((v^2 - \sigma(t))^2 + h^2 (v')^2 \right) dt \colon v \in W^{1,2}((-L, L), \mathbb{R}) \quad \text{with} \quad v \geq 0 \text{ a.e.} \right\} \\
&= 2L^2 h M_1^h.
\end{aligned}
$$

The first inequality follows by replacing $\partial_2 y$ with a function w which depends only on x_1 and taking the infimum amongst $W^{1,2}$ functions, and the second follows by noting $(|w|')^2 \leq |w'|^2$. Finally, we simply replace $|w|$ by a function $v \geq 0$ for the first equality, and the second equality follows by a change of variables $v(t) = u(t/L)$. This completes the proof. □

Definition for Case 2: In this case, we again relabel so that $\alpha = 1$ and $\beta = 2$. We note that $\tilde{n}_{02} \neq 0$ (otherwise, following the definition of Case 2, $\tilde{n}_{01} = 0$ and therefore $\tilde{n}_{01} = \tilde{n}_{02}$ which is not allowed). Hence, we again fix a global Cartesian frame so that $e_2 = \tilde{n}_{02}/|\tilde{n}_{02}|$ and e_1 points in the direction of region ω_2. Next, for some $R > 0$, we find a ball $B_R \subset \omega_1 \cup \omega_2$ whose center intersects the interface g_{12}. Note that $R = R(\omega)$ depends only on ω. We set $\theta \in (0, \pi/2]$ to be the acute angle between $\tilde{n}_{02}$ and g_{12} (which is non-zero by definition of this case) and define

$$
L_1 := R\cos(\theta), \qquad \tau := L_1 \frac{\tan(\theta)}{1 + \tan(\theta)}.
$$

We note that by their very definition, L_1 and τ depend only on ω and n_0. Further, $\tau \in (0, L_1]$. In particular, it cannot be zero since $\theta \neq 0$. A schematic of this case is provided in Figure 9b. We make the following observation for this case:

Proposition 4.2. *If ω and n_0 have an interface as in the definition of Case 2 (see Figure 9b), then for any $y \in W^{2,2}(\omega, \mathbb{R}^3)$,*

$$
\tilde{\mathcal{I}}^h_{n_0}(y) \geq L_1 h \int_{-\tau}^{\tau} M_2^h(s) ds
$$

where

$$
M_2^h(s) := \left\{ \int_{-1}^{1} \left((u^2 - \sigma(s,t))^2 + \frac{h^2}{L_1^2}(u')^2 \right) dt \colon u \in W^{1,2}((-1, 1), \mathbb{R}) \quad \textit{with} \quad u \geq 0 \textit{ a.e.} \right\}
$$

$$\sigma(s,t) = \begin{cases} \sigma_1 & \textit{if } t < \max\{-1+\tau/L_1, (1-\tau/L_1)(s/\tau)\} \\ \sigma_2 & \textit{if } t > \min\{1-\tau/L_1, (1-\tau/L_1)(s/\tau)\}. \end{cases}$$

Here $\sigma_1, \sigma_2 \geq 0$ and $\sigma_1 \neq \sigma_2$.

Proof. Let $y \in W^{2,2}(\omega, \mathbb{R}^3)$. Akin to the estimate in (4.1), we reason that

$$\tilde{\mathcal{I}}^h_{n_0}(y) \geq 2h \int_{-\tau}^{\tau} \int_{-L_1}^{L_1} \left(||\partial_2 y|^2 - \sigma(\tilde{x})|^2 + h^2 |\partial_2^2 y|^2 \right) d\tilde{x} \tag{4.2}$$

for $\sigma(\tilde{x})$ depending on both coordinates and given by

$$\sigma(\tilde{x}) = \begin{cases} r^{-1/3}(1+(r-1)\frac{(\tilde{n}_{01}\cdot\tilde{n}_{02})^2}{|\tilde{n}_{02}|^2}) & \text{if } x_2 < \max\{-L_1+\tau, (L_1-\tau)(x_1/\tau)\} \\ r^{-1/3}(1+(r-1)|\tilde{n}_{02}|^2) & \text{if } x_2 > \min\{L_1-\tau, (L_1-\tau)(x_1/\tau)\}. \end{cases}$$

Since $\tilde{n}_{01} \neq \pm\tilde{n}_{02}$ by definition, $(\tilde{n}_{01}\cdot\tilde{n}_{02}) \neq |\tilde{n}_{02}|^2$. Therefore, $\sigma_1 := r^{-1/3}(1+(r-1)|\tilde{n}_{02}|^{-2}(\tilde{n}_{01}\cdot\tilde{n}_{02})^2)$ does not equal $\sigma_2 := r^{-1/3}(1+(r-1)|\tilde{n}_{02}|^2)$. Moreover, $\sigma_1, \sigma_2 > 0$ since $r > 0$.

Now, given the inequality in (4.2), we again see that in this case

$$\begin{aligned} \mathcal{I}^h_{n_0}(y) &\geq h \int_{-\tau}^{\tau} \inf \left\{ \int_{-L_1}^{L_1} \left((|w|^2 - \sigma(s,t))^2 + h^2 |w'|^2 \right) dt \colon w \in W^{1,2}((-L_1, L_1), \mathbb{R}^3) \right\} ds \\ &\geq L_1 h \int_{-\tau}^{\tau} M_2^h(s) ds \end{aligned}$$

as desired. This part of the argument is completely analogous to that of Proposition 4.1. This completes the proof. $\square$

4.2 The Modica-Mortola analog and proof of optimal scaling

We have shown that given any design described by flat sheet $\omega \subset \mathbb{R}^2$ and $n_0 \colon \omega \to \mathbb{S}^2$ satisfying (1.9), (1.10) and (1.11), the problem of deducing a lowerbound on the energy (1.18) reduces to a unit cell problem which has at most two flavors: Case 1 and Case 2 in Section 4.1. Actually though, following Proposition 4.1 and 4.2, we find for the lowerbound that one only needs to consider the variational problem given by the one dimensional functionals

$$\mathcal{I}^h_s(u) := \int_{-1}^{1} \frac{1}{h} \left((u^2 - \sigma(s,t))^2 + c_1 h (u')^2 \right) dt, \quad s \in [-c_2, c_2]$$

minimized amongst the functions $\{u \in W^{1,2}((-1,1), \mathbb{R}) \colon u \geq 0\}$ where

$$\sigma(s,t) = \begin{cases} \sigma_1 & \text{if } t < \max\{-1+c_3, c_4 s\} \\ \sigma_2 & \text{if } t > \min\{1-c_3, c_4 s\} \end{cases}$$

for $c_1, c_2 > 0, c_3 \in (0,1]$ and $c_4 \in [0, (1-c_3)/c_2)$. In fact, the proof of Theorem 1.6 follows from the observation that the infimum of $\mathcal{I}^h_s$ is bounded away from zero. Precisely:

Lemma 4.3. *For any $c_1, c_2 > 0, c_3 \in (0,1]$ and $c_4 \in [0, (1-c_3)/c_2)$, and for $h > 0$ sufficiently small*

$$\inf \left\{ \mathcal{I}^h_s(u) \colon u \in W^{1,2}((-1,1), \mathbb{R}) \textit{ with } u \geq 0 \textit{ a.e.} \right\} \geq c_L$$

where $c_L = c_L(c_1, c_2, c_3, c_4) > 0$ is independent of s and h.

This is the crucial observation for the theorem. Indeed:

Proof of Theorem 1.6. We note following Section 4.1 that it suffices to restrict to the unit cell problem given by the two cases in Figure 9. From Proposition 4.1 and 4.2, we have that for any $y \in W^{2,2}(\omega, \mathbb{R}^3)$,

$$\mathcal{I}^h_{n_0}(y) \geq \begin{cases} 2L^2 h M_1^h & \text{for Case 1} \\ L_1 h \int_{-\tau}^{\tau} M_2^h(s) dx & \text{for Case 2.} \end{cases}$$

In addition, we observe that

$$\begin{aligned} M_1^h &= h \inf \left\{ \mathcal{I}^h_s(u) \colon u \in W^{1,2}((-1,1), \mathbb{R}) \text{ with } u \geq 0 \text{ a.e.} \right\} \\ &\quad \text{when} \quad c_1 = L^{-1},\ c_3 = 1,\ c_4 = 0; \\ M_2^h(s) &= h \inf \left\{ \mathcal{I}^h_s(u) \colon u \in W^{1,2}((-1,1), \mathbb{R}) \text{ with } u \geq 0 \text{ a.e.} \right\} \\ &\quad \text{when} \quad c_1 = L_1^{-1},\ c_2 = \tau,\ c_3 = \tau/L_1,\ c_4 = (1/\tau - 1/L_1). \end{aligned}$$

Thus, by these observations and given Lemma 4.3,

$$\mathcal{I}^h_{n_0}(y) \geq \begin{cases} 2L^2 c_L h^2 & \text{for Case 1} \\ 2L_1 \tau c_L h^2 & \text{for Case 2} \end{cases}$$

for $c_L = c_L(c_1, c_2, c_3, c_4) > 0$ as in the lemma. This completes the proof. □

To close the argument, it remains to prove Lemma 4.3. We do this now:

Proof of Lemma 4.3. By the direct methods in the calculus of variations (see, for instance, Dacorogna [19]), we find that for any $s \in [-c_2, c_2]$ and $h > 0$, there exists a minimizer to $\mathcal{I}^h_s$ in the space $\{u \in W^{1,2}((-1,1), \mathbb{R}) \colon u \geq 0 \text{ a.e.}\}$. For the lowerbound, it suffices to restrict our attention to any such minimizer, which we label as u^h_s. Further, we may assume for some fixed constant $M > 0$ that

$$\mathcal{I}^h_s(u^h_s) < M. \tag{4.3}$$

Indeed, if for some $s \in [-c_2, c_2]$ and $h > 0$ this does not hold, then we immediately establish a lowerbound for this case since the reverse inequality holds.

Now, since $c_4 \in [0, (1 - c_3)/c_2)$, we have that $\sigma(s,t) = \sigma_1$ when $t < -1 + c_3$ and $\sigma(s,t) = \sigma_2$ when $t > 1 - c_3$. Without loss of generality, we assume $\sigma_1 < \sigma_2$. We let $\langle \sigma \rangle = (\sigma_1 + \sigma_2)/2$, and we claim that for $h > 0$ sufficiently small,

$$\begin{cases} \text{for some} \quad t \in [-1, -1 + c_3/2], & u^h_s(t)^2 \in (\frac{1}{2}\sigma_1, \frac{1}{2}(\sigma_1 + \langle \sigma \rangle)); \\ \text{for some} \quad t \in [1 - c_3/2, 1], & u^h_s(t)^2 \in (\frac{1}{2}(\sigma_2 + \langle \sigma \rangle), \frac{3}{2}\sigma_2). \end{cases} \tag{4.4}$$

Indeed, suppose the first condition does not hold. Then $(u^h_s(t)^2 - \sigma_1)^2 \geq \frac{1}{4} \min\{\sigma_1^2, (\langle \sigma \rangle - \sigma_1)^2\} > 0$ on the interval $[-1, -1 + c_3/2]$ which gives

$$\mathcal{I}^h_s(u^h_s) \geq \int_{-1}^{-1+c_3/2} \frac{1}{h}((u^h_s)^2 - \sigma_1)^2 dt \geq \frac{c_3}{8h} \min\{\sigma_1^2, (\langle \sigma \rangle - \sigma_1)^2\}.$$

Taking $h > 0$ sufficiently small, we eventually arrive at a contradiction to (4.3). The second condition in (4.4) holds by an identical argument.

