Fintushel–Stern knot surgery in torus bundles

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Abstract

Suppose that X is a torus bundle over a closed surface with homologically essential fibers. Let X_K be the manifold obtained by Fintushel–Stern knot surgery on a fiber using a knot $K \subset S^3$. We prove that X_K has a symplectic structure if and only if K is a fibered knot. The proof uses Seiberg–Witten theory and a result of Friedl–Vidussi on twisted Alexander polynomials.

1 Introduction

One important question in 4–dimensional topology is to determine which smooth closed 4–manifolds admit symplectic structures. There are some topological constructions of symplectic 4–manifolds. For example, Thurston [23] showed that most surface bundles over surfaces are symplectic, and Gompf [11] generalized this result to Lefschetz fibrations. On the other hand, there are obvious obstructions to the existence of symplectic structures from algebraic topology. Moreover, Taubes' results [20, 21] provide more constraints in terms of the Seiberg–Witten invariants of the 4–manifold.

However, very little obstruction to the existence of symplectic structures is known besides the above mentioned ones. For example, given a symplectic manifold X, a symplectic torus $T \subset X$ with $[T]^2 = 0$, and a knot $K \subset S^3$, Fintushel and Stern [4] introduced a construction called knot surgery to get a new manifold X_K. They showed that X_K is symplectic if K is fibered, and X_K can often be proven to be non-symplectic when the Alexander polynomial of K is not monic. (See Section 4 for more details.) However, if the Alexander polynomial of K is monic, the obstruction from Seiberg–Witten theory does not exclude the possibility that X_K has a symplectic structure. Nevertheless, one can mention the following folklore conjecture.

Conjecture 1.1. Suppose that X^4 is a closed 4–manifold admitting a Lefschetz fibration whose regular fibers are tori. Let $T \subset X$ be a regular fiber of the fibration, and suppose that $[T] \neq 0$ in $H_2(X;\mathbb{R})$. (Hence X is symplectic by [23].) Let X_K be a manifold obtained by Fintushel–Stern knot surgery on T

using a knot $K \subset S^3$. Then X_K has a symplectic structure if and only if K is a fibered knot.

As we remarked before, the "if" part of the above conjecture was proved by Fintushel and Stern. The most interesting case of Conjecture 1.1 is when $\pi_1(X\backslash T)$ (and hence $\pi_1(X)$ and $\pi_1(X_K)$) is trivial, as X_K is then homeomorphic to X by Freedman's theorem. In this case, the Lefschetz fibration of X must contain singular fibers. Our main result in this paper is the case of the above conjecture when X is a genuine torus bundle, namely, there are no singular fibers in the Lefschetz fibration.

Theorem 1.2. Conjecture 1.1 is true when the Lefschetz fibration of X^4 is a torus bundle.

Friedl and Vidussi [6] proved that a closed four-manifold $S^1 \times N$ is symplectic if and only if N is a surface bundle over S^1. Their result implies the special case of Theorem 1.2 when X is a trivial torus bundle $T^2 \times F = S^1 \times (S^1 \times F)$, where F is a closed surface. Our proof uses a similar strategy as in [6]. Namely, If X_K has a symplectic structure, then any finite cover of X_K also has a symplectic structure. We can then use the constraints from Seiberg–Witten theory to study the existence of symplectic structures on finite covers of X_K. The Seiberg–Witten invariants of finite covers of X_K can be expressed in terms of twisted Alexander polynomials of K. We then use a vanishing theorem for twisted Alexander polynomials due to Friedl–Vidussi [7] to get our conclusion. Of course, this strategy works only if the fundamental group of the 4–manifold we consider contains many finite index subgroups.

A major difference between [6] and our case is that any finite cover $\widetilde{N} \to N$ gives rise to a finite cover $S^1 \times \widetilde{N} \to S^1 \times N$, but the construction of finite covers of X_K is not so obvious. The main technical part of this paper is devoted to constructing finite covers of X_K. We also need the full strength of the gluing theorem for Seiberg–Witten invariants along essential T^3 from [22].

Throughout this paper, the manifolds we consider are all smooth and oriented. Suppose that M is a submanifold of a manifold N, then $\nu(M)$ denotes a closed tubular neighborhood of M in N, and $\nu^\circ(M)$ denotes the interior of $\nu(M)$.

This paper is organized as follows. In Section 2 we will review the definition of twisted Alexander polynomials and state a vanishing theorem of Friedl–Vidussi [7]. In Section 3 we will review the Seiberg–Witten invariants for 4–manifolds with boundary consisting of copies of T^3, and state the gluing formula for Seiberg–Witten invariants when glued along essential tori. In Section 4 we will review several constructions of symplectic 4–manifolds, and state the constraints on symplectic 4–manifolds from Seiberg–Witten theory. In Section 5, we will analyze the topology of torus bundles and construct certain covers of X_K. Our main theorem will then be proved in Section 6.

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2 Twisted Alexander polynomials

Twisted Alexander polynomials were introduced by Xiao-Song Lin [15] in 1990. Many authors [12, 24, 13, 2] have since generalized this invariant in various ways. We will follow the treatment in [5].

Let N be a compact 3–manifold with $b_1(N) > 0$,

$$H = H(N) = H^2(N, \partial N)/\mathrm{Tors} \cong H_1(N)/\mathrm{Tors}.$$

Let F be a free abelian group, and $\phi \in Hom(H, F)$. Then $\pi_1(N)$ acts on F by translation via ϕ. Let $\alpha\colon \pi_1(N) \to GL(n, \mathbb{Z})$ be a representation. Then there is an induced representation

$$\alpha \otimes \phi\colon \pi_1(N) \to GL(n, \mathbb{Z}[F])$$

defined as follows. For $g \in \pi_1(N)$, $\alpha \otimes \phi(g)$ sends $\sum_{f \in F} a_f f \in (\mathbb{Z}[F])^n$ to

$$\sum_{f \in F} (\alpha(g)(a_f))(f\phi(g)),$$

where each $a_f \in \mathbb{Z}^n$, and the elements in F are written multiplicatively. Thus $(\mathbb{Z}[F])^n$ is a left $\mathbb{Z}[\pi_1(N)]$–module, whose left $\mathbb{Z}[\pi_1(N)]$ multiplication commutes with the right $\mathbb{Z}[F]$–module structure.

