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The Capacity Region of Broadcast Channels with Memory[1]

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Abstract — **We derive the two-user capacity region of a broadcast channel with memory (ISI), assuming additive white Gaussian noise (AWGN) and an input power constraint. The results can be extended to any finite number of users.**

I. Introduction

In [1, Chapter 8], Gallager derives the water-pouring formula for the capacity of a single-user channel with memory under an input power constraint. In [2], Cover derives the capacity region of a stochastically-degraded memoryless broadcast channel using superposition codes with successive decoding. In this paper we obtain the capacity region of the broadcast channel with memory, using a coding strategy which combines Gallager's water-pouring with Cover's superposition codes.

Our channel model is a two-user broadcast channel with input $x(t)$ and output $y_i(t) = x(t)*h_i(t)+n_i(t)$ for the ith user. The finite-duration impulse response of each user's channel is $h_i(t)$, and the $n_i(t), i = 1,2$, are independent white Gaussian noise processes with spectral densities $N_0/2$. We assume that the input is constrained to an average power P. The two-user capacity region defines the convex hull of simultaneously-achievable rate pairs (R_1, R_2) for the two users.

For a single-user channel with memory, Gallager's water-pouring capacity formula is obtained by using the Karhunen-Loeve decomposition over a finite observation interval of the input and output. This decomposition reduces the continuous-time channel to a countable collection of parallel, independent, discrete-time, AWGN channels. The capacity of the parallel channel set is the sum of the individual channel capacities with a jointly-optimized power allocation. The water-pouring formula is obtained by letting the observation interval at the input and output increase to infinity.

For the broadcast channel with memory we use a Fourier series decomposition over a finite observation interval for the input and output. This reduces the broadcast channel to a countable set of parallel, independent, discrete-time, stochastically-degraded AWGN broadcast channels, which we will refer to as the *set of parallel broadcast channels.* The achievable rate region for each channel is thus described by Cover's channel capacity equation for stochastically-degraded memoryless broadcast channels [2]. We then derive the capacity region of the parallel channels by jointly optimizing the power allocation P_k assigned to channel k and the subal-location of P_k between the the two users on channel k. Taking the limit as the observation interval for the input and output grows to infinity yields the final capacity region.

II. Results

There are three main theorems which lead to the final capacity result. We first prove that the Fourier series decomposition reduces the two-user broadcast channel to a set of two-user parallel broadcast channels.

Theorem 1: Assume the channel input $x(t)$ is time-limited to $[-T_0/2, T_0/2]$ and the outputs $y_1(t)$ and $y_2(t)$ are time-limited to $[-T/2, T/2]$. An equivalent model for this channel is a set of two-user discrete-time channels. The kth channel in this set has input X_k and outputs $Y_{1,k} = X_k + N_{1,k}$ and $Y_{2,k} = X_k + N_{2,k}$, where the $N_{i,k}$ are independent AWGN random variables with mean zero and variance $.5N_0/|H_i(k/T)|^2$, for $H_i(f) = \mathcal{F}[h_i(t)]$.

This theorem is proved by following a similar argument to that of the single-user channel decomposition in [1]. The Fourier series provides an orthonormal basis which can be used to decompose both channels $h_1(t)$ and $h_2(t)$ to obtain an equivalent set of independent parallel two-user channels. It is then easily shown that each two-user channel in the parallel set is a memoryless, stochastically-degraded, AWGN broadcast channel. A similar result is obtained for discrete-time channels using a DFT decomposition.

We next obtain a closed-form solution for the capacity region of a set of two-user parallel broadcast channels.

Theorem 2: The capacity region of a countable set of parallel broadcast channels with inputs $\{X_k\}$ and outputs $\{Y_{1,k}\}$ and $\{Y_{2,k}\}$ under a total power constraint P is given by all rate pairs beneath the following curve:

$$(R_1, R_2) = (c_1 - c_2 \log(1/(\beta - 1)), c_3 - c_4 \log(\beta - 1) + c_5 \log \beta)$$

for $\beta \geq 1$ and by

$$(R_1, R_2) = (d_1 + d_2 \log(\beta - 1), d_3 - d_4 \log(\beta) - d_5 \log(\beta - 1))$$

for $\beta < 1$, where β is the slope of the rate region and the constants c_i and d_i depend on P, β, and the noise variances in each of the parallel channels. The optimal power allocation P_k^* in each channel is obtained using a modified water-pouring formula.

Finally, let $(R_1, R_2)_T$ denote the capacity region of the set of parallel broadcast channels. This region is obtained from the Fourier series decomposition of the broadcast channel with memory for an observation interval $[-T/2, T/2]$ of the input and output. The final theorem is the coding theorem and converse for the capacity region $C = \lim_{T \to \infty} (R_1, R_2)_T$.

Theorem 3: Any rate pair (R_1, R_2) inside the convex region C is an achievable rate pair, and any rate pair outside this region has probability of error bounded away from zero.

References

[1] R. G. Gallager. *Information Theory and Reliable Communication.* John Wiley & Sons, Inc., New York, 1968.

[2] T. M. Cover. Broadcast channels. *IEEE Transactions on Information Theory*, IT-18(1):2–14, January 1972.

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