Now, by the Sobolev embedding theorem $u_s^h \in W^{1,2}((-1,1),\mathbb{R})$ has a continuous representative. This continuity and the observation that (4.4) holds leads to the non-zero lowerbound on the energy. Indeed, we have the estimate

$$\mathcal{I}_s^h(u_s^h) \geq 2\sqrt{c_1}\int_{-1}^{1} |(u_s^h)^2 - \sigma(s,t)||(u_s^h)'|dt. \tag{4.5}$$

Hence, we define

$$a := \max\big\{t \in [-1,1] \colon u_s^h(t)^2 = \frac{1}{2}(\sigma_1 + \langle\sigma\rangle)\big\}, \quad b := \min\big\{t \in (a,1] \colon u_s^h(t)^2 = \frac{1}{2}(\sigma_2 + \langle\sigma\rangle)\big\}.$$

7 By the continuity of u_s^h and the observation (4.4), these quantities (as asserted) do, in fact, exist. Moreover,

$$\begin{aligned}\int_{-1}^{1} |(u_s^h)^2 - \sigma(s,t)||(u_s^h)'|dt &\geq \int_a^b |(u_s^h)^2 - \sigma(s,t)||(u_s^h)'|dt \geq \frac{1}{2}\min_{1,2}\{|\langle\sigma\rangle - \sigma_i|\}\int_a^b |(u_s^h)'|dt \\ &\geq \frac{1}{2}\min_{1,2}\{|\langle\sigma\rangle - \sigma_i|\}\Big|\int_a^b (u_s^h)'dt\Big| = \frac{1}{2}\min_{1,2}\{|\langle\sigma\rangle - \sigma_i|\}|u_s^h(b) - u_s^h(a)| \\ &= \frac{1}{2\sqrt{2}}\min_{1,2}\{|\langle\sigma\rangle - \sigma_i|\}|(\sigma_2 + \langle\sigma\rangle)^{1/2} - (\sigma_1 - +\langle\sigma\rangle)^{1/2}| \end{aligned} \tag{4.6}$$

by the fundamental theorem of calculus. Since this lowerbound is positive and independent of s and h, combining (4.5) and (4.6) completes the proof. □

5 Compactness for bending configurations and the metric constraint

In this section, we prove that the metric constraint (1.6) is necessary for a configuration in pure bending when Frank elasticity is comparable to entropic elasticity at the bending scale, Theorem 1.12. In Section 5.1, we address some key preliminary results for this compactness, including a crucial lemma which is a consequence of Geometric Rigidity. In Section 5.2, we prove Theorem 1.12.

5.1 Preliminaries for compactness

The key lemma which enables a proof of compactness in this setting is based on the seminal result of Geometric Rigidity by Friesecke, James and Müller [23], and generalization to non-Euclidean plates by Lewicka and Pakzad [30].

Lemma 5.1. *Let $\omega \subset \mathbb{R}^2$ bounded and Lipschitz, and $r_f, r_0 > 1$ and $\tau \geq 0$. There exists a $C = C(\omega, r_f, r_0, \tau) > 0$ with the following property: For every $h > 0$, $\Omega_h := \omega \times (-h/2, h/2)$, $Y^h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$, $N^h \in W^{1,2}(\Omega_h, \mathbb{S}^2)$ and N_0^h as in (1.7) with $n_0 \in W^{1,2}(\omega, \mathbb{S}^2)$, there exists an associated matrix field $G^h \colon \omega \to \mathbb{R}^{3\times 3}$ satisfying the estimates*

$$\begin{aligned}&\frac{1}{h}\int_{\Omega_h} |G^h - (\ell_{N^h}^f)^{-1/2}(\nabla Y^h)(\ell_{N_0^h}^0)^{1/2}|^2 dx \\ &\quad \leq \frac{C}{h}\int_{\Omega_h}\Big(\mathrm{dist}^2((\ell_{N^h}^f)^{-1/2}(\nabla Y^h)(\ell_{N_0^h}^0)^{1/2}, SO(3)) + h^2(|\nabla N^h|^2 + |\tilde{\nabla} n_0|^2 + 1)\Big)dx,\end{aligned}$$

$$
\begin{aligned}
&\int_\omega |\tilde\nabla G^h|^2 d\tilde x \\
&\qquad \le \frac{C}{h^3}\int_{\Omega_h}\Big(\mathrm{dist}^2((\ell^f_{N^h})^{-1/2}(\nabla Y^h)(\ell^0_{N^h_0})^{1/2}, SO(3)) + h^2(|\nabla N^h|^2 + |\tilde\nabla n_0|^2 + 1)\Big)\, dx.
\end{aligned}
$$

We address this result in Appendix E. For similar results related to non-Euclidean plates in a different context, see Lewicka et al. [10], [29] .

Now, in terms of the rescaled variables W^h and M_0^h, we have:

Proposition 5.2. *Let $\omega \subset \mathbb{R}^2$ bounded and Lipschitz, $r_f, r_0 > 1$, $\tau \ge 0$, and ε_h as in (1.31). Let $W^h \in W^{1,2}(\Omega,\mathbb{R}^3)$, $M^h \in W^{1,2}(\Omega,\mathbb{S}^2)$ and M_0^h as in (1.32) for $n_0 \in W^{1,2}(\omega,\mathbb{S}^2)$. There exists an associated matrix field $G^h\colon \omega \to \mathbb{R}^{3\times 3}$ such that*

$$
\int_\Omega |G^h - (\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M_0^h})^{1/2}|^2 dz \le Ch^2(\mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h) + \|\tilde\nabla n_0\|^2_{L^2(\omega)} + 1) \tag{5.1}
$$

$$
\int_\Omega |\tilde\nabla G^h|^2 d\tilde z \le C(\mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h) + \|\tilde\nabla n_0\|^2_{L^2(\omega)} + 1) \tag{5.2}
$$

for δ_h, c_l as in (1.31) and some uniform $C = C(\omega, r_f, r_0, c_l\tau)$ which is independent of h.

Proof. Using Proposition A.3 and the identity (3.13), we find that

$$
\begin{aligned}
\int_\Omega \mathrm{dist}^2((\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M_0^h})^{1/2}, SO(3))dz &\le \int_\Omega \widehat{W}_{nH}((\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M_0^h})^{1/2})dz \\
&\le \int_\Omega \widehat{W}^e(\nabla_h W^h, M^h, M_0^h)dz
\end{aligned}
$$

where $\widehat{(\cdot)} = (2/\mu)(\cdot)$. Since ε_h as in (1.31), we also find that

$$
\int_\Omega h^2|\nabla_h M^h|^2 dz \le \frac{1}{c_l^2}\int_\Omega \varepsilon_h^2 |\nabla_h M^h|^2 dz.
$$

Combining these two estimates, we find that

$$
\int_\Omega \Big(\mathrm{dist}^2((\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M_0^h})^{1/2}, SO(3)) + h^2|\nabla_h M^h|^2\Big)\, dz \le Ch^2\mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h) \tag{5.3}
$$

for some uniform $C = C(c_l)$.

To obtain the desired estimates (5.1) and (5.2), we change variables via

$$
z(x) := (x_1, x_2, x_3/h), \quad x \in \Omega_h, \tag{5.4}
$$

and (1.29), we apply Lemma 5.1, and the estimates follow from the bound (5.3). □

5.2 Compactness for comparable entropic and Frank elasticity in bending

We turn now to the proof of Theorem 1.12. For clarity, we break up the proof into several steps.

Recall that for this theorem, we suppose M_0^h as in (1.32) with $n_0 \in W^{1,2}(\omega,\mathbb{S}^2)$ and ε_h as in (1.31). We consider a sequence $\{W^h\} \subset W^{1,2}(\Omega,\mathbb{R}^3)$ such that

$$
\mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h) \le C < \infty \tag{5.5}
$$

as $h \to 0$. The convergences stated in each step are for a suitably chosen subsequence as $h \to 0$.

Step 1. $M_0^h \to n_0$ in $L^2(\Omega, \mathbb{S}^2)$, and $(\ell^0_{M_0^h})^{\pm 1/2} \to (\ell^0_{n_0})^{\pm 1/2}$ in $L^2(\Omega, \mathbb{R}^{3\times 3})$.

The first convergence is a trivial consequence of the definition of M_0^h in (1.32). The second follows from the estimate $|(\ell^0_\alpha)^{\pm 1/2} - (\ell^0_\beta)^{\pm 1/2}| \leq C(r_0)|\alpha - \beta|$ for $\alpha, \beta \in \mathbb{S}^2$ and the first convergence. □

Step 2. The functions

$$M^h := N(\nabla_h W^h, M_0^h) \quad \text{on } \Omega$$

are in $W^{1,2}(\Omega, \mathbb{S}^2)$. Moreover, $M^h \rightharpoonup n$ in $W^{1,2}(\Omega, \mathbb{S}^2)$ for some n independent of z_3 and $(\ell^f_{M^h})^{\pm 1/2} \to (\ell^f_n)^{\pm 1/2}$ in $L^2(\Omega, \mathbb{R}^{3\times 3})$.

M^h is in $W^{1,2}(\Omega, \mathbb{S}^2)$ since the energy $\mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h)$ is finite. In particular, for $h \ll 1$ we have

$$\frac{1}{c_l^2}\int_\Omega |\nabla M^h|^2 dz \leq \frac{1}{c_l^2}\int_\Omega \left(|\tilde{\nabla} M^h|^2 + \frac{1}{h^2}|\partial_3 M^h|^2\right) dz \leq \mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h) \leq C \tag{5.6}$$

for C independent of h by (5.5). Thus, up to a subsequence $M^h \rightharpoonup n$ in $W^{1,2}(\Omega, \mathbb{R}^3)$. By Rellich's theorem, taking a further subsequence (if necessary), we have strong convergence, $M^h \to n$ in $L^2(\Omega, \mathbb{R}^3)$. Since $M^h \in \mathbb{S}^2$ a.e., we deduce that $n \in \mathbb{S}^2$ a.e. by this strong convergence. Further, n is independent of z_3 since by the estimate (5.6), we find $\partial_3 M^h \to 0$ in $L^2(\Omega, \mathbb{R}^3)$, and therefore $\partial_3 n = 0$ a.e. by the uniqueness of the weak $W^{1,2}$ limit. The convergences of $(\ell^f_{M^h})^{\pm 1/2}$ follow by an argument similar to the convergences of $(\ell^0_{M_0^h})^{\pm 1/2}$ in Step 1. □

Step 3. $(W^h - \frac{1}{|\Omega|}\int_\Omega W^h dz) \rightharpoonup y$ in $W^{1,2}(\Omega, \mathbb{R}^3)$ for some y independent of z_3. Also, $h^{-1}\partial_3 W^h \rightharpoonup b$ in $L^2(\Omega, \mathbb{R}^3)$.

We have

$$\frac{1}{c}\int_\Omega (|\tilde{\nabla} W^h|^2 + |h^{-1}\partial_3 W^h|^2 - 1) dz \leq \int_\Omega W^e(\nabla_h W^h, M^h, M_0^h) dz \leq \mathcal{J}^{h,\varepsilon_h}_{M_0^h}(W^h) \leq C \tag{5.7}$$

by Proposition A.1 and (5.5). Thus, since $|\nabla W^h| \leq |\nabla_h W^h|$ for $h \ll 1$, we conclude the first convergence (up to a subsequence) given the estimate (5.7) and an application of the Poincaré inequality. We again use (5.7) to conclude that up to a subsequence, $h^{-1}\partial_3 W^h \rightharpoonup b$ in $L^2(\Omega, \mathbb{R}^3)$ for some vector valued function b, and that the limit y is independent of z_3 (exactly the same argument as in Step 2 for n independent of z_3). □

Step 4. There exists a sequence of matrix fields $\{G^h\}$ with $G^h \colon \omega \to \mathbb{R}^{3\times 3}$ such that

$$\int_\Omega |G^h - (\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M_0^h})^{1/2}|^2 dz \leq Ch^2, \qquad \int_\omega |\tilde{\nabla} G^h|^2 d\tilde{z} \leq C \tag{5.8}$$

for C independent of h. Moreover, $G^h \rightharpoonup R$ in $W^{1,2}(\omega, \mathbb{R}^{3\times 3})$ with $R \in SO(3)$ a.e.

To obtain the estimates in (5.8), we first apply Proposition 5.2 to obtain each matrix field G^h, and then observe that the estimates follow from the bound on the energy (5.5) and the fact that by hypothesis $n_0 \in W^{1,2}(\omega, \mathbb{S}^2)$.