Let $\widetilde{N}$ be the universal cover of N, then $\pi_1(N)$ acts on the left of $\widetilde{N}$ as group of deck transformations. The chain group $C_*(\widetilde{N})$ is a right $\mathbb{Z}[\pi_1(N)]$–module, with the right action defined via $\sigma \cdot g := g^{-1}(\sigma)$. We can form the chain complex

$$C_*(\widetilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} (\mathbb{Z}[F])^n,$$

and define $H_*(N; \alpha \otimes \phi)$ be its homology group, which is also a $\mathbb{Z}[F]$–module. We call $H_1(N; \alpha \otimes \phi)$ the *(first) twisted Alexander module*.

Since $\mathbb{Z}[F]$ is Noetherian, $H_1(N; \alpha \otimes \phi)$ is a finitely generated module over $\mathbb{Z}[F]$. There exists a free resolution

$$(\mathbb{Z}[F])^m \xrightarrow{\ f\ } (\mathbb{Z}[F])^n \longrightarrow H_1(N; \alpha \otimes \phi) \longrightarrow 0,$$

where m, n are positive integers. We can always arrange that $m \geq n$. Let A be an $n \times m$ matrix over $\mathbb{Z}[F]$ representing f.

Definition 2.1. The *twisted Alexander polynomial* of (N, α, ϕ), denoted by $\Delta^{\alpha}_{N,\phi}$, is the greatest common divisor of all the $n \times n$ minors of A. It is well defined only up to multiplication by a unit in $\mathbb{Z}[F]$.

When $F = H$ and ϕ is the identity on H, we simply write Δ_N^α. When α is the trivial representation to $GL(1,\mathbb{Z})$, we omit the superscript α. In particular, $\Delta_N \in \mathbb{Z}[H]$ is the usual Alexander polynomial of N. When $\alpha\colon \pi_1(N) \to G$ is a representation into a finite group, we get an induced representation into $\mathrm{Aut}(\mathbb{Z}[G])$, which is denoted by α as well. In that case, the twisted Alexander polynomials are essentially determined by the untwisted Alexander polynomials of the covers of N correponding to $\ker\alpha$. More precisely, we recall [5, Proposition 3.6]:

Proposition 2.2 (Friedl–Vidussi)**.** Let N be a 3–manifold with $b_1(N) > 0$ and let $\alpha\colon \pi_1(N) \to G$ be an epimorphism onto a finite group. Let N_G be the covering space of N corresponding to $\ker\alpha$. Let $\pi_*\colon H(N_G) \to H(N)$ be the map induced by the covering map. Then Δ_N^α and Δ_{N_G} satisfy the following relations:

- If $b_1(N_G) > 1$, then

$$\Delta_N^\alpha = \begin{cases} \pi_*(\Delta_{N_G}), & \text{if } b_1(N) > 1; \\ (a-1)^2\pi_*(\Delta_{N_G}), & \text{if } b_1(N) = 1, \mathrm{im}\pi_* = \langle a\rangle. \end{cases}$$

- If $b_1(N_G) = 1$, then $b_1(N) = 1$ and

$$\Delta_N^\alpha = \pi_*(\Delta_{N_G}).$$

Given $\phi \in H^1(N)$, we say ϕ is *fibered* if ϕ is dual to a fiber of a fibration of N over S^1. A key ingredient in this paper is the following vanishing theorem of Friedl–Vidussi [7] concerning non-fibered cohomology classes.

Theorem 2.3 (Friedl–Vidussi)**.** Let N be a compact, orientable, connected 3–manifold with (possibly empty) boundary consisting of tori. If $\phi \in H^1(N)$ is not fibered, then there exsits an epimorphism $\alpha\colon \pi_1(N) \to G$ onto a finite group G such that $\Delta_{N,\phi}^\alpha = 0$.

3 Seiberg–Witten invariants and gluing formula along essential tori

In this section, we will review the Seiberg–Witten theory for 4–manifolds with boundary consisting of tori, and the gluing formula for cutting along essential tori. We will follow the treatment in [22].

First, let us recall the usual Seiberg–Witten invariants for closed 4–manifolds [26]. Given a closed, oriented, connected, smooth, 4–manifold X with $b_2^+(X) > 0$, let $\mathrm{Spin}^c(X)$ be the set of Spin^c structures on X. One can define the Seiberg–Witten invariant $sw_X\colon \mathrm{Spin}^c(X) \to \mathbb{Z}/\{\pm 1\}$. The sign can be fixed by choosing an orientation on

$$L_X = \Lambda^{\mathrm{top}} H^1(X;\mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X;\mathbb{R}).$$

In order to construct sw_X, we need to start with a riemannian metric on X. It turns out that sw_X does not depend on the choice of the metric when $b_2^+(X) > 1$.

When $b_2^+(X)=1$, there are two chambers in the space of metrics corresponding to two orientations on $H_2^+(X;\mathbb{R})$, sw_X only depends on the chamber the metric lies in.

From now on in this section, we assume that X is a compact, oriented, connected, smooth 4–manifold such that ∂X is a (possibly empty) disjoint union of T^3, and there exists a cohomology class $\varpi \in H^2(X;\mathbb{R})$ whose pull-back is non-zero in the cohomology of each component of ∂X. When $\partial X=\emptyset$, we do not need such ϖ to define sw_X, but we still assume the existence of ϖ in order to state the gluing formula. Moreover, we assume $b_2^+(X)>0$ when $\partial X=\emptyset$.

