On the Achievable Region for Multiple Description Source Codes on Gaussian Sources

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Abstract — **We demonstrate inconsistencies in prior results on the achievable region for multiple description (MD) source codes on iid Gaussian sources with the squared error distortion measure. We then describe the complete region.**

I. Previous results

Suppose that an MD code for source $X \sim \mathcal{N}(0, \sigma^2)$ has two side reconstructions Y_1 and Y_2 with expected distortions D_1 and D_2, respectively, and a joint reconstruction Y_0 with expected distortion D_0. Prior authors use two distinct approaches for building the reproductions needed to apply El Gamal and Cover's 2DSC-achievable rate-distortion result [1]. To discuss those results, we define

$$R(D) = \tfrac{1}{2}\log\tfrac{\sigma^2}{D}$$
$$D_L(D_1, D_2) = D_1 + D_2 - \sigma^2$$
$$D_H(D_1, D_2) = \left(\tfrac{1}{D_1} + \tfrac{1}{D_2} - \tfrac{1}{\sigma^2}\right)^{-1}$$
$$\mathcal{D}_1 = \{(D_0, D_1, D_2) : D_0 \in [0, D_L(D_1, D_2))\}$$
$$\mathcal{D}_2 = \{(D_0, D_1, D_2) : D_0 \in (D_L(D_1, D_2), D_H(D_1, D_2))\}$$
$$\mathcal{D}_3 = \{(D_0, D_1, D_2) : D_0 \in [D_H(D_1, D_2), \infty)\}$$
$$L_{G0} = \tfrac{1}{2}\log\tfrac{(\sigma^2 - D_0)^2}{(\sigma^2 - D_0)^2 - (A-B)^2}$$
$$L_{G12} = \tfrac{1}{2}\log\tfrac{(\sigma^2 - D_0)^2 D_1 D_2}{(\sigma^2 - D_0)^2 D_1 D_2 - (\sigma^2 B - D_0 A)^2},$$

where $A = \sqrt{(\sigma^2 - D_1)(\sigma^2 - D_2)}$ and $B = \sqrt{(D_1 - D_0)(D_2 - D_0)}$.

The approach used in [1], which we call the "joint decoder first" approach, sets up $X \to Y_0 \to (Y_1, Y_2)$ as a Markov chain. The resulting achievable region is

$$R_1 \geq R(D_1), R_2 \geq R(D_2), R_1 + R_2 \geq R(D_0) + L_0, \quad (1)$$

where $L_0 = 0$ in $\mathcal{D}_1$ and $L_0 = L_{G0}$ in $\mathcal{D}_2 \cup \mathcal{D}_3$.

The approach of [2, 3], which we call the "side decoders first" approach, sets up $X \to (Y_1, Y_2) \to Y_0$ as a Markov chain. The resulting achievable region is

$$\begin{aligned} R_1 \geq R(D_1), \qquad & R_2 \geq R(D_2), \\ R_1 + R_2 \geq R(D_1) &+ R(D_2) + L_{12}, \end{aligned} \quad (2)$$

where $L_{12} = \frac{1}{2}\log\frac{D_1 D_2}{\sigma^2(D_1 + D_2 - \sigma^2)}$ in $\mathcal{D}_1$, $L_{12} = L_{G12}$ in $\mathcal{D}_2$, and $L_{12} = 0$ in $\mathcal{D}_3$.

While the authors claim agreement with each other and with Ozarow's converse [2], the results actually differ for some (D_0, D_1, D_2) values. In Section II, we discuss the derivation for the complete achievable region, pointing out the discrepancies with prior characterizations. The given achievable region is tight for all (D_0, D_1, D_2).

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II. Results

We treat regions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ separately.

Theorem 1 *In $\mathcal{D}_1$, the achievable region is*

$$R_1 \geq R(D_1), \quad R_2 \geq R(D_2), \quad R_1 + R_2 \geq R(D_0). \quad (3)$$

Theorem 1, which relies on the joint decoder first approach, differs from (2) in region $\mathcal{D}_1$. The side decoders first approach gives the smaller achievable region $R_1 \geq R(D_1)$, $R_2 \geq R(D_2)$, and $R_1 + R_2 \geq R(D_1) + R(D_2) + L_{G12}$ in $\mathcal{D}_1$. This implies that the derivation in [3] that uses this approach to get a Shannon type inner bound in $\mathcal{D}_1$ is problematic.

Theorem 2 *In $\mathcal{D}_2$, the achievable region is*

$$\begin{aligned} R_1 &\geq R(D_1), \quad R_2 \geq R(D_2), \\ R_1 + R_2 &\geq R(D_0) + L_{G0} = R(D_1) + R(D_2) + L_{G12}. \end{aligned}$$

Theorem 2 can be proven with either approach; therefore it agrees with both (1) and (2) in region $\mathcal{D}_2$.

Theorem 3 *In $\mathcal{D}_3$, the achievable region is*

$$R_1 \geq R(D_1), R_2 \geq R(D_2). \quad (4)$$

Theorem 3 differs from (1) in $\mathcal{D}_3$, where the joint decoder first approach additionally forces $R_1 + R_2 \geq R(D_0) + L_{G0}$.

III. Conclusions

Neither the joint decoder first nor the side decoders first approach is sufficient to find the complete achievable region. The *no excess rate sum* case given by (3) is achievable only in $\mathcal{D}_1$ using the joint decoder first approach. The *no excess marginal rate* case given by (4) is achievable only in $\mathcal{D}_3$ using the side decoders first approach. Both approaches yield the identical optimal result in $\mathcal{D}_2$.

References

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