The first estimate in (5.8) implies

$$\int_\omega |G^h|^2 d\tilde{z} \leq Ch^2 + 2c(r_f, r_0)\int_\Omega |\nabla_h W^h|^2 dz$$

The constant $c(r_f, r_0)$ is from estimating the step-length tensors. From Step 3, $\nabla_h W^h$ is bounded uniformly in L^2, and therefore using the above estimate and the second estimate in (5.8), we conclude that up to a subsequence $G^h \rightharpoonup R$ in $W^{1,2}(\omega, \mathbb{R}^{3\times 3})$. Now, to deduce that $R \in SO(3)$ a.e., we estimate via two applications of the triangle inequality

$$\begin{aligned}\int_\omega \operatorname{dist}^2(R, SO(3))d\tilde{z} &\leq 2\int_\Omega \left(\operatorname{dist}^2(G^h, SO(3))dz + |G_h - R|^2\right)dz \\ &\leq C\left(h^2 + \int_\Omega (\operatorname{dist}^2((\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M^h_0})^{1/2}, SO(3)) + |G_h - R|^2)dz\right) \\ &\leq C\left(h^2 + h^2 \mathcal{J}^{h,\varepsilon_h}_{M^h_0}(W^h) + \int_\omega |G_h - R|^2 d\tilde{z}\right).\end{aligned}$$

In the second estimate, we also use the first estimate in (5.8). For the third estimate, we recall (5.3). Now, by Rellich's theorem, we have $G^h \to R$ in $L^2(\Omega, \mathbb{R}^{3\times 3})$ for a subsequence. Thus, it is clear given (5.5) that the upperbound above vanishes as $h \to 0$. This implies $R \in SO(3)$ a.e. as desired. □

Step 5. $(\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M^h_0})^{1/2} \to R$ *in* $L^2(\Omega, \mathbb{R}^{3\times 3})$ *for* R *from Step 4.*

Since

$$\begin{aligned}&\int_\Omega |(\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M^h_0})^{1/2} - R|^2 dz \\ &\quad\leq 2\int_\Omega \left(|G^h - R|^2 + |G^h - (\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M^h_0})^{1/2}|^2\right)dz,\end{aligned}$$

we conclude that $(\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M^h_0})^{1/2} \to R$ in $L^2(\Omega, \mathbb{R}^{3\times 3})$ using Step 4. □

Step 6. Actually, $R = (\ell^f_n)^{-1/2}(\tilde{\nabla} y|b)(\ell^0_{n_0})^{-1/2}$ *a.e. for the limiting fields above. In particular,* $(\ell^f_n)^{-1/2}(\tilde{\nabla} y|b)(\ell^0_{n_0})^{1/2} \in W^{1,2}(\omega, SO(3))$.

We observe that $\|(\ell^f_{M^h})^{-1/2}(\nabla_h W^h)\|_{L^2(\Omega)} \leq c(r_f)\|\nabla_h W^h\|_{L^2(\Omega)} \leq C$ by the compactness of the step-length tensor on $\mathbb{S}^2$ and following Step 3. So up to a subsequence $(\ell^f_{M^h})^{-1/2}(\nabla_h W^h)$ converges weakly in $L^2(\Omega, \mathbb{R}^{3\times 3})$. In addition, the results of Step 2 and 3 imply $(\ell^f_{M^h})^{-1/2}(\nabla_h W^h) \rightharpoonup (\ell^f_n)^{-1/2}(\tilde{\nabla} y|b)$ in $L^1(\Omega, \mathbb{R}^{3\times 3})$. Hence, in combination and by the uniqueness of the L^1 limit, we also have weak convergence to this limiting field in L^2 (rather than just L^1).

Given the weak-L^2 convergence just established and the convergence in Step 1, we deduce

$$(\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M^h_0})^{1/2} \rightharpoonup (\ell^f_n)^{-1/2}(\tilde{\nabla} y|b)(\ell^0_{n_0})^{1/2} \quad \text{in } L^1(\Omega, \mathbb{R}^{3\times 3}).$$

By the convergence in Step 5 and the uniqueness of the weak-L^1 limit $R = (\ell^f_n)^{-1/2}(\tilde{\nabla} y|b)(\ell^0_{n_0})^{1/2}$ a.e. To complete the proof, we recall from Step 4 that $R \in W^{1,2}(\omega, \mathbb{R}^{3\times 3})$ and that $R \in SO(3)$ a.e. □

Step 7. The sequences in Step 3 actually converge strongly in their respective spaces. In addition, we have improved regularity: $y \in W^{2,2}(\omega, \mathbb{R}^3)$ *and* b *is independent of* z_3 *and in* $W^{1,2}(\omega, \mathbb{R}^3)$.

For the strong L^2 convergence, we have the estimate

$$\begin{aligned}&\int_\Omega |\nabla_h W^h - (\tilde{\nabla} y|b)|^2 dz \\ &\leq 2\int_\Omega \left(|\nabla_h W^h - (\ell^f_{M^h})^{1/2} R(\ell^0_{M^h_0})^{-1/2}|^2 + |(\ell^f_{M^h})^{1/2} R(\ell^0_{M^h_0})^{-1/2} - (\tilde{\nabla} y|b)|^2\right)dz\end{aligned}$$

$$\leq C\int_{\Omega}\Big(|(\ell^f_{M^h})^{-1/2}(\nabla_h W^h)(\ell^0_{M^h_0})^{-1/2}-R|^2+|(\ell^f_{M^h})^{1/2}-(\ell^f_n)^{1/2}|^2+|(\ell^0_{M^h_0})^{-1/2}-(\ell^0_{n_0})^{-1/2}|^2\Big)\,dz$$

using that $R=(\ell^f_n)^{-1/2}(\tilde\nabla y|b)(\ell^0_{n_0})^{1/2}$ a.e. from Step 6, and that the step-length tensors are compact and invertible on $\mathbb{S}^2$. It's clear that the upper bound $\to 0$ as $h\to 0$ due to the strong-L^2 convergences of each term (established in the previous steps). Thus, $\nabla_h W^h\to(\tilde\nabla y|b)$ in $L^2(\Omega,\mathbb{R}^{3\times 3})$ as desired.

For the improved regularity, we see that

$$(\tilde\nabla y|b)=(\ell^f_n)^{1/2}R(\ell^0_{n_0})^{-1/2}\qquad\text{a.e. on }\omega.$$

Note that $R\in W^{1,2}$ from Step 4, $n\in W^{1,2}$ from Step 2, and $n_0\in W^{1,2}$ by assumption. By the structure of the step-length tensors, we also have that $(\ell^0_{n_0})^{-1/2},(\ell^f_n)^{1/2}\in W^{1,2}$. Thus, the improved regularity is clear from differentiating the right side using the product rule for these Sobolev functions. Finally, b is independent of z_3 since $(\ell^f_n)^{1/2}R(\ell^0_{n_0})^{-1/2}e_3$ is independent of z_3. □

Step 8. Actually,

$$n=N((\tilde\nabla y|b),n_0)=\frac{(\tilde\nabla y|b)n_0}{|(\tilde\nabla y|b)n_0|}\qquad\text{a.e. on }\omega. \tag{5.9}$$

Since both $\nabla_h W^h$ and M^h_0 strongly converge in L^2, they converge (up to a subsequence) pointwise a.e. to the functions $(\tilde\nabla y|b)$ and n_0 respectively. $N(F,m_0)$ is continuous is on the set $\{(F,m_0)\in\mathbb{R}^{3\times3}\times\mathbb{S}^2\colon \det F=1\}$. By this continuity and the pointwise a.e. convergence, we conclude $N(\nabla_h W^h,M^h_0)\to N((\tilde\nabla y|b),n_0)$ in $L^2(\Omega,\mathbb{R}^{3\times3})$ by Lebesgue dominated convergence. Considering the convergence to n in Step 2, the equality follows. □

Step 9. Finally,

$$(\tilde\nabla y)^T\tilde\nabla y=\ell_{n_0}\ \text{a.e. on }\omega. \tag{5.10}$$

From Step 6, $(\ell^f_n)^{-1/2}(\tilde\nabla y|b)(\ell^0_{n_0})^{1/2}\in W^{1,2}(\omega,SO(3))$, and from Step 7, n as in (5.9). Hence,

$$\int_\omega W^e((\tilde\nabla y|b),n,n_0)d\tilde z=\int_\omega W_{nH}((\ell^f_n)^{-1/2}(\tilde\nabla y|b)(\ell^0_{n_0})^{1/2})d\tilde z=0$$

using the definitions of W^e and W_{nH}, and since W_{nH} vanishes on $SO(3)$. The conclusion (5.10) follows by Proposition A.4. □

Proof of Theorem 1.12. The proof follows by the collection of steps above. In particular, Step 7 shows the strong convergence of $\{W^h\}$ and the desired regularity of the limiting field y as a consequence of (5.5). Step 9 shows that the metric constraint must also be satisfied. This is the proof. □

Appendices

A Some facts about the entropic energy

A.1 Some estimates on the energy densities

Proposition A.1. *If $F\in\mathbb{R}^{3\times3}$ such that $\det F=1$ and $n_0\in\mathbb{S}^2$, then*

$$\frac{1}{c}(|F|^2-1)\leq W^e(F,n,n_0)\leq c(|F|^2+1) \tag{A.1}$$

for some $c = c(r_f, r_0) > 0$.

Proof. Since $\det F = 1$ and $n_0 \in \mathbb{S}^2$, we have $n = Fn_0/|Fn_0| \in \mathbb{S}^2$ and

$$\begin{aligned}W^e(F, n, n_0) &= W_{nH}((\ell_n^f)^{-1/2} F (\ell_{n_0}^0)^{1/2}) \\ &\geq \frac{\mu}{2}\left(\frac{1}{3}\sigma_{min}^2((\ell_{n_0}^0)^{1/2})|(\ell_n^f)^{-1/2}F|^2 - 3\right) \\ &\geq \frac{\mu}{2}\left(\frac{1}{9}\sigma_{min}^2((\ell_{n_0}^0)^{1/2})\sigma_{min}^2((\ell_n^f)^{-1/2})|F|^2 - 3\right).\end{aligned}$$

We note that $\sigma_{min}((\ell_{n_0}^0)^{1/2})$ is nonzero and depends only on r_0. Similarly, $\sigma_{min}((\ell_n^f)^{-1/2})$ is nonzero and depends only on r_f. Thus, the lowerbound in (A.1) follows. The upperbound is similar. □

Proposition A.2. *Let* $A \in \mathbb{R}^{3\times 3}$ *such that* $\det(I_{3\times 3} + A) = 1$. *We find*

$$W_{nH}(I_{3\times 3} + A) \leq C(|A|^2 + |A|^3), \tag{A.2}$$

for W_{nH} *in (3.14) and for some uniform constant* $C > 0$.

Proof. For the inequality on W_{nH}, we note that since $\det(I_{3\times 3} + A) = 1$,

$$\operatorname{Tr}(A) = -\operatorname{Tr}(\operatorname{cof} A) - \det(A)$$

and W_{nH} is finite. Hence,

$$\begin{aligned}W_{nH}(I + A) &= \frac{\mu}{2}\left(|I + A|^2 - 3\right) \\ &= \frac{\mu}{2}\left(|A|^2 + 2\operatorname{Tr}(A)\right) \\ &= \frac{\mu}{2}\left(|A|^2 - 2\operatorname{Tr}(\operatorname{cof} A) - 2\det(A)\right).\end{aligned} \tag{A.3}$$

Since there exists a $C > 0$ independent of A such that $|\operatorname{Tr}(\operatorname{cof} A)| \leq C|A|^2$ and $|\det(A)| \leq C|A|^3$, we conclude (A.2) following the identity (A.3). □

A.2 Relating the metric and the step-length tensor

Proposition A.3. *The energy density* W_{nH} *in (3.14) satisfies* $W_{nH}(A) \geq \frac{\mu}{2}\operatorname{dist}^2(A, SO(3))$ *for all* $A \in \mathbb{R}^{3\times 3}$.