Let $\mathrm{Spin}^c(X)$ be the set of Spin^c structures on X, and $\mathrm{Spin}^c_0(X)\subset \mathrm{Spin}^c(X)$ be the subset consisting of $\mathfrak{s}$ such that the pull-back of $c_1(\mathfrak{s})$ is zero in $H^2(\partial X)$. By the exact sequence

$$H^2(X,\partial X) \xrightarrow{\pi^*} H^2(X) \xrightarrow{\iota^*} H^2(\partial X),$$

if $\mathfrak{s}\in \mathrm{Spin}^c_0(X)$, then $c_1(\mathfrak{s})\in H^2(X)$ is in the image of π^*. Let

$$\mathrm{Spin}^c_0(X,\partial X)=\left\{(\mathfrak{s},z)\,\middle|\,\mathfrak{s}\in\mathrm{Spin}^c_0(X), z\in H^2(X,\partial X), \pi^*(z)=c_1(\mathfrak{s})\right\}.$$

One can define the relative Seiberg–Witten invariant

$$sw_X\colon\ \mathrm{Spin}^c_0(X,\partial X)\to \mathbb{Z}/\{\pm 1\}.$$

The sign can be fixed by choosing an orientation on

$$L_X=\Lambda^{\mathrm{top}}H^1(X,\partial X;\mathbb{R})\otimes\Lambda^{\mathrm{top}}H^{2+}(X,\partial X;\mathbb{R}).$$

When $\partial X=\emptyset$, sw_X is just the usual Seiberg–Witten invariant. When $\partial X\neq\emptyset$, sw_X is an invariant of the pair (X,ϖ), and it is unchanged under continuous deformation of ϖ in $H^2(X;\mathbb{R})$ through classes with non-zero restriction in the cohomology of each component of ∂X.

Suppose that $M\subset X$ is a 3–torus such that the restriction of ϖ to $H^2(M;\mathbb{R})$ is nontrivial. We will consider the gluing formula for sw when X is cut open along M. There are two cases. In the first case, X is split by M into two parts X_1, X_2. In the second case, $X_1=X\setminus\nu^\circ(M)$ is connected.

When M is separating, there is a canonical isomorphism

$$L_X\cong L_{X_1}\otimes L_{X_2}. \tag{1}$$

One can define a map

$$\wp\colon\ \mathrm{Spin}^c_0(X_1,\partial X_1)\times\mathrm{Spin}^c_0(X_2,\partial X_2)\to\mathrm{Spin}^c_0(X,\partial X).$$

When M is non-separating, there is a canonical isomorphism

$$L_X\cong L_{X_1}. \tag{2}$$

One can define

$$\wp\colon\ \mathrm{Spin}^c_0(X_1,\partial X_1)\to\mathrm{Spin}^c_0(X,\partial X).$$

In any case, if $(\mathfrak{s},z)\in\mathrm{im}\wp$, then $c_1(\mathfrak{s})|_M=0$.

Theorem 3.1 (Taubes)**.** Let $M \subset X$ be a three-dimensional torus satisfying that the pull-back of ϖ in $H^2(M;\mathbb{R})$ is nontrivial.
• If M splits X into two parts X_1, X_2, we orient L_X using (1). Then

$$sw_X(\mathfrak{s}, z) = \sum_{((\mathfrak{s}_1,z_1),(\mathfrak{s}_2,z_2))\in\wp^{-1}(\mathfrak{s},z)} sw_{X_1}(\mathfrak{s}_1, z_1) sw_{X_2}(\mathfrak{s}_2, z_2).$$

• If M does not split X, let $X_1 = X \setminus \nu^\circ(M)$, we orient L_X using (2). Then

$$sw_X(\mathfrak{s}, z) = \sum_{(\mathfrak{s}_1,z_1)\in\wp^{-1}(\mathfrak{s},z)} sw_{X_1}(\mathfrak{s}_1, z_1).$$

Theorem 3.1 implies the more general case of the gluing formula when we cut X open along more than one tori. Suppose that $M = M_1 \cup M_2 \cup \cdots \cup M_m$ is a disjoint union of three-dimensional tori in X such that the restriction of ϖ to $H^2(M_i;\mathbb{R})$ is nontrivial for every i. Let $X_1, \dots, X_n$ be the components of $X \setminus \nu^\circ(M)$. Let $\mathcal{G}$ be the graph with vertices $v_1, \dots, v_n$ and edges $e_1, \dots, e_m$. The incidence relation in $\mathcal{G}$ is as follows: if M_k is adjacent to X_i and X_j, the edge e_k connects v_i and v_j. Let $\mathcal{T}$ be a spanning tree of $\mathcal{G}$, then $\mathcal{T}$ has exactly $n-1$ edges. Without loss of generality, we may assume the edges in $\mathcal{G} \setminus \mathcal{T}$ are $e_1, \dots, e_{m-n+1}$. We consider a sequence of manifolds $X^{(i)}$, $i = 0, \dots, m$:

$$X^{(0)} = X, \quad X^{(i)} = X^{(i-1)} \setminus \nu^\circ(M_i), i > 0.$$

Clearly, M_i is non-separating in $X^{(i-1)}$ when $1 \le i \le m-n+1$, and M_i is separating in $X^{(i-1)}$ when $m-n+2 \le i \le m$. Thus we can apply Theorem 3.1 inductively to get the gluing formula when we cut open along M.

More precisely, applying (2) and (1) consecutively, we get a canonical isomorphism

$$L_X \cong \bigotimes_{i=1}^n L_{X_i},$$

which will be used to orient L_X. We can also define a map

$$\wp\colon \prod_{i=1}^n \mathrm{Spin}_0^c(X_i, \partial X_i) \to \mathrm{Spin}_0^c(X, \partial X).$$

Theorem 3.2. Under the above settings, we have

$$sw_X(\mathfrak{s}, z) = \sum_{((\mathfrak{s}_1,z_1),\dots,(\mathfrak{s}_n,z_n))\in\wp^{-1}(\mathfrak{s},z)} \prod_{i=1}^n sw_{X_i}(\mathfrak{s}_i, z_i).$$

In practice, it is more convenient to consider the following version of Seiberg–Witten invariant:

$$SW_X\colon\ H^2(X, \partial X) \to \mathbb{Z}$$

defined by letting

$$SW_X(z) = \sum_{(\mathfrak{s},z)\in\mathrm{Spin}_0^c(X,\partial X)} sw_X(\mathfrak{s}, z).$$

Let

$$\rho_i\colon\ H^2(X_i,\partial X_i)\to H^2(X,\partial X),\quad i=1,\dots,n$$

be the natural maps, and

$$\rho=\rho_1+\cdots+\rho_n\colon\ \bigoplus_{i=1}^n H^2(X_i,\partial X_i)\to H^2(X,\partial X).$$

Then Theorem 3.2 implies

Theorem 3.3. Under the condition of Theorem 3.2, we have

$$SW_X(z)=\sum_{(z_1,\dots,z_n)\in\rho^{-1}(z)}\prod_{i=1}^n SW_{X_i}(z_i).$$

It is often convenient to represent the Seiberg–Witten invariants in the following more compact form.

