Improved Bounds for the Rate Loss of Multi-Resolution Source Codes

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Abstract — **In this paper, we present new bounds for the rate loss of multi-resolution source codes. Consider an M-resolution code with ith-resolution rate and distortion R_i and D_i. The ith-resolution rate loss, defined as $L_i = R_i - R(D_i)$, describes the performance degradation of the multi-resolution code compared to the best single-resolution code with the same distortion. For 2-resolution codes, there are three scenarios of particular interest: (i) both resolutions are equally important; (ii) the rate loss at the first resolution is 0; (iii) the rate loss at the second resolution is 0. Lastras and Berger give constant upper bounds for the rate loss of an arbitrary i.i.d. source in scenarios (i) and (ii) and an asymptotic bound for scenario (iii) as $D_2 \to 0$. In this paper, we: (a) prove that $L_2 \leq 1.1610$ for all $D_2 < D_1$ in scenario (iii); (b) tighten the Lastras-Berger bound from $L_1 \leq 1$ to $L_1 \leq 0.7250$ in scenario (ii); (c) tighten the Lastras-Berger bound from 0.5 to 0.3801 in scenario (i); and (d) generalize the bound for scenario (ii) to M-resolution codes.**

I. Introduction

The benefit of multi-resolution source coding over single-resolution source coding is the flexibility attained by using a data description that can be decoded at a variety of rates. We here bound the price of that advantage. For any $R_2 > R_1 \geq 0$ and any $D_1 > D_2 \geq 0$, we call the quadruple (R_1, R_2, D_1, D_2) *achievable* by multi-resolution coding on source X if there exists a two-resolution code that uses R_1 bits per symbol to describe X with distortion D_1 and then uses an additional $R_2 - R_1$ bits per symbol to refine the description to distortion D_2. (Rate and distortion values are expectations.) The *rate loss* of the given two-resolution code is defined as $L_i = R_i - R(D_i)$ (i=1,2), where $R(D)$ is the rate-distortion function for X. Rate loss describes the performance degradation associated with using a multi-resolution code rather than the best single-resolution code with the same distortion.

A source is *successively refinable* if there always exists a multi-resolution code that achieves the rate-distortion bound at *both* stages, i.e., $L_1 = L_2 = 0$ for all (D_1, D_2) [1, 2]. Necessary and sufficient conditions for successive refinability appear in [1]. Examples of discrete and continuous sources that are not successively refinable appear in [1, 3]. The achievable rate-distortion region for multi-resolution coding appears in [4, 5]. In [6], Lastras and Berger consider the question of whether there exists a source in which the rate loss can be made arbitrarily large. They demonstrate that for any i.i.d. source and the mean squared error (MSE) distortion measure, there exists an achievable quadruple for any $D_1 > D_2 \geq 0$ such that $L_1, L_2 \leq 1/2$. Moreover, for any $D_1 > D_2 \geq 0$, an achievable quadruple can be found with $L_1 = 0$ and $L_2 \leq 1$. They also show that as $D_2 \to 0$, $L_1 \leq 0.5$ and $L_2 = 0$ is achievable.

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In this paper we: (a) present a non-asymptotic bound for L_1 when $L_2 = 0$; (b) tighten the bound when $L_1 = 0$ from $L_2 \leq 1$ to $L_2 \leq 0.7250$; (c) tighten the simultaneous bound on L_1 and L_2 from 0.5 to 0.3801; (d) generalize the $L_1 = 0$ result from 2-resolution to M-resolution codes for any $M \geq 2$.

II. Main Results

We assume an arbitrary i.i.d. source with variance σ^2. We use the MSE distortion measure.

Theorem 1 *For any $D_1 > D_2 \geq 0$, there exists an achievable quadruple with $L_2 = 0$ and $L_1 \leq 0.5 \log 5 \approx 1.1610$.*

Theorem 2 *For any $D_1 > D_2 \geq 0$, there exists an achievable quadruple with $L_1 = 0$ and $L_2 \leq 0.5 \log(\sqrt{3}+1) \approx 0.7250$.*

Theorem 3 *For any $D_1 > D_2 \geq 0$, there exists an achievable quadruple with $L_1 = 0$ and*

$$L_2 \leq \frac{1}{2} \log \left[\left(\sqrt{\frac{6\sigma^2 - 5D_1}{2\sigma^2 - D_1}} + 1 \right) \left(1 - \frac{D_1}{2\sigma^2} \right) \right].$$

Remarks: This bound is a decreasing function of D_1 for fixed σ^2. Its maximum, $L_2 \leq 0.7250$ when $D_1 = 0$, is consistent with Theorem 2. Theorem 3 is tighter than Theorem 2 for all $D_1 > 0$, achieving a perfectly tight result when $D_1 = \sigma^2$.

Theorem 4 *For any $D_1 > D_2 \geq 0$, there exists an achievable quadruple with $L_1 = L_2 \leq 0.3801$.*

In [6], it is shown that for any $M \geq 2$, there exists an achievable $2M$-tuple $(R_1, \ldots R_M, D_1, \ldots, D_M)$ with $L_i \leq 1/2$, $i \in \{1, \ldots, M\}$. This solution suggests approximately identical priorities at all resolutions. But what if we minimize the rate loss at the first resolution, then minimize the rate loss at the second resolution subject to the constraint on the first rate loss and so on? This apparently maximizes the rate loss at the last resolution. The next theorem provides an upper bound for this scenario; it can also be regarded as a generalization of Theorem 2.

Theorem 5 *For any $D_1 > \cdots > D_M \geq 0$, there exists an achievable $2M$-tuple with $L_1 = 0$, $L_i = I(X; U_1, \ldots, U_i) - R(D_i)$ for all $1 < i < M$, and $L_M \leq M/2 - 0.2750$, where random variables $U_1, \ldots, U_{M-1}$ are sequentially defined as*

$$U_i = \arg\min_{U: Ed(X,U) \leq D_i} I(X; U_1, \ldots, U_{i-1}, U).$$

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