Proof. We may assume $\det A = 1$ as the bound holds trivially otherwise. Consequently and by the polar decomposition theorem, $A = RU$ for $R \in SO(3)$ and U positive definite. Hence, we find $\operatorname{dist}(A, SO(3)) = |U - I_{3\times 3}|$. In addition, since $\det U = 1$ we conclude

$$\begin{aligned}\frac{\mu}{2}\operatorname{dist}^2(A, SO(3)) &= \frac{\mu}{2}|U - I_{3\times 3}|^2 \\ &= \frac{\mu}{2}(|U|^2 - 2\operatorname{Tr}(U) + 3) \\ &\leq \frac{\mu}{2}(|U|^2 - \inf\{\operatorname{Tr}(B) \colon B \text{ pos. def.}, \det B = 1\} + 3) \\ &= \frac{\mu}{2}(|U|^2 - 3) = W_{nH}(A).\end{aligned}$$

Here, we used that the infimum above is attained at $B = I$. □

Proposition A.4. *The energy density W^e in (1.2) satisfies $W^e(F, Fn_0/|Fn_0|, n_0) = 0$ if and only if $\det F = 1$ and $F^T F = \ell_{n_0}$ for ℓ_{n_0} defined in (1.5). In addition, if these identities hold, then $Fn_0/|Fn_0| = Rn_0$ where $R \in SO(3)$ is the unique rotation associated with the polar decomposition of F.*

Proof. We first assume $W^e(F, Fn_0/|Fn_0|, n_0) = 0$. Then $\det F = 1$, $n_0 \in \mathbb{S}^2$ and $|Fn_0| \neq 0$. We set $n := Fn_0/|Fn_0| \in \mathbb{S}^2$ and observe from (3.13),

$$0 = W^e(F, Fn_0/|Fn_0|, n_0) = W_{nH}((\ell_n^f)^{-1/2} F (\ell_{n_0}^0)^{1/2}).$$

Thus, we deduce from Proposition A.3 that $(\ell_n^f)^{-1/2} F (\ell_{n_0}^0)^{1/2} = R$ for some $R \in SO(3)$. Evidently then,

$$F = (\ell_n^f)^{1/2} R (\ell_{n_0}^0)^{-1/2}. \tag{A.4}$$

Further,

$$\begin{aligned} r_f^{1/6} n = (\ell_n^f)^{-1/2} n &= (\ell_n^f)^{-1/2} \left(\frac{Fn_0}{|Fn_0|} \right) \\ &= (\ell_n^f)^{-1/2} \left(\frac{(\ell_n^f)^{1/2} R (\ell_{n_0}^0)^{-1/2} n_0}{|Fn_0|} \right) = r_0^{-1/6} \left(\frac{Rn_0}{|Fn_0|} \right). \end{aligned}$$

Here, we used the definition of n, the result in (A.4) and properties of the step-length tensors (1.3). Since both n and $Rn_0 \in \mathbb{S}^2$, it follows from this equality chain that actually $n = Rn_0$. Substituting this into (A.4) yields

$$F = R(\ell_{n_0}^f)^{1/2} R^T R (\ell_{n_0}^0)^{-1/2} = R(\ell_{n_0}^f)^{1/2} (\ell_{n_0}^0)^{-1/2} = R\ell_{n_0}^{1/2}$$

noting that $(\ell_{Rn_0}^f)^{1/2} = R(\ell_{n_0}^f)^{1/2} R^T$ and $(\ell_{n_0}^f)^{1/2} (\ell_{n_0}^0)^{-1/2} = \ell_{n_0}^{1/2}$. Consequently, $F^T F = \ell_{n_0}$ as desired.

For the other direction, we assume $\det F = 1$ and $F^T F = \ell_{n_0}$. This implies $F = R\ell_{n_0}^{1/2}$ for some $R \in SO(3)$ and $n := Fn_0/|Fn_0| \in \mathbb{S}^2$. Thus,

$$n = \frac{Fn_0}{|Fn_0|} = \frac{R\ell_{n_0}^{1/2} n_0}{|Fn_0|} = \bar{r}^{-1/6} \frac{Rn_0}{|Fn_0|},$$

and since both n and $Rn_0 \in \mathbb{S}^2$, we deduce $n = Rn_0$. Then by definition (1.2), $W^e(F, Fn_0/|Fn_0|, n_0 = W^e(F, Rn_0, n_0)$ and clearly this is finite. Further given (3.13), we find

$$\begin{aligned} W^e(F, Rn_0, n_0) &= W_{nH}((\ell_{Rn_0}^f)^{-1/2} F (\ell_{n_0}^0)^{1/2}) \\ &= W_{nH}(R(\ell_{n_0}^f)^{-1/2} R^T R \ell_{n_0}^{1/2} (\ell_{n_0}^0)^{1/2}) \\ &= W_{nH}(R(\ell_{n_0}^f)^{-1/2} (\ell_{n_0}^f)^{1/2} (\ell_{n_0}^0)^{-1/2} (\ell_{n_0}^0)^{1/2}) = W_{nH}(R) \end{aligned}$$

For this, we have exploited properties of the step-length tensors (see previous paragraph). Hence since $R \in SO(3)$, by Proposition A.3 we find $W_{nH}(R) = 0$ as desired.

Finally for the implication, we note that in the proof of both directions, we found $F = R\ell_{n_0}^{1/2}$ and $n = Rn_0$ for $R \in SO(3)$. Consequently, since $\ell_{n_0}^{1/2}$ is positive definite, R is actually the unique rotation in the polar decomposition of F. $\square$

Proposition A.5. *If $\tilde{F} \in \mathbb{R}^{3\times 2}$ and $n_0 \in \mathbb{S}^2$ such that $\tilde{F}^T\tilde{F} = \tilde{\ell}_{n_0}$, then there exists a $b \in \mathbb{R}^3$ such that*

$$(\tilde{F}|b)^T(\tilde{F}|b) = \ell_{n_0}, \quad \det(\tilde{F}|b) = 1.$$

In particular,

$$\begin{aligned} &b = \bar{b}_1\tilde{F}e_1 + \bar{b}_2\tilde{F}e_2 + \bar{b}_3(\tilde{F}e_1 \times \tilde{F}e_2), \\ &\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix} = (\tilde{\ell}_{n_0})^{-1} I_{2\times 3}\ell_{n_0}e_3, \quad \bar{b}_3 = \det((\tilde{\ell}_{n_0})^{-1}), \\ &(\tilde{\ell}_{n_0})^{-1} = \bar{r}^{1/3}\left(I_{2\times 2} + \left(\frac{1-r}{1+|\tilde{n}_0|^2(r-1)}\right)\tilde{n}_0 \otimes \tilde{n}_0\right). \end{aligned} \tag{A.5}$$

for $\tilde{n}_0 = (n_0 \cdot e_1, n_0 \cdot e_2) \in B_1(0) \subset \mathbb{R}^2$.

Proof. We remark that $\det(\tilde{\ell}_{n_0}) = r^{-2/3}(1 + (r-1)|\hat{n}_0|^2) > 0$ for $r > 0$. Thus $\operatorname{rank}\tilde{F} = 2$ since by hypothesis $\tilde{F}^T\tilde{F} = \tilde{\ell}_{n_0}$. Therefore, $\operatorname{span}\{\tilde{F}e_1, \tilde{F}e_2, \tilde{F}e_1 \times \tilde{F}e_2\} = \mathbb{R}^3$. Hence, (A.5) simply rewrites $b \in \mathbb{R}^3$ equivalently in terms of $(\bar{b}_1, \bar{b}_2, \bar{b}_3) \in \mathbb{R}^3$. The proof follows by explicitly verifying the formulae. □

B Proof of Proposition 1.10

Here we prove Proposition 1.10 which gives an equivalent rewriting of the metric constraint in a form that is more useful for design. In fact we prove slightly more, as detailed in the following

Proposition B.1. *Let $r \in (0,1)$ or > 1. $\tilde{F} \in \mathbb{R}^{3\times 2}$ and $m \in \mathbb{S}^2$ satisfy $\tilde{F}^T\tilde{F} = \tilde{\ell}_m$ if and only if*

$$\tilde{F} \in \mathcal{D}_r, \quad m \in \mathcal{N}^r_{\tilde{F}}$$

for $\mathcal{D}_r$ defined in (1.24) and $\mathcal{N}^r_{\tilde{F}}$ defined in (1.25). Further, if $\tilde{F} \in \mathcal{D}_r$, then there exists an $m \in \mathcal{N}^r_{\tilde{F}}$.

Proof. Let $\tilde{F} \in \mathbb{R}^{3\times 2}$ and $m \in \mathbb{S}^2$ satisfy $\tilde{F}^T\tilde{F} = \tilde{\ell}_m$. Equivalently,

$$\begin{pmatrix} |\tilde{F}\tilde{e}_1|^2 & (\tilde{F}\tilde{e}_1 \cdot \tilde{F}\tilde{e}_2) \\ (\tilde{F}\tilde{e}_1 \cdot \tilde{F}\tilde{e}_2) & |\tilde{F}\tilde{e}_2|^2 \end{pmatrix} = r^{-1/3}\begin{pmatrix} 1 + (r-1)(m\cdot e_1)^2 & (r-1)(m\cdot e_1)(m\cdot e_2) \\ (r-1)(m\cdot e_1)(m\cdot e_2) & 1 + (r-1)(m\cdot e_2)^2 \end{pmatrix} \tag{B.1}$$

for $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{e_1, e_2, e_3\}$ the standard basis on $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Now, since $m \in \mathbb{S}^2$, $(m \cdot e_\alpha)^2 \in [0,1]$ and

$$\begin{aligned} |\tilde{F}\tilde{e}_\alpha|^2 &\in [r^{-1/3}, r^{2/3}] \quad \text{if} \quad r > 1, \\ &\in [r^{2/3}, r^{-1/3}] \quad \text{if} \quad r < 1, \end{aligned} \tag{B.2}$$

from (B.1) for $\alpha = 1, 2$. In addition, $(m \cdot e_1)^2 + (m \cdot e_2)^2 \leq 1$, and so

$$\begin{aligned} |\tilde{F}|^2 = |\tilde{F}\tilde{e}_1|^2 + |\tilde{F}\tilde{e}_2|^2 &\leq r^{2/3} + r^{-1/3} \quad \text{if} \quad \bar{r} > 1, \\ &\geq r^{2/3} + r^{-1/3} \quad \text{if} \quad r < 1, \end{aligned} \tag{B.3}$$

also from (B.1). Now note that substituting the diagonal terms into the square of the off diagonal term in (B.1) results in

$$(\tilde{F}\tilde{e}_1 \cdot \tilde{F}\tilde{e}_2)^2 = (|\tilde{F}\tilde{e}_1|^2 - r^{-1/3})(|\tilde{F}\tilde{e}_2|^2 - r^{-1/3}). \tag{B.4}$$

(a) A single interface

(b) Three-faced junction

Figure 10: Schematic of interfaces and junctions in nonisometric origami

Combining (B.2), (B.3) and (B.4), we conclude $\tilde{F} \in \mathcal{D}_r$ as desired. To prove $m \in \mathcal{N}^r_{\tilde{F}}$, note that since $r \neq 1$, rearranging the diagonal terms in (B.1) gives

$$(m \cdot e_\alpha)^2 = \frac{|\tilde{F}\tilde{e}_\alpha|^2 - r^{-1/3}}{r^{2/3} - r^{-1/3}}, \quad \alpha = 1, 2. \tag{B.5}$$

Further, since $r > 0$ and $\neq 1$, taking the sign of the off diagonal term in (B.1) gives

$$\operatorname{sgn}((m \cdot e_1)(m \cdot e_2)) = \operatorname{sgn}(r - 1)\operatorname{sgn}(\tilde{F}\tilde{e}_1 \cdot \tilde{F}\tilde{e}_2). \tag{B.6}$$

Since $m \in \mathbb{S}^2$, combining (B.5) and (B.6) yields $m \in \mathcal{N}^r_{\tilde{F}}$.

Now, let $\tilde{F} \in \mathcal{D}_r$ and $m \in \mathcal{N}^r_{\tilde{F}}$. To prove $\tilde{F}^T\tilde{F} = \tilde{\ell}_m$, we need to show (B.1). By hypothesis, we have (B.5). By rearranging this formula, we obtain the diagonal terms in (B.1). For the off diagonal term, we note that in addition to (B.5), we have (B.4) by hypothesis. Combining these relations, we find

$$(\tilde{F}\tilde{e}_1 \cdot \tilde{F}\tilde{e}_2)^2 = (r^{2/3} - r^{-1/3})^2(m \cdot e_1)^2(m \cdot e_2)^2.$$

Taking the square root, we have the off diagonal term up to the sign. The correct choice of sign is guaranteed since m and $\tilde{F}$ satisfy (B.6), again by hypothesis.