Let $H(X)=H^2(X,\partial X)/\mathrm{Tors}$. Given $z\in H^2(X,\partial X)$, let $[z]\in H(X)$ be the reduction of z. We define

$$\underline{SW}_X=\sum_{z\in H^2(X,\partial X)} SW_X(z)[z],$$

which lies either in $\mathbb{Z}[H(X)]$, or, in certain cases, an extension of this group ring which allows semi-infinite power series.

For example, let $t\in H(D^2\times T^2)$ be the Poincaré dual to the fundamental class of the torus, then

$$\underline{SW}_{D^2\times T^2}=\frac{t}{1-t^2}=t+t^3+\cdots. \tag{3}$$

The invariant $\underline{SW}_X$ is related to the Alexander polynomial of a 3–manifold. Let N be a compact, oriented, connected 3–manifold with $b_1(N)>0$ such that ∂N is a (possibly empty) disjoint union of T^2. Let $p^*\colon\ H^2(N,\partial N)\to H^2(S^1\times N,\partial(S^1\times N))$ be the map on cohomology induced by the projection $p\colon\ S^1\times N\to N$. Let

$$\Phi_2\colon\ \mathbb{Z}[H(N)]\to\mathbb{Z}[H(S^1\times N)]$$

be the map induced by $2p^*$. Meng and Taubes [17] proved the following theorem.

Theorem 3.4 (Meng–Taubes)**.** Let N be a compact, oriented, connected 3–manifold with $b_1(N)>0$ such that ∂N is a (possibly empty) disjoint union of T^2. When $b_1(N)=1$, let t be a generator of $H(N)\cong\mathbb{Z}$, and let $|\partial N|=0$ or 1 be the number of boundary components of ∂N. Then, there exits an element $\xi\in\pm p^*(H(N))$, such that

$$\underline{SW}_{S^1\times N}=\begin{cases}\xi\Phi_2(\Delta_N), & \text{if } b_1(N)>1;\\ \xi\Phi_2((1-t)^{|\partial N|-2}\Delta_N), & \text{if } b_1(N)=1.\end{cases}$$

As a corollary, we prove the following gluing result for the Alexander polynomial.

Corollary 3.5. Let N be as in Theorem 3.4, and $K \subset N$ be a knot such that $[K]$ is nontorsion. Let $\kappa \in H[N]$ be the coset of the Poincaré dual of $[K]$. Let $M = N \setminus \nu^\circ(K)$, and let

$$\pi^*\colon\ H^2(M,\partial M) \cong H^2(N,\nu(K)\cup\partial N) \to H^2(N,\partial N)$$

be the natural map induced by the inclusion $(N,\partial N)\subset(N,\nu(K)\cup\partial N)$. We also use π^* to denote the induced map $\mathbb{Z}[H(M)]\to\mathbb{Z}[H(N)]$. Then there exists an element $\xi\in\pm H(N)$, such that

$$\Delta_N = \begin{cases} \xi\pi^*(\Delta_M), & \text{if } b_1(N)=1;\\ \xi(1-\kappa)^{-1}\pi^*(\Delta_M), & \text{if } b_1(N)>1.\end{cases}$$

Proof. Let $p_N^*\colon\ H^2(N,\partial N)\to H^2(S^1\times N, S^1\times\partial N)$, and define p_M^* similarly. We first consider the case $b_1(N)>1$. By Theorem 3.4, there exist $\zeta\in\pm p_N^*(H(N))$ and $\eta=\pm p_M^*(H(M))$ such that

$$\zeta\Phi_2(\Delta_N) = \sum_{z\in H^2(N,\partial N)} SW_{S^1\times N}(p_N^*(z))[p_N^*(z)], \tag{4}$$

and

$$\eta\Phi_2(\Delta_M) = \sum_{w\in H^2(M,\partial M)} SW_{S^1\times M}(p_M^*(w))[p_M^*(w)]. \tag{5}$$

Let $a\in H^2(T^2\times D^2, T^2\times\partial D^2)$ be the positive generator. Using Theorem 3.3 and (3), we get

$$\begin{aligned} & SW_{S^1\times N}(p_N^*(z)) \\ = & \sum_{n\in\mathbb{Z}} SW_{T^2\times D^2}((2n+1)a) \sum_{\substack{w\in H^2(M,\partial M)\\ \rho((2n+1)a),p_M^*(w))=p_N^*(z)}} SW_{S^1\times M}(p_M^*(w)) \\ = & \sum_{n\geq 0}\ \sum_{\substack{w\in H^2(M,\partial M)\\ \rho((2n+1)a,p_M^*(w))=p_N^*(z)}} SW_{S^1\times M}(p_M^*(w)). \end{aligned}$$

Using (4), (5), and the fact that

$$\rho((2n+1)a, p_M^*(w)) = p_N^*((2n+1)\mathrm{PD}([K])+\pi^*(w)),$$

we get

$$\begin{aligned} & \zeta\Phi_2(\Delta_N) \\ = & \sum_{z\in H^2(N,\partial N)}\sum_{n\geq 0}\ \sum_{\substack{w\in H^2(M,\partial M)\\ \rho((2n+1)a,p_M^*(w))=p_N^*(z)}} SW_{S^1\times M}(p_M^*(w))[p_N^*(z)] \\ = & \sum_{n\geq 0}\sum_{w\in H^2(M,\partial M)} SW_{S^1\times M}(p_M^*(w))\kappa^{2n+1}[\pi^*(w)] \\ = & \frac{\kappa}{1-\kappa^2}\pi^*(\eta\Phi_2(\Delta_M)). \end{aligned}$$

So our result holds.

When $b_1(N) = 1$, the proof is similar. $\square$

4 Symplectic geometry

In this section, we will review some topological constructions of symplectic 4–manifolds, and state the constraints on the Seiberg–Witten invariants of symplectic manifolds.

Thurston [23] found a very general topological construction of symplectic manifolds:

Theorem 4.1 (Thurston)**.** Let $M^{2n+2} \to N^{2n}$ be a fiber bundle over a symplectic manifold. If the homology class of the fiber is nonzero in $H_2(M;\mathbb{R})$, then M has a symplectic structure such that each fiber is a symplectic submanifold. Moreover, if $\rho\colon\ N \hookrightarrow M$ is a section, then the image of ρ is a symplectic submanifold.

In dimension 4, Thurston's construction was generalized by Gompf [11] to the extent that if a 4–manifold X admits a Lefschetz fibration (or a Lefschetz pencil) such that the homology class of the fiber is nontorsion, then X has a symplectic structure. This construction, together with the celebrated theorem of Donaldson [3] that all closed symplectic manifolds have Lefschetz pencils, gives us a topological characterization of closed symplectic 4–manifolds.

The above characterization of symplectic 4–manifolds is not always practical. When we construct symplectic 4–manifolds, we often need the following construction due to Gompf [10] and McCarthy–Wolfson [16]. Suppose that X_1, X_2 are two smooth four-manifolds, $F_i \subset X_i$, $i = 1, 2$, are two 2–dimensional closed connected submanifolds such that F_1 is homeomorphic to F_2 and $[F_1]^2 = -[F_2]^2$. Let $N(F_i), \nu(F_i)$ be two tubular neighborhoods of F_i in X_i, $i = 1, 2$, such that $\nu(F_i)$ is contained in the interior of $N(F_i)$. Let $W_i = N(F_i) \setminus \nu^\circ(F_i)$, $i = 1, 2$, regarded as an annulus bundle over F_i. Suppose that $f\colon\ F_1 \to F_2$ is a diffeomorphism, then there exists an orientation preserving diffeomorphism $\bar{f}\colon\ W_1 \to W_2$ such that $\bar{f}(\partial N(F_1)) = \partial\nu(F_2)$, and $\bar{f}$ is a bundle map covering f. Let X be the manifold obtained by gluing $X_1 \setminus \nu^\circ(F_1)$ and $X_2 \setminus \nu^\circ(F_2)$ together via the diffeomorphism $\bar{f}$. Then X is called the *normal connected sum* of (X_1, F_1) and (X_2, F_2), denoted $X_1 \#_f X_2$. If X_i is symplectic, F_i is a symplectic submanifold, $i = 1, 2$, and $f, \bar{f}$ are chosen to be symplectomorphisms, then X also has a symplectic structure, and the operation is called a *symplectic normal connected sum* or simply *symplectic sum*.

Suppose that X is a smooth 4–manifold containing a smooth 2–torus T with $[T]^2 = 0$. Let $K \subset S^3$ be a knot, and let $K' \subset S^3_0(K)$ be the dual knot in the zero surgery. We can perform the normal connected sum of (X, T) and $(S^1 \times S^3_0(K), S^1 \times K')$ to get a new manifold X_K. (This X_K is usually not unique, since it depends on the choice of a homeomorphism f and $\bar{f}$.) This procedure was investigated by Fintushel and Stern [4], who called it *knot surgery*. By

Theorems 3.1 and 3.4, we know that

$$\underline{SW}_{X_K} = \underline{SW}_X \cdot \Delta_K(\mathrm{PD}([T])^2), \tag{6}$$

where Δ_K is the Alexander polynomial of K. (Clearly, X_K has the same homology type as X, so we can identify $H(X_K)$ with $H(X)$.) This construction is particularly interesting when $\pi_1(X \setminus T) = 1$, since X_K is then homeomorphic to X by Freedman's theorem, but X_K is not diffeomorphic to X if $\Delta_K \neq 1$.

When K is fibered, $S^1 \times S^3_0(K)$ is a surface bundle over T^2 with $S^1 \times K'$ being a section, and the fiber is homologically essential. Theorem 4.1 implies that $S^1 \times S^3_0(K)$ has a symplectic structure such that $S^1 \times K'$ is a symplectic submanifold. Hence the symplectic sum construction implies the following theorem.

Theorem 4.2 (Fintushel–Stern)**.** Suppose that X is a symplectic 4–manifold, $T \subset X$ is a symplectic torus with $[T]^2 = 0$. Then X_K is symplectic if K is fibered.

It is natural to guess that a converse to Theorem 4.2 should be true in many cases. More precisely, one can mention the folklore Conjecture 1.1. Evidence to this conjecture is a famous theorem of Taubes [20, 21].

Theorem 4.3 (Taubes)**.** Suppose that (X, ω) is a closed symplectic 4–manifold with $b_2^+ > 1$, $\mathfrak{k}$ is the canonical Spin^c structure on X, and $\bar{\mathfrak{k}}$ is the conjugate of $\mathfrak{k}$. Then

$$SW_X(\mathfrak{k}) = \pm 1.$$

Moreover, if $\mathfrak{s} \in \mathrm{Spin}^c(X)$ satisfies that $SW_X(\mathfrak{s}) \neq 0$, then

$$|c_1(\mathfrak{s}) \smile [\omega]| \leq c_1(\mathfrak{k}) \smile [\omega],$$

and the equality holds if and only if $\mathfrak{s} = \mathfrak{k}$ or $\bar{\mathfrak{k}}$.

In particular, if X is the K3 surface, and Δ_K is not monic, Taubes' theorem implies that X_K is not symplectic.

We will also need the following theorem proved by Bauer [1] and Li [14].

Theorem 4.4 (Bauer, Li)**.** Suppose that X is a closed symplectic 4–manifold with $c_1(\mathfrak{k})$ torsion. Then $b_1(X) \leq 4$.