Finally, we let $\tilde{F} \in \mathcal{D}_r$, and show $\mathcal{N}^r_{\tilde{F}}$ is non-empty. Indeed by definition, $\tilde{F}$ satisfies (B.2) and (B.3). Thus, the right side of (B.5) is non-negative. From this, we may find an $m \in \mathbb{R}^3$ satisfying (B.5) and (B.6). Further by (B.3), $(m \cdot e_1)^2 + (m \cdot e_2)^2 \leq 1$. Thus, we can choose $(m \cdot e_3)$ such that $m \in \mathbb{S}^2$. It follows that $\mathcal{N}^r_{\tilde{F}}$ is non-empty. $\square$

C On nonisometric origami

The actuation of complex origami shape stems from satisfying the nonisometric condition (1.13), hence the term *nonisometric origami*. In particular, the compatibility of interfaces separating regions of distinct constant director (Figure 10a) combined with the compatibility of junctions where these interfaces merge at a single point (Figure 10b) play the key role in actuation. To address this with mathematical precision, we note that (1.13) is equivalent to

$$\tilde{F}_\alpha = R_\alpha(\ell^{1/2}_{n_{0\alpha}})_{3\times 2} \quad \text{for some } R_\alpha \in SO(3)\ , \quad \alpha \in \{1, \dots, N\}$$

where $(\ell_{n_{0\alpha}}^{1/2})_{3\times 2} = r^{-1/6}(I_{3\times 2} + (r^{1/2} - 1) n_{0\alpha} \otimes \tilde{n}_{0\alpha})$ for the projection $\tilde{n}_{0\alpha} \in B_1(0) \subset \mathbb{R}^2$. Thus for compatibility, the deformation y in (1.12) must be continuous across each interface separating regions of distinct constant director. This occurs if and only if

$$R_\alpha(\ell_{n_{0\alpha}}^{1/2})_{3\times 2}\tilde{g}_{\alpha\beta} = R_\beta(\ell_{n_{0\beta}}^{1/2})_{3\times 2}\tilde{g}_{\alpha\beta} \tag{C.1}$$

for every interface tangent $\tilde{g}_{\alpha\beta} \in \mathbb{S}^1$. Explicitly, $\tilde{g}_{\alpha\beta}$ represents the tangent vector to the interface separating regions ω_α with director $n_{0\alpha}$ and ω_β with director $n_{0\beta}$ as depicted in Figure 10. This condition is akin to the rank-one condition studied in the context of fine-scale twinning during the austenite martensite phase transition and actuation active martensitic sheets [2],[8],[9]. More recently, this compatibility has been appreciated as a means of actuation for nematic elastomer and glass sheets [32],[33] using planar programming of the director. Here though, (C.1) describes the most general case of compatibility in thin nematic sheets as $n_{0\alpha}, n_{0\beta} \in \mathbb{S}^2$ need not be planar.

While (C.1) encodes a complete description of nonisometric origami as defined by (1.12) and (1.13), more useful criterion are gleamed from examining necessary and sufficient conditions associated with this constraint. In particular, taking the norm of both sides of (C.1) yields, after some manipulation, a necessary condition for nonisometric origami,

$$|\tilde{n}_{0\alpha} \cdot \tilde{g}_{\alpha\beta}| = |\tilde{n}_{0\beta} \cdot \tilde{g}_{\alpha\beta}| \tag{C.2}$$

for every interface tangent $\tilde{g}_{\alpha\beta}$ (when $r \neq 1$). We emphasize again that $\tilde{n}_{0\alpha} \in B_1(0) \subset \mathbb{R}^2$ is the projection of $n_{0\alpha}$ onto the tangent plane of ω. That this need not be a unit vector is a direct consequence of allowing for non-planar programming.

A director program satisfying (C.2) is not, however, sufficient to ensure the existence of a continuous piecewise affine deformation y satisfying (1.12) and (1.13). To illustrate this point, consider the design in Figure 11a. Here, we have a junction with three sectors of equal angle $2\pi/3$, and the director is programmed to bisect the sector angle (respectively, perpendicular to the bisector) on heating (respectively, cooling). This program satisfies the necessary condition (C.2). However in this case, due to the stretching part of the deformation upon actuation, the base of each triangle expands while the height contracts. Thus, it is clear geometrically that no series of rotations and/or translations of the three deformed triangles can bring about a continuous piecewise affine deformation of the entire junction. Conversely, if thermal actuation is reversed, as illustrated in Figure 11b with the color change of the director program, then the base of each triangle contracts and the height expands. In this case, a continuous piecewise affine deformation is realized by rotating each of the deformed triangles out-of-plane to form a 3-sided pyramid.

Figure 11b, by way of example, also highlights a simple scheme to form a compatible pyramidal junction. Indeed, if a junction has $K \geq 3$ sectors of equal angle $2\pi/K$ as in Figure 12, then programming the director to bisect this angle upon cooling (respectively, perpendicular to the bisector on heating) alway leads to a compatible K-sided pyramid. There are, of course, an infinite number of these types of junction, as emphasized with the designs in the right part of Figure 12. Most importantly though, these junctions can be used as unit cells to actuate more complex structures from nematic sheets. This is shown in Figure 13 with designs for actuating a box (a), rhombic triacontehedron (b) and azimuthally periodic structures (c). Each design incorporates a unit cell in Figure 12 as the building block.

The examples highlighted in Figure 13 illustrate that for even the simplest of building blocks, there is a richness of shape changing deformations of nematic elastomer sheets described by nonisometric origami. It should be noted, however, that these structure are in general degenerate. This is shown in Figure 13d where we design a program to actuate a rhombic dodecahedron upon cooling. Here though, we have done nothing to break the reflection symmetry associated with the building

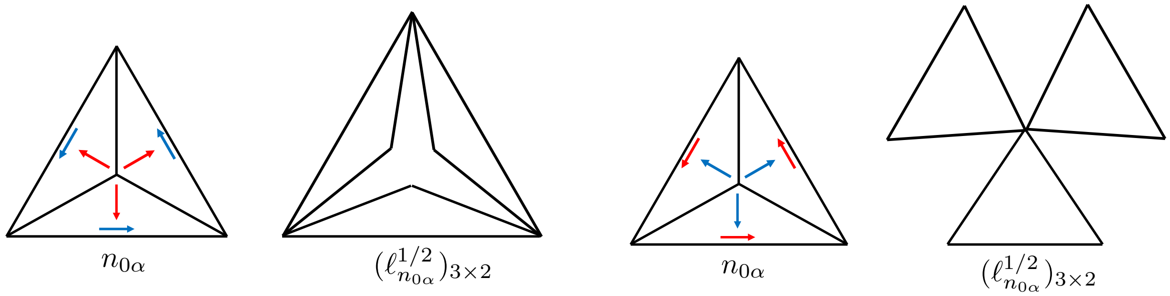


Figure 11: Two junctions with director programs satisfying (C.2). Blue represents the design for cooling and red represents the design for heating. The stretch part of the deformation upon thermal actuation is plotted.

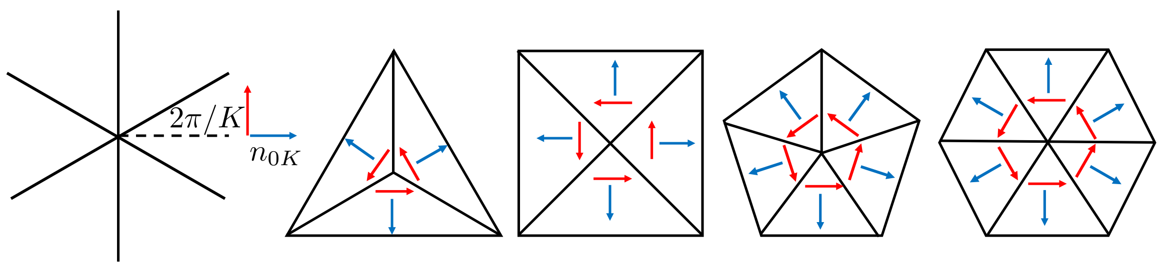
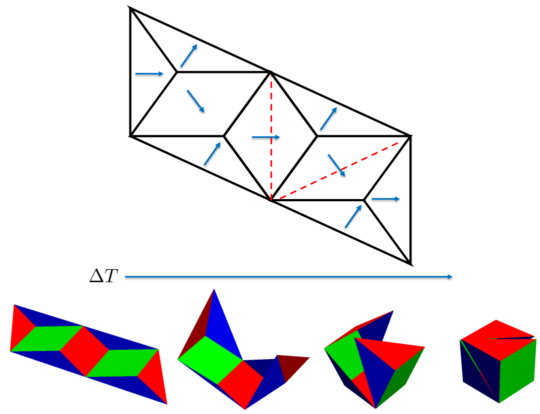


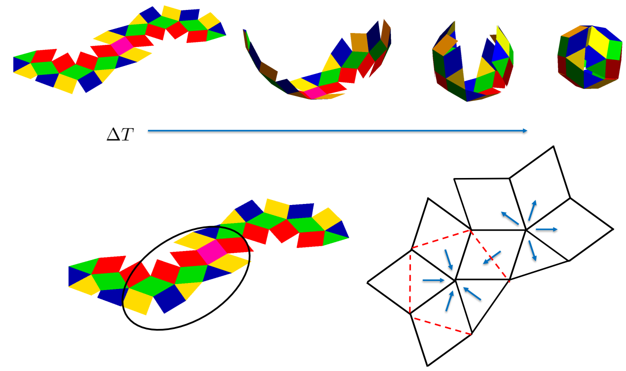
Figure 12: Simple scheme for compatible junction.

block. Thus, each interior junction is free to actuate either up or down. Therefore, in addition to possibly actuating the rhombic dodecahedron, the actuation of four alternative surfaces is a completely equivalent outcome given this framework. Such degeneracy was observed actuating conical defects by Ware et al. [41], where it was shown that each defect could actuate either up or down. However, it may be possible to suppress these degeneracies by introducing a slight bias in the thru thickness director orientation via twisted nematic prescription. This was intimated in another work by Fuchi et al. [25], where actuation of a box like structure was achieved through folds biased in the appropriate direction using such prescription. Thus, biasing would appear a promising means of breaking the reflection symmetry. Nevertheless, we do not address this here as it is difficult to analyze to the level of rigor intended for this work.

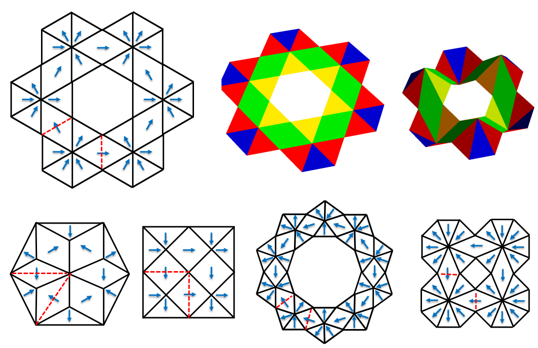
We now make one final comment on the design landscape for these constructions before proceeding to the discussion of our main result regarding the energy of nonisometric origami. Recall that the relations associated with (C.1) provide a complete, but not particularly transparent, description of nonisometric origami. Further, the more useful condition (C.2) is only necessary as we provided a counterexample to sufficiency in Figure 11a. In fact, to our knowledge, a complete characterization of the geometry of configurations satisfying (C.1) remains open. Nevertheless, we do expect an immense richness to such a characterization. For instance, in [38] a more general, but by no means complete, characterization of compatible three-faced junctions is worked out, and numerous non-trivial examples of compatibility emerge from the analysis. For these reasons, we feel a further pursuit in this direction appealing, though we do not probe deeper herein due to length considerations.



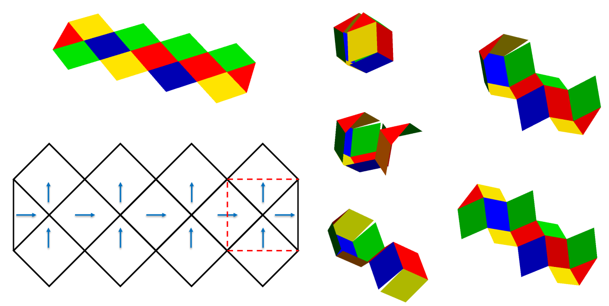
(a) Box construction



(b) Rhombic tricontahedron construction



(c) Azimuthally Periodic Construction



(d) Degeneracies

Figure 13: Examples of nonisometric origami. The design for cooling is shown. For heating, each director field is replaced by its respective perpendicular in the plane. The unit cells which form the building blocks for these constructions are highlighted in red.