5 Constructing covering spaces of X_K

Before we state the main result in this section, we set up the basic notations we will use. Let X be a torus bundle over a closed surface F. Let T be a fiber of X, and let $E = X \setminus \nu^\circ(T)$. Let $K \subset S^3$ be a nontrivial knot, $N = S^3 \setminus \nu^\circ(K)$, $N_0 = S^3_0(K)$ be the zero surgery on K, and $K' \subset N_0$ be the dual knot of the surgery. Let $f\colon\, S^1 \times K' \to T$ be a diffeomorphism, and let $X_K = X \#_f (S^1 \times N_0)$.

The goal in this section is to construct covering spaces of X_K. More precisely, we will prove the following proposition.

Proposition 5.1. Suppose that $\alpha\colon \pi_1(N_0) \to G$ is an epimorphism, where G is a finite group. Let $p_0\colon \widetilde{N}_0 \to N_0$ be the covering map corresponding to $\ker\alpha$, and let $\widetilde{N} = p_0^{-1}(N)$. Suppose that $p_0^{-1}(K')$ has r components. Since p_0 is a regular cover, the restriction of p_0 on each component of $p_0^{-1}(K')$ has the same degree l. If the genus of F is positive, then there exists an rl^3–fold cover $\widetilde{X}_K$ of X_K, such that $\widetilde{X}_K$ contains a submanifold diffeomorphic to $S^1 \times \widetilde{N}$, and $\widetilde{X}_K$ admits a retraction onto the complete bipartite graph $K_{r,l}$.

In order to prove this proposition, we will need some preliminary material. We start by analyzing the topology of torus bundles. The structural group of a torus bundle is $\mathrm{Diff}^+(T^2)$, which is homotopy equivalent to its subgroup $\mathrm{Aff}^+(T^2) \cong T^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$. If the structural group is contained in $\mathrm{SL}(2,\mathbb{Z})$, we say this torus bundle is an $\mathrm{SL}(2,\mathbb{Z})$*–bundle*.

Each torus bundle $X \to F$ is uniquely determined up to isomorphism by the homotopy type of its classifying map $F \to B\mathrm{Diff}^+(T^2) \simeq B\mathrm{Aff}^+(T^2)$. From the short split exact sequence

$$1 \to T^2 \to \mathrm{Aff}^+(T^2) \to \mathrm{SL}(2,\mathbb{Z}) \to 1$$

we get a fiber bundle

$$BT^2 \to B\mathrm{Aff}^+(T^2) \to B\mathrm{SL}(2,\mathbb{Z})$$

which has a section. Since $BT^2 = \mathbb{C}P^\infty \times \mathbb{C}P^\infty = K(\mathbb{Z}^2, 2)$ and $B\mathrm{SL}(2,\mathbb{Z}) = K(\mathrm{SL}(2,\mathbb{Z}), 1)$, we have

$$\begin{aligned} \pi_1(B\mathrm{Aff}^+(T^2)) &\cong \mathrm{SL}(2,\mathbb{Z}), \\ \pi_2(B\mathrm{Aff}^+(T^2)) &\cong \mathbb{Z}^2. \end{aligned}$$

Hence the homotopy type of a map $F \to B\mathrm{Aff}^+(T^2)$ is determined by a representation $\rho\colon \pi_1(F) \to \pi_1(B\mathrm{Aff}^+(T^2)) \cong \mathrm{SL}(2,\mathbb{Z})$ (called the *monodromy*) and a pair of integers $(m,n) \in H^2(F; \pi_2(B\mathrm{Aff}^+(T^2))) \cong \mathbb{Z}^2$ (called the *Euler class*).

Remark 5.2. In particular, when $F = S^2$, X is completely determined by the Euler class. It is easy to see $[T] \neq 0 \in H_2(X;\mathbb{R})$ if and only if $(m,n) = (0,0)$. In this case, $X = T^2 \times S^2$. As we mentioned before, this case is covered by Friedl and Vidussi's work [6]. Hence, in order to prove Theorem 1.2, we only need to consider the case when the genus of F is positive.

Remark 5.3. In general, $[T] \neq 0 \in H_2(X;\mathbb{R})$ if and only if $E^\infty_{0,2} \cong \mathbb{Z}$, where $\{E^i_{*,*}\}_{i=1}^\infty$ is the Leray–Serre spectral sequence for the fiber bundle $X \to F$. (See [9, Section 4] or [25, Lemma 4.6] for more detail.) When F is a torus, Geiges explicitly described the cases when $[T] \neq 0$ [9, Theorem 1], using Sakamoto–Fukuhara's classification of torus bundles over torus [19].

Definition 5.4. Let $T \subset Y^4$ be a torus with trivial neighborhood. We fix a product structure $S^1 \times S^1$ on T and identify S^1 with $\mathbb{R}/\mathbb{Z}$. We can remove a neighborhood $\nu(T) \cong T^2 \times D^2$ then glue it back to $Y \setminus \nu^\circ(T)$ via the homeomorphism $f\colon T^2 \times \partial D^2 \to \partial(Y \setminus \nu^\circ(T))$ which sends (x,y,θ) to $(x+m\theta, y+n\theta, \theta)$. This procedure is called the (m,n)*–framed surgery* on T.

Given ρ and (m,n), as in [25, Section 4], we can reconstruct $X \to F$ by first constructing an $\mathrm{SL}(2,\mathbb{Z})$–bundle over F using the monodromy ρ then doing (m,n)–framed surgery on a fiber. Suppose that

$$\pi_1(F) = \langle a_1, a_2, \ldots, a_{2g-1}, a_{2g} | \prod_{k=1}^{g} [a_{2k-1}, a_{2k}] \rangle$$

and that $\rho(a)$ acts on $\mathbb{Z}^2 = \langle s_1, s_2 | [s_1, s_2] \rangle$ for every $a \in \pi_1(F)$. We can write down a presentation of $\pi_1(X)$ from the construction of X as follows:

$$\pi_1(X) = \left\langle s_1, s_2, t_1, \ldots, t_{2g} \;\middle|\; \begin{array}{l} [s_1, s_2], \\ t_i s_j t_i^{-1} (\rho(a_i)(s_j))^{-1}, (1 \le i \le 2g, j = 1, 2) \\ s_1^m s_2^n (\prod_{k=1}^{g} [t_{2k-1}, t_{2k}])^{-1} \end{array} \right\rangle. \tag{7}$$

Proposition 5.5. Let $X \to F$ be a torus bundle over a closed surface with positive genus. For any integer $l > 0$, there exists a torus bundle $\widetilde{X}$ and an l^3–fold cover $p\colon \widetilde{X} \to X$, such that for any fiber $T \subset X$ and any component $\widetilde{T}$ of $p^{-1}(T)$, the map $p|_{\widetilde{T}}\colon \widetilde{T} \to T$ is the covering map corresponding to the characteristic subgroup $(l\mathbb{Z}) \times (l\mathbb{Z}) \subset \pi_1(T)$.

Proof. Let $\overline{F} \to F$ be an l–fold cover, and $\overline{X} \to \overline{F}$ be a torus bundle over $\overline{F}$ which is the pull-back of $X \to F$. Suppose that the genus of $\overline{F}$ is $\bar{g}$, and the monodromy of $\overline{X}$ is $\bar{\rho}$. Suppose that the Euler class of $X \to F$ is (m,n), then the Euler class of $\overline{X} \to \overline{F}$ is (ml, nl). By (7),

$$\pi_1(\overline{X}) = \left\langle s_1, s_2, t_1, \ldots, t_{2\bar{g}} \;\middle|\; \begin{array}{l} [s_1, s_2], \\ t_i s_j t_i^{-1} (\bar{\rho}(a_i)(s_j))^{-1}, (1 \le i \le 2\bar{g}, j = 1, 2) \\ s_1^{ml} s_2^{nl} (\prod_{k=1}^{\bar{g}} [t_{2k-1}, t_{2k}])^{-1} \end{array} \right\rangle.$$

Let Γ_l be the subgroup of $\Gamma = \pi_1(\overline{X})$ generated by $s_1^l, s_2^l, t_1, \ldots, t_{2\bar{g}}$, we claim that $[\Gamma : \Gamma_l] = l^2$. If this claim is true, let $\widetilde{X}$ be the covering space of $\overline{X}$ corresponding to Γ_l, then $\widetilde{X}$ is the covering space of X we want.

The rest of this proof is devoted to proving $[\Gamma : \Gamma_l] = l^2$. Any element in Γ can be written as a word st, where s is a word in $s_1^{\pm 1}, s_2^{\pm 1}$, t is a word in $t_i^{\pm 1}$, $1 \le i \le 2\bar{g}$. Since the subgroup $\Sigma_l = \langle s_1^l, s_2^l \rangle$ of $\langle s_1, s_2 \rangle \cong \mathbb{Z}^2$ is preserved by any $\bar{\rho}(a_i)$, $st \in \Gamma_l$ if and only if $s \in \Sigma_l$. Let $(u_1, v_1), (u_2, v_2) \in \{0, 1, \ldots, l-1\}^2$, then it follows that $s_1^{u_1} s_2^{v_1} \in s_1^{u_2} s_2^{v_2} \Gamma_l$ if and only if $(u_1, v_1) = (u_2, v_2)$. So

$$s_1^u s_2^v \Gamma_l, \quad (u,v) \in \{0, 1, \ldots, l-1\}^2$$

are distinct left cosets of Γ_l in Γ. Clearly, the union of these cosets is Γ, so $[\Gamma : \Gamma_l] = l^2$. $\square$

Proof of Proposition 5.1. By Proposition 5.5, there exists a degree l^3 covering map $p_X\colon \widetilde{X} \to X$, such that for any fiber $T \subset X$ and any component $\widetilde{T}$ of $p_X^{-1}(T)$, the map $p_X|_{\widetilde{T}}\colon \widetilde{T} \to T$ is the covering map corresponding to $(l\mathbb{Z}) \times$

$(l\mathbb{Z}) \subset \pi_1(T)$. By the construction of $\widetilde{X}$, $p_X^{-1}(T)$ has l components. Let $\widetilde{E} = p_X^{-1}(E)$.

There is a covering map

$$q_N = q_l \times p_0\colon\ S^1 \times \widetilde{N}_0 \to S^1 \times N_0, \tag{8}$$

where $q_l\colon\ S^1 \to S^1$ is the l–fold cyclic cover. There are r components in $q_N^{-1}(S^1 \times K')$, and the restriction of q_N on each component is the covering map corresponding to $(l\mathbb{Z}) \times (l\mathbb{Z}) \subset \pi_1(S^1 \times K')$.

Since $(l\mathbb{Z}) \times (l\mathbb{Z})$ is a characteristic subgroup of $\mathbb{Z} \times \mathbb{Z}$, for any component $\widetilde{T}$ of $p_X^{-1}(T)$ and any component $\widetilde{S}$ of $q_N^{-1}(S^1 \times K')$, the map $f\colon\ S^1 \times K' \to T$ lifts to a map $\tilde{f}\colon\ \widetilde{S} \to \widetilde{T}$. Hence we can use $\tilde{f}$ to perform a normal connected sum of $\widetilde{X}$ and $S^1 \times \widetilde{N}_0$.

Recall that $p_X^{-1}(T) \subset \widetilde{X}$ has l components, and $q_N^{-1}(S^1 \times K') \subset S^1 \times \widetilde{N}_0$ has r components. Take r copies of $\widetilde{X}$ and l copies of $S^1 \times \widetilde{N}_0$. For any copy of $\widetilde{X}$ and any copy of $S^1 \times \widetilde{N}_0$, we can perform a normal connected sum of these two manifolds along a component of $p_X^{-1}(T)$ and a component of $q_N^{-1}(S^1 \times K')$, such that each component of $p_X^{-1}(T)$ or $q_N^{-1}(S^1 \times K')$ is used exactly once. The new manifold we get, denoted by $\widetilde{X}_K$, is clearly an rl^3–fold cover of X_K.