D Proof of Proposition 2.3

Recall that this calculation encapsulates an approximation to the boundary deformations $y \in \{y_{box}, y_{rd}, y_{rt}\}$ given in (2.23) of Section 2.3. We show this in three steps:

Step 1. There exists a $y_1^\delta \in C^\infty(\omega, \mathbb{R}^3)$ which approximates y as in (2.25) and satisfies the derivative estimates (2.26) outside of the ball $B_{\delta/2}(O)$.

We set $y_0^\delta \colon \omega \to \mathbb{R}^3$ as in Proposition 2.1 with $\rho_0^\delta \geq \delta / \sin(\theta_K)$ to be chosen later. In addition, we set $y_c \colon \omega \to \mathbb{R}^3$ as

$$y_c(\tilde{x}) := \frac{r^{5/6}}{c_K^r}\rho\left(\cos(\frac{\tilde{\theta}_K}{\theta_K}\theta)e_2 + \sin(\frac{\tilde{\theta}_K}{\theta_K}\theta)e_2\right) \tag{D.1}$$

where $x_1 = \rho\cos(\theta)$, $x_2 = \rho\sin(\theta)$ and $\tilde{\theta}_K$, θ_K defined in (2.23) and (2.24). This deformation is compared to y in Figure 7. The important features are that $y_c(x_1, 0) = y_{e_1}(x_1, 0)$ for y_{e_1} in (2.22) (due to the prefactor $r^{5/6}/c_K^r$) and the θ_K sector in Figure 4b is deformed to $\tilde{\theta}_K$ for both y_c and y.

Hence, we define $y_1^\delta \colon \omega \to \mathbb{R}^3$ such that

$$y_1^\delta = y_0^\delta + \psi_\delta(y_c - y_0^\delta) \tag{D.2}$$

for $\psi_\delta \in C^\infty(\mathbb{R}, [0,1])$ a smooth radial cutoff function $\psi_\delta = \psi_\delta(\rho)$ taking the value 1 in $B_{\rho_0^\delta}(O)$ and 0 outside $B_{\rho_1^\delta}(O)$ for some ρ_1^δ to be chosen subsequently.

We claim that after suitable choice of ρ_0^δ ρ_1^δ and ψ_δ, this deformation satisfies the desired properties for Step 1. First, we observe that $y_1^\delta \in C^\infty(\omega, \mathbb{R}^3)$ since $y_c \in C^\infty(\omega, \mathbb{R}^3)$ and y_0^δ is smooth outside of the ball $B_{\rho_0^\delta}(O)$. Note that it is not smooth up to the boundary since $|\tilde{\nabla}\tilde{\nabla} y_c| \to \infty$ as $\rho \to 0$. Next, we observe that y_1^δ satisfies (2.25) and restricted to the set $\omega \setminus B_{\rho_1^\delta}(O)$, it satisfies (2.26). This follows from the fact that $y_1^\delta = y_0^\delta$ on $\omega \setminus B_{\rho_1^\delta}(O)$ and by Proposition 2.1, y_0^δ enjoys these properties. Next, by an identical argument to that of Proposition 2.2, y_1^δ satisfies the upper bound estimates in (2.26) on the punctured disk $B_{\rho_1^\delta}(O) \setminus B_{\rho_0^\delta}(O)$. Finally, on the punctured disk $B_{\rho_0^\delta}(O) \setminus B_{\delta/2}(O)$, $y_1^\delta = y_c$, y_c is independent of δ and $\tilde{\nabla} y_c$ is full-rank. So we conclude that is satisfies the estimates in (2.26) on this set.

Hence, it remains only to prove that $|\partial y_1^\delta \times \partial y_1^\delta| \geq c > 0$ on the punctured disk $B_{\rho_1^\delta}(O) \setminus B_{\rho_0^\delta}(O)$ for c depending only on y. This will follow after a suitable choice of ρ_0^δ and ρ_1^δ, the result of an explicit calculation which we outline below.

We remark that we need only to show the result on the set ω_{e_1} in Figure 4b. This is due to the fact that y_1^δ enjoys the same underlying periodicity as y and so each $\omega_{\tilde{R}_K^\alpha e_1}$ sector can be mapped back to ω_{e_1}, deformed within this sector and subsequently rotated Q_K^α to achieve y_1^h on $\omega_{\tilde{R}_K^\alpha e_1}$. The transformation rule is made explicit in (2.23). Further, it suffices to focus on the projection of y_1^δ onto the $\{e_1, e_2\}$ plane, denoted here as $\tilde{y}_1^\delta \colon \omega \to \mathbb{R}^2$, since $|\partial_1 y_1^\delta \times \partial_2 y_1^\delta| \geq |\det(\tilde{\nabla}\tilde{y}_1^\delta)|$.

We compute this planar gradient in the global Cartesian basis, and evaluate its determinant parameterized in polar coordinates so that

$$\det(\tilde{\nabla}\tilde{y}_1^\delta) = \det(\partial_1 \tilde{y}_1^\delta \otimes e_1 + \partial_2 \tilde{y}_1^\delta \otimes e_2) =: f_{K,\delta}^r(\rho, \theta). \tag{D.3}$$

Note, as suggested, $f_{K,\delta}^r$ depend on $r \in [1, r_c]$, θ_K and δ. We focus first on the region where y_0^δ is affine (i.e. $= \tilde{F}^\pm \tilde{x}$ on ω_{e_1} in Figure 4b). This corresponds to $(\rho, \theta) \in (\omega_{e_1} \setminus \Gamma_\delta) \cap (B_{\rho_1^\delta}(O) \setminus B_{\rho_0^\delta}(O))$. We find as the result of an explicit calculation that

$$\begin{aligned} f_{K,\delta}^r(\rho,\theta) = \frac{r^{5/6}}{c_K^r}\Big(& A_{1,K}^r + A_{2,K}^r(\theta)\rho\psi_\delta'(\rho) \\ & + \psi_\delta(\rho)\big(A_{3,K}^r(\theta) + A_{4,K}^r(\theta)\rho\psi_\delta'(\rho) + A_{4,K}^r(\theta)\psi_\delta(\rho)\big)\Big) \end{aligned} \tag{D.4}$$

in this regime. Here we denote the cases $K = 3, 4, 5$ for the box, rhombic dodecahedron, and rhombic triacontahedron respectively. This is also the notation above in (2.23). In addition for $r \in (1, r_c]$, we find $A_{1,K}^r > 0$ independent of θ as shown, $A_{3,K}^r(\theta) > 0$ and $A_{2,K}^r(\theta), A_{4,K}(\theta) \geq 0$ with $A_{j,K}^r(\theta) = A_{j,K}^r(-\theta)$ for $\theta \in [-\theta_K, \theta_K]$ and for $j = 2, 3, 4$. We also find for $\epsilon > 0$ sufficiently small that

$$\max_{K \in \{3,4,5\}} \max_{r \in (1, r_c]} \max_{\theta \in [0, \theta_K]} \left\{ \frac{A_{2,K}^r(\theta)}{A_{1,K}^r}, \frac{A_{4,K}^r(\theta)}{A_{3,K}^r(\theta)} \right\} \leq \frac{3}{4}(1 - \sqrt{\epsilon}). \tag{D.5}$$

This is shown in Figure 14. In addition, we choose our cutoff function ψ_δ such that

$$\|\psi_\delta'\|_{L^\infty} \leq \frac{1 + \sqrt{\epsilon}}{\rho_1^h - \rho_0^h}$$

for said ϵ. This can always be done given the properties for ψ_h below (D.2). Hence, since $\rho\psi_h'(\rho) \le 0$, we see given (D.4),

$$f_{K,\delta}^r(\rho,\theta) \ge \frac{\bar{r}^{5/6}}{c_K^r}\left(A_{1,K}^r + \psi_\delta(\rho)A_{3,K}^r(\theta)\right)\left(1 + \frac{3}{4}(1-\sqrt{\epsilon})\rho\psi_\delta'(\rho)\right)$$
$$\ge C\left(1 - \frac{3}{4}(1-\sqrt{\epsilon})\rho_1^\delta\|\psi_\delta'\|_{L^\infty}\right)$$
$$\ge C\left(1 - \frac{3}{4}(1-\epsilon)\frac{\rho_1}{\rho_1^\delta - \rho_0^\delta}\right). \tag{D.6}$$

Thus, requiring that

$$\frac{\rho_1^\delta}{\rho_1^\delta - \rho_0^\delta} \le \frac{4}{3}, \tag{D.7}$$

we satisfy $\det \tilde{\nabla}\tilde{y}_1^\delta \ge C\epsilon$ in this regime combining (D.6) with (D.3).

It remains to verify that y_1^δ is full-rank in the $\delta-$strip region in Figure 4b, $(\rho,\theta) \in (\Gamma_\delta \cap \omega_{e_1}) \cap (B_{\rho_1^\delta}(O) \setminus B_{\rho_0^\delta}(O))$. Note, we are still free to choose ρ_0^δ. Then ρ_1^δ will be set by the constraint (D.7). Therefore, to simplify the verification, we will choose $\rho_0^\delta = m\delta$ for a sufficiently large fixed constant $m > 0$ such that in this regime $\theta \ll 1$. Now,

$$\tilde{\nabla}\tilde{y}_1^\delta = (1-\psi_\delta)\tilde{\nabla}\tilde{y}_c + \psi_\delta\tilde{\nabla}\tilde{y}_0^\delta + \psi_\delta'(\tilde{y}_c - \tilde{y}_0^\delta) \otimes e_\rho$$

where $e_\rho = (\cos(\theta), \sin(\theta))$. With $\theta \ll 1$ due to our choice of m, it will become clear that the first two terms taken together have positive determinant and the last term is small and therefore does not effect the positivity of $\det \tilde{\nabla}\tilde{y}_1^\delta$. Indeed, we find

$$(1-\psi_\delta)\tilde{\nabla}\tilde{y}_c + \psi_\delta\tilde{\nabla}\tilde{y}_0^\delta = \frac{r^{5/6}}{c_K^r}\Bigg((1-\psi_\delta)\begin{pmatrix} 1 & 0 \\ 0 & \frac{\tilde{\theta}_K}{\theta_K}\end{pmatrix}$$
$$+ \psi_\delta\begin{pmatrix} 1 & (2\lambda_\delta - 1)\frac{(r-1)}{r\tan(\theta_K)} \\ 0 & \frac{c_K^r}{r}\end{pmatrix} + O(\theta)\Bigg).$$

This is the result of an explicit calculation using the formulas (2.7), (2.10), (2.22) and (D.1) under the assumption that $\theta \ll 1$. Hence, since $\tilde{\theta}_K/\theta_K \ge 1$ and $c_K^r \ge 1$ for $r \ge 1$, we find that

$$\det\left((1-\psi_\delta)\tilde{\nabla}\tilde{y}_c + \psi_\delta\tilde{\nabla}\tilde{y}_0^\delta\right) \ge \frac{r^{5/3}}{(c_K^r)^2}\left(\frac{1}{3} + O(\theta)\right). \tag{D.8}$$

Note, the 1/3 term comes from setting $r = 3$ as the worst case when the corners of the box merge. Now for the remaining term, with $\theta \ll 1$

$$|\psi_\delta'(\tilde{y}_c - \tilde{y}_0^\delta) \otimes e_\rho| = |\psi_\delta'(\rho O(\theta^2) - \gamma_\delta\tilde{F}_{2\times 2}^- e_2 + (x_2 - \gamma_h)\tilde{F}_{2\times 2}^+ e_2) \otimes (e_1 + O(\theta))|$$
$$\le |\psi_\delta'|\left(\rho_1^\delta|O(\theta^2)| + \frac{\delta}{2}|O(1)|\right)(1 + |O(\theta)|)$$
$$\le \frac{\rho_1^\delta}{\rho_1^\delta - \rho_0^\delta}|O(\theta^2)| + \frac{\delta}{\rho_1^\delta - \rho_0^\delta}|O(1)|$$
$$\le \frac{4}{3}|O(\theta^2)| + \frac{|O(1)|}{M-m}$$

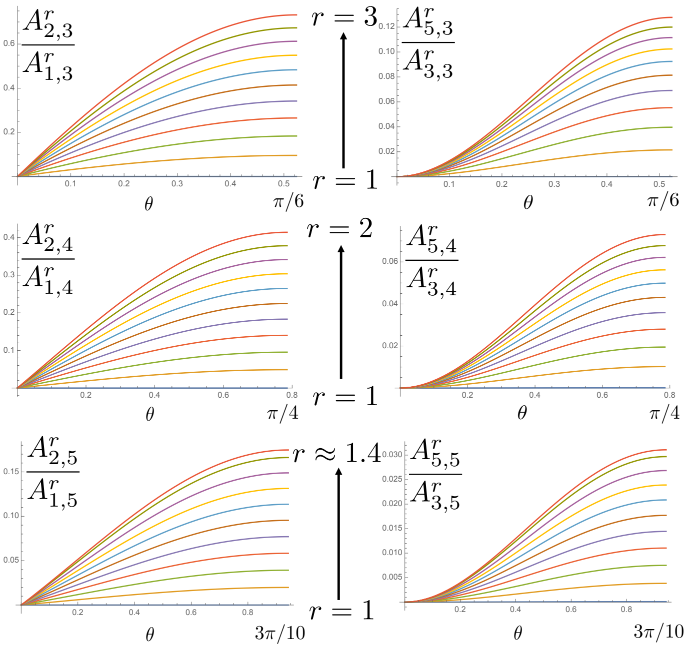


Figure 14: The coefficients associated with the maximization in (D.5) are plotted for the box (top), rhombic dodecahedron (middle) and rhombic triacontehedron (bottom). Multiple values for $r \in [1, r_c]$ are plotted, and we see monotonicity in the functions with respect to r. The worst case is given by the top-left graph. Thus, we see that in all cases, the range of these graphs is bounded from above by 3/4 as asserted in (D.5).

where we set $\rho_1^\delta = M\delta$. Here, we again used (2.7) with $\tilde{F}^{\pm}_{2\times 2}$ the 2×2 submatrix of $\tilde{F}^{\pm}$ given in (2.22) associated to the projection $\tilde{y}$. The estimate 4/3 stems from the requirement (D.7). The important point here is that for any $\epsilon > 0$, we may choose m, M independent of h and satisfying $M/(M-m) \leq 4/3$ such that $|\psi_\delta'(\tilde{y}_c - \tilde{y}_0^\delta) \otimes e_\rho| \leq \epsilon$ since the $|O(\theta^2)|$ term in the estimate above is governed by the largeness of m. Thus, we can assure $\det \tilde{\nabla}\tilde{y}_1^\delta > 0$ given (D.8) and an appropriate choice of m and M. This completes the proof. $\square$

Step 2. There exists a $y_2^\delta \in C^3(\bar{\omega}, \mathbb{R}^3)$ which approximates y in (2.25) and satisfies the upperbound derivative estimates in (2.26). Further, $\tilde{\nabla} y_2^\delta$ is full-rank everywhere but the origin.

For y_2^δ, we modify y_1^δ in the ball $B_{\delta/2}(O)$. Specifically, we replace ρ in y_c defined in (D.1) with

$$f_\delta(\rho) := \begin{cases} \frac{160\rho^4}{\delta^3} - \frac{720\rho^5}{\delta^4} + \frac{1152\rho^6}{\delta^5} - \frac{640\rho^7}{\delta^6} & \text{if } \rho \in [0, \delta/2) \\ \rho & \text{otherwise.} \end{cases} \tag{D.9}$$

This choice is the result of a polynomial fit in which the function behaves like ρ at $h/2$, i.e. by setting $f_\delta(\delta/2) = \delta/2$, $f_\delta'(\delta/2) = 1$ and $f_\delta''(\delta/2)$, $f_\delta^{(3)}(\delta/2) = 0$, and this fit ensures that the gradients satisfy

$$|f_\delta'| \leq C, \quad |f_\delta''| \leq C\delta^{-1}, \quad |f_\delta^{(3)}| \leq C\delta^{-2} \quad \text{on } B_{\delta/2}(O).$$

Hence, we let

$$y_2^\delta = y_0^\delta + \psi_\delta(y_c^\delta - y_0^\delta) \tag{D.10}$$

where y_c^δ is y_c in (D.1) with this replacement and ψ_δ is the cutoff function of Step 1. Given that f_h has the properties we have described, it follows that $\bar{y}_2^\delta \in C^3(\bar{\omega}, \mathbb{R}^3)$ and satisfies the upperbounds in (2.26) on this set. On the exceptional set, it satisfies both (2.25) and (2.26) since $y_2^\delta = y_1^\delta$ on this set, and we established this for y_1^δ in Step 1.

It remains to show that y_2^δ is full-rank everywhere but the origin. For this calculation, we need only consider this rank condition on $B_{\delta/2}(O)$ where y_2^δ is modified from y_1^δ. For this case, we find

$$|\partial_1 y_2^\delta \times \partial_2 y_2^\delta| = |\partial_1 y_c^\delta \times \partial_2 y_c^\delta| = \frac{r^{5/3}}{(c_K^r)^2} \frac{\tilde{\theta}_K}{\theta_K} |\frac{f_\delta' f_\delta}{\rho}|.$$

Hence noting that the quantity $|f_\delta' f_\delta|/\rho$ does not vanish anywhere save the origin, the proof is complete. $\square$

Step 3. Proof of Proposition 2.3.

To satisfy the lower bound condition on the cross-product, we now replace y_c^δ in (D.10) with

$$y_c^{\delta,\epsilon} = y_c^\delta + \epsilon(x_1 e_1 + x_2 e_3).$$

We claim that with this replacement, the function $y_\epsilon^\delta \in C^3(\bar{\omega}, \mathbb{R}^3)$ given by

$$y_\epsilon^\delta = y_0^\delta + \psi_\delta(y_c^{\delta,\epsilon} - y_0^\delta)$$

has all the desired properties of Proposition 2.3 for $\epsilon > 0$ chosen sufficiently small. Indeed, this perturbation does not modify y_2^δ outside $B_{\rho_1^\delta}(O)$ and so (2.25) holds since it holds for y_2^δ in Step 2. In addition, the upperbound estimates in (2.26) are unmodified by this perturbation. So we need only to show the lowerbound on the cross-product. This will be the case after an appropriate choice of ϵ.

We are assured this lowerbound outside $B_{\rho_1^\delta}(O)$, so we need only consider two cases. First, on the set $B_{\rho_1^\delta}(O) \setminus B_{\delta/2}(O)$, we find

$$\tilde{\nabla} y_\epsilon^\delta = \tilde{\nabla} y_1^\delta + \tilde{\nabla}(\psi_\delta \epsilon(x_1 e_1 + x_2 e_3)).$$

This last term is estimated as $|\tilde{\nabla}(\psi_\delta \epsilon(x_1 e_1 + x_2 e_3))| \leq C\epsilon$. Hence since $|\partial_1 y_1^\delta \times \partial_2 y_2^\delta| \geq c > 0$ for c depending only on y, and since $\tilde{\nabla} y_\epsilon^\delta$ is ϵ close to $\tilde{\nabla} y_1^\delta$ on this set, we conclude by continuity that there exists an $\bar{\epsilon} > 0$ such that $|\partial_1 y_\epsilon^\delta \times \partial_2 y_\epsilon^\delta| \geq c/2 > 0$ on $B_{\rho_1^\delta}(O) \setminus B_{\delta/2}(O)$ for all $\epsilon \in (0, \bar{\epsilon})$.

Now, we set $\hat{\theta}_K = \tilde{\theta}_K/\theta_K$. Hence, in the ball $B_{h/2}(O)$, we find explicitly that

$$\begin{aligned}(\partial_1 y_\epsilon^\delta \times \partial_2 y_\epsilon^\delta) \cdot e_3 &= \frac{\hat{\theta}_K r^{5/3}}{(c_K^r)^2} \frac{f_\delta' f_h}{\rho} + \epsilon g(\theta, f_\delta', f_\delta/\rho) \\ &= \bar{f}(\rho/\delta) + \epsilon \bar{g}(\theta, \rho/\delta)\end{aligned}$$

where $\bar{f}(\rho/\delta)$ in non-negative and vanishes only at $\rho/\delta = 0$. Further, $\bar{g}$ is bounded. Thus, we choose an $\epsilon \in (0, \bar{\epsilon})$ such that this quantity is positive for all $\rho/\delta \geq c$ where we take $c \ll 1$. Note this ϵ only depends on ρ/δ and not on δ itself. Hence, with this ϵ we set $\tilde{\nabla} y_\epsilon^\delta \equiv \tilde{\nabla} y^\delta$, and note that this is full rank outside of $\rho/\delta \leq c \ll 1$.

Now in the set $\rho/\delta \leq c \ll 1$, we have the simplification

$$f_\delta/\rho = 160(\rho/\delta)^3 + O((\rho/\delta)^4), \quad f_\delta' = 640(\rho/\delta)^3 + O((\rho/\delta)^4)$$

following the definition in (D.9). Using this simplification, we find that

$$(\partial_1 y^\delta \times \partial_2 y^\delta)\cdot e_1 = \epsilon\left(\frac{80r^{5/6}}{c_K^r}\left(\frac{\rho}{\delta}\right)^3 f_1(\hat\theta_K,\theta) + O((\rho/\delta)^4)\right) \quad \text{for} \tag{D.11}$$

$$f_1(\hat\theta_K,\theta) := (4-\hat\theta_K)\sin\Big((1+\hat\theta_K)\theta\Big) + (4+\hat\theta_K)\sin\Big((\hat\theta_K-1)\theta\Big).$$

Similarly, we find that

$$(\partial_1 y^\delta \times \partial_2 y^\delta)\cdot e_2 = -\epsilon^2 - \epsilon\left(\frac{80r^{5/6}}{c_K^r}\left(\frac{\rho}{\delta}\right)^3 f_2(\hat\theta_K,\theta) + O((\rho/\delta)^4)\right) \quad \text{for} \tag{D.12}$$

$$f_2(\hat\theta_K,\theta) := (4-\hat\theta_K)\cos\Big((1+\hat\theta_K)\theta\Big) + (4+\hat\theta_K)\cos\Big((\hat\theta_K-1)\theta\Big). \tag{D.13}$$

We now verify that the quantities in (D.11) and (D.12) are never simultaneously 0. For this, we first note that in the calculations of interest $\hat\theta_K \in [1,2]$ (i.e. the box, rhombic dodecahedron and rhombic triacontehedron constructions in Figure 6). Hence, we find that

$$f_1(\hat\theta_K,\theta)^2 + f_2(\hat\theta_K,\theta)^2 = 2(17 + (\hat\theta_K-1)(\hat\theta_K+1) - (\hat\theta_K-4)(\hat\theta_K+4)\cos(2\theta)) \geq 4\hat\theta_K^2. \tag{D.14}$$

Now we suppose $(\partial_1 y^\delta \times \partial_2 y^\delta)\cdot e_1 = 0$. Then either $\rho = 0$ which gives $(\partial_1 y^\delta \times \partial_2 y^\delta)\cdot e_2 = -\epsilon^2$ or $f_1(\hat\theta_K,\theta) = O(\rho/h)$. Thus, the rank of $\tilde\nabla y^\delta$ vanishes on the set $\rho/\delta \leq c \ll 1$ only if $f_1(\hat\theta_K,\theta) = O(\rho/\delta)$. But if this is the case, we also have

$$f_2(\hat\theta_K,\theta) \geq 2\hat\theta_K + O(\rho/\delta) \quad \text{ or } \quad f_2(\hat\theta_K,\theta) \leq -2\hat\theta_K + O(\rho/\delta) \tag{D.15}$$

given the inequality in (D.14). In fact though, the latter inequality is impossible for the deformations of interest.

To see this, we first remark that the relevant sets to consider for these deformations are

$$\begin{aligned} y \equiv y_{box}&\colon \quad \theta \in [-\pi/2,\pi/2], \quad \hat\theta_K \in [1,2];\\ y \equiv y_{rd}&\colon \quad \theta \in [-\pi/2,\pi], \quad \hat\theta_K \in [1,7/6];\\ y \equiv y_{rt}&\colon \quad \theta \in [-6\pi/10,6\pi/5], \quad \hat\theta_K \in [1,10/9]. \end{aligned} \tag{D.16}$$

This can be seen directly from Figure 6, or alternatively it follows from the definitions of $\hat\theta_K$ and so $\tilde\theta_K$ in (2.24). Now, we assume f_2 in (D.13) satisfies the second inequality in (D.15). We will arrive at a contradiction given (D.16). Indeed, $\cos((\hat\theta_K-1)\theta) \geq 0$ for these deformations. Thus by a crude estimate for f_2, we have for these deformations

$$f_2(\hat\theta_K,\theta) \geq -4 + \hat\theta_K.$$

Hence, for the second inequality in (D.15), it must be that $\hat\theta_K \leq 4/3 + O(\rho/\delta)$. Therefore, actually $\cos((\hat\theta_K-1)\theta)$ is approximately bounded from below by $\cos(\pi/6) = \sqrt{3}/2$. Hence, in this regime, we have

$$f_2(\hat\theta_K,\theta) \geq -4 + \hat\theta_K + \frac{\sqrt{3}}{2}(4+\hat\theta_K) + O(\rho/\delta) > 0.$$

This is the desired contradiction.

Finally, we can show that $\tilde\nabla y^\delta$ satisfies the cross product bound everywhere. Indeed, assume it does not. Then $f_1(\hat\theta_K,\theta) = O(\rho/\delta)$. This implies $f_2(\hat\theta_K,\theta) \geq 2\hat\theta_K + O(\rho/\delta)$. Hence from (D.12) with f_2 positive, we conclude

$$|\partial_1 y^\delta \times \partial_2 y^\delta| \geq \epsilon^2$$

whenever $(\partial_1 y^\delta \times \partial_2 y^\delta)\cdot e_1 = 0$. This completes the proof of Proposition 2.3. □

E Geometric rigidity and nematic elastomers

First, we derive the key estimate which relates Geometric Rigidity [23] to the setting of nematic elastomers.

Proposition E.1. *Let $\omega \subset \mathbb{R}^3$ bounded and Lipschitz. There exists a constant $C = C(r_0, r_f, \tau)$ with the following property: for all $h > 0$, $Q_{\tilde{x}^*,h} := (-h/2, h/2)^3 \subset \Omega_h$, $W^h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$, $N^h \in W^{1,2}(\Omega_h, \mathbb{S}^2)$ and N_0^h as in (1.7) with $n_0 \in W^{1,2}(\omega, \mathbb{R}^3)$, there exists an associated constant rotation $R^h_{\tilde{x}^*} \in SO(3)$ such that*

$$\int_{Q_{\tilde{x}^*,h}} |(\ell^f_{N^h})^{-1/2} \nabla Y^h (\ell^0_{N_0^h})^{1/2} - R^h_{\tilde{x}^*}|^2 dx$$
$$\leq C \int_{Q_{\tilde{x}^*,h}} \Big(\operatorname{dist}^2((\ell^f_{N^h})^{-1/2} \nabla Y^h (\ell^0_{N_0^h})^{1/2}, SO(3)) + h^2(|\nabla N^h|^2 + |\tilde{\nabla} n_0|^2 + 1) \Big) dx$$

Proof. Let $Y^h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$, $N^h \in W^{1,2}(\Omega_h, \mathbb{S}^2)$ and $n_0 \in W^{1,2}(\omega, \mathbb{S}^2)$ with N_0^h as in (1.7). we fix $\tilde{x}^*$ such that $Q_{\tilde{x}^*,h} \subset \Omega_h$ and set

$$A^f_h := \frac{1}{|Q_{\tilde{x}^*,h}|} \int_{Q_{\tilde{x}^*,h}} (\ell^f_{N^h})^{1/2} dx, \quad A^0_h := \frac{1}{|Q_{\tilde{x}^*,h}|} \int_{Q_{\tilde{x}^*,h}} (\ell^0_{n_0})^{-1/2} dx. \tag{E.1}$$

Because of the structure of the step-length tensors, these averages are positive definite, and each of the eigenvalues lives in a compact set of the positive real numbers depending only on r_f and r_0 (in particular, this set does not depend on h). Hence, these linear maps belong to a family of h-indepdent Bilipschtiz maps with controlled Lipschitz constant, and so we write $A^f \equiv A^f_h$ and $A^0 \equiv A^0_h$ in sequel.

Now, we set

$$V^h(s) = (A^f)^{-1} Y^h((A^0)^{-1} s), \quad s \in (A^0) Q_{\tilde{x}^*,h}.$$

We observe that $V^h \in W^{1,2}((A^0)Q_{\tilde{x},h}, \mathbb{R}^3)$ by the regularity of Y^h. Therefore by Geometric Rigidity [23], there exists a constant rotation $R^h_{\tilde{x}} \in SO(3)$ such that

$$\begin{aligned}
\int_{Q_{\tilde{x}^*,h}} |(A^f)^{-1} \nabla Y^h(x) (A^0)^{-1} - R^h_{\tilde{x}^*}|^2 dx &= |\det A^0|^{-1} \int_{(A^0)Q_{\tilde{x}^*,h}} |\nabla V^h(s) - R^h_{\tilde{x}^*}|^2 ds \\
&\leq |\det A^0|^{-1} C((A^0)Q_{\tilde{x}^*,h}) \int_{(A^0_\omega)Q_{\tilde{x}^*,h}} \operatorname{dist}^2(\nabla V^h(s), SO(3)) ds \\
&= C((A^0)Q_{\tilde{x}^*,h}) \int_{Q_{\tilde{x}^*,h}} \operatorname{dist}^2((A^f)^{-1} \nabla Y^h(x) (A^0)^{-1}, SO(3)) dx
\end{aligned} \tag{E.2}$$

The constant $C((A^0)Q_{\tilde{x}^*,h})$ can be chosen uniformly for a family of domains which are Bilipschitz equivalent with controlled Lipschitz constant. Hence, actually we can choose $C(r_0, Q_{\tilde{x}^*,h}) \geq C((A^0)Q_{\tilde{x}^*,h})$. Moreover, the constant is invariant under translation and dilatation. Hence, actually we have $C(r_0, Q_{\tilde{x}^*,h}) = C(r_0)$ for any $Q_{\tilde{x}^*,h} \subset \Omega_h$. These properties are given in Friesecke, James and Müller, Theorem 9 [24]. Since r_0 is fixed in this calculation, we write $C(r_0) \equiv C$, and thus

$$\int_{Q_{\tilde{x}^*,h}} |(A^f)^{-1} \nabla Y^h(x) (A^0)^{-1} - R^h_{\tilde{x}^*}|^2 dx \leq C \int_{Q_{\tilde{x}^*,h}} \operatorname{dist}^2((A^f)^{-1} \nabla Y^h(x)(A^0)^{-1}, SO(3)) dx \tag{E.3}$$

from (E.2).

Since we will no longer be dealing with a change of variables in this proof, we now drop the explicit dependence on x inside the integrals. We observe by the key estimate (E.3) that

$$\begin{aligned}
\int_{Q_{\tilde{x}^*,h}} |\nabla Y^h - (A^f) R^h_{\tilde{x}^*}(A^0)|^2 dx &\leq C \int_{Q_{\tilde{x}^*,h}} |(A^f)^{-1}\nabla Y^h (A^0)^{-1} - R^h_{\tilde{x}^*}|^2 dx \\
&\leq C \int_{Q_{\tilde{x}^*,h}} \mathrm{dist}^2((A^f)^{-1}\nabla Y^h(A^0)^{-1}, SO(3)) dx \leq C \int_{Q_{\tilde{x}^*,h}} \mathrm{dist}^2(\nabla Y^h, (A^f)SO(3)(A^0)) dx \\
&\leq C \int_{Q_{\tilde{x}^*,h}} \Big(\mathrm{dist}^2(\nabla Y^h, (\ell^f_{N^h})^{1/2} SO(3)(\ell^0_{N^h_0})^{-1/2}) + |(\ell^f_{N^h})^{1/2} - A^f|^2 + |(\ell^0_{N^h_0})^{-1/2} - A^0|^2\Big)\, dx \\
&\leq C \int_{Q_{\tilde{x}^*,h}} \Big(\mathrm{dist}^2((\ell^f_{N^h})^{-1/2}\nabla Y^h (\ell^0_{N^h_0})^{1/2}, SO(3)) + h^2|\nabla N^h|^2 \\
&\qquad\qquad + |(\ell^0_{n_0})^{-1/2} - A^0|^2 + |(\ell^0_{N^h_0})^{-1/2} - (\ell^0_{n_0})^{-1/2}|^2\Big) dx \\
&\leq C \int_{Q_{\tilde{x}^*,h}} \Big(\mathrm{dist}^2((\ell^f_{N^h})^{-1/2}\nabla Y^h (\ell^0_{N^h_0})^{1/2}, SO(3)) + h^2(|\nabla N^h|^2 + |\tilde{\nabla} n_0|^2 + 1)\Big)\, dx
\end{aligned} \tag{E.4}$$

Here, the constant $C = C(r_0, r_f, \tau)$ is due to several applications of the triangle inequality and the fact that the norm of the step-length tensors, inverses and averages are compact and this depends only on r_f,r_0. We have also applied the standard Poincaré inequality given the averages (E.1), and used that the diameter of $Q_{\tilde{x}^*,h}$ is h and that the gradients of the step-length tensors are controlled by the gradients of the directors. Finally, from the assumed control of non-idealities for N^h_0 in (1.7), we have the estimate $\|(\ell^0_{N^h_0})^{-1/2} - (\ell^0_{n_0})^{-1/2}\|_{L^\infty} \leq c(r_0)\tau h$. This gives the dependence on τ in the constant.

Now using (E.4), we find that

$$\begin{aligned}
\int_{Q_{\tilde{x}^*,h}} |(\ell^f_{N^h})^{-1/2}\nabla Y^h (\ell^0_{N^h_0})^{1/2} - R^h_{\tilde{x}^*}|^2 dx &\leq C \int_{Q_{\tilde{x}^*,h}} |\nabla Y^h - (\ell^f_{N^h})^{1/2} R^h_{\tilde{x}^*} (\ell^0_{N^h_0})^{-1/2}|^2 dx \\
&\leq C \int_{Q_{\tilde{x}^*,h}} \Big(|\nabla Y^h - (\ell^f_{N^h})^{1/2} R^h_{\tilde{x}^*}(\ell^0_{n_0})^{-1/2}|^2 + |(\ell^0_{N^h_0})^{-1/2} - (\ell^0_{n_0})^{-1/2}|^2\Big)\, dx \\
&\leq C \int_{Q_{\tilde{x}^*,h}} \Big(|\nabla Y^h - (A^f) R^h_{\tilde{x}^*}(A^0)|^2 + h^2 + |(\ell^f_{N^h})^{1/2} - A^f|^2 + |(\ell^0_{n_0})^{-1/2} - A^0|^2\Big)\, dx \\
&\leq C \int_{Q_{\tilde{x}^*,h}} \Big(\mathrm{dist}^2((\ell^f_{N^h})^{-1/2}\nabla Y^h (\ell^0_{n_0})^{1/2}, SO(3)) + h^2(|\nabla N^h|^2 + |\tilde{\nabla} n_0|^2 + 1)\Big)\, dx
\end{aligned}$$

as desired. □

Now we note that the approximations in Lemma 5.1 are not new. They essentially follow from the same argument as that of Theorem 10 in Friesecke, James and Müller [24], modified appropriately for nematic elastomers using the estimate in Proposition E.1. In the general context of non-Euclidean plates, there is a recent body of literature on such estimates (e.g. Lewicka and Pakzad (Lemma 4.1) [30] and Lewicka et al. (Theorem 1.6) [29], (Lemma 2.3) [10]). Thus briefly:

Proof of Lemma 5.1. We repeat steps 1-3 in the proof of Theorem 10 in [24] with some modification due to our nematic elastomer setting. The lemma follows by the estimate in Proposition E.1. □

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