By the construction, $\widetilde{X}_K$ is obtained by gluing r copies of $\widetilde{E}$ and l copies of $S^1 \times \widetilde{N}$ together, such that any copy of $\widetilde{E}$ and any copy of $S^1 \times \widetilde{N}$ are glued along a T^3. Hence there is a retraction of $\widetilde{X}_K$ onto $K_{r,l}$. □

6 Proof of the main theorem

In this section, we will prove Theorem 1.2. By Remark 5.2, we only consider the case that X is a torus bundle over a closed surface F with positive genus. Assume that K is a nontrivial knot in S^3 and X_K is a symplectic manifold.

Lemma 6.1. There exists a finite cover of X_K with $b_1 > 4$.

Proof. Let $\widehat{\Sigma} \subset N_0$ be the closed surface obtained from a minimal Seifert surface Σ of K by capping off $\partial\Sigma$ with a disk. By [8], N_0 is irreducible and $\widehat{\Sigma}$ is incompressible in N_0. Since $\pi_1(N_0)$ is residually finite, we can find an epimorphism α from $\pi_1(N_0)$ onto a finite group G, such that $\pi_1(\widehat{\Sigma}) \not\subset \ker\alpha$. Hence $p_0\colon\ \widetilde{N}_0 \to N_0$, the covering map corresponding to $\ker\alpha$, is not a cyclic covering map. As a result, $p_0^{-1}(K')$ has $r > 1$ components. Suppose that each component of $p_0^{-1}(K')$ is an l–fold cyclic cover of K'. We may assume $l > 5$, since we can always take a large cyclic cover of N_0 first.

We construct a cover $\widetilde{X}_K$ of X_K as in Proposition 5.1. Since there is a retraction of $\widetilde{X}_K$ onto $K_{r,l}$,

$$b_1(\widetilde{X}_K) \geq b_1(K_{r,l}) = (r-1)(l-1) \geq l-1 > 4.$$ □

Corollary 6.2. Let $\mathfrak{k}$ be the canonical Spin^c structure of X_K. Then $c_1(\mathfrak{k})$ is nontorsion.

Proof. By Lemma 6.1, there exists a finite cover $\widetilde{X}_K$ of X_K with $b_1 > 4$. Assume that $c_1(\mathfrak{k})$ is torsion, then $c_1(\widetilde{X}_K)$ is also torsion since it is the pull-back of $c_1(\mathfrak{k})$ by the covering map. By Theorem 4.4, $b_1(\widetilde{X}_K) \le 4$, a contradiction. $\square$

In order to apply Theorem 4.3, we need the following lemma.

Lemma 6.3. If $(r-1)(l-1) > 2$, then $b_2^+(\widetilde{X}_K) > 1$.

Proof. The Euler characteristic of X is zero since the fiber has zero Euler characteristic. It is well known that the signature of X is zero [18]. Since X_K has the same homology type as X, both the Euler characteristic and the signature of X_K are zero, and the same is true for $\widetilde{X}_K$. It follows that

$$b_2^+(\widetilde{X}_K) = b_1(\widetilde{X}_K) - 1 \ge (r-1)(l-1) - 1 > 1. \qquad \square$$

Proof of Theorem 1.2. Assume that K is not fibered. By [8], N_0 is not fibered. Let ϕ be the positive generator of $H^1(N_0) \cong \mathbb{Z}$, and let $\psi \in H^1(N)$ be the restriction of ϕ. We can regard ϕ as a map $\pi_1(N_0) \to \mathbb{Z}$. By Theorem 2.3, there exists a surjective homomorphism $\alpha\colon \pi_1(N_0) \to G$, where G is a finite group, such that

$$\Delta^{\alpha}_{N_0} = \Delta^{\alpha}_{N_0,\phi} = 0. \tag{9}$$

As in Proposition 5.1, let $p_0\colon \widetilde{N}_0 \to N_0$ be the covering map corresponding to $\ker\alpha$, and let $\widetilde{N} = p_0^{-1}(N)$. We may assume $r > 1, l > 3$. Otherwise, as in the proof of Lemma 6.1, we can take a regular finite cover M_0 of N_0 satisfying $r > 1, l > 3$, and let $\beta\colon \pi_1(N_0) \to G_1$ be an epimorphism onto a finite group such that $\ker\beta = \ker\alpha \cap \pi_1(M_0)$. It follows from [7, Lemma 2.2] that $\Delta^{\beta}_{N_0} = 0$. So we can use β instead of α.

Since $r > 1, l > 3$, we have $b_2^+(\widetilde{X}_K) > 1$ by Lemma 6.3.

Let $(p_0)_*\colon \pi_1(\widetilde{N}_0) \to \pi_1(N_0)$ be the induced map on π_1, and let

$$\widetilde{\phi} = \phi \circ (p_0)_*\colon \pi_1(\widetilde{N}_0) \to \mathbb{Z}.$$

Let $\widetilde{\phi}_*\colon \mathbb{Z}[H(\widetilde{N}_0)] \to \mathbb{Z}[\mathbb{Z}]$ be the induced ring homomorphism. By Proposition 2.2, (9) implies $(p_0)_*(\Delta_{\widetilde{N}_0}) = 0$, hence

$$\widetilde{\phi}_*(\Delta_{\widetilde{N}_0}) = 0. \tag{10}$$

Let $(p_0|\widetilde{N})_*\colon \pi_1(\widetilde{N}) \to \pi_1(N)$ be the induced map on π_1, and let

$$\widetilde{\psi} = \psi \circ (p_0|\widetilde{N})_*\colon \pi_1(\widetilde{N}) \to \mathbb{Z}.$$

Let $\widetilde{\psi}_*\colon \mathbb{Z}[H(\widetilde{N})] \to \mathbb{Z}[\mathbb{Z}]$ be the induced ring homomorphism. We also regard $\widetilde{\psi}$ as a cohomology class in $H^1(\widetilde{N})$, then $\widetilde{\psi} \in H^1(\widetilde{N})$ is the pull-back of $\psi \in H^1(N)$ by the covering map. Clearly, for any component $\widetilde{K'}$ of $p_0^{-1}(K')$, we have $\widetilde{\psi}([\widetilde{K'}]) \ne 0$. Hence we can use Corollary 3.5 and (10) to conclude

$$\widetilde{\psi}_*(\Delta_{\widetilde{N}}) = 0. \tag{11}$$

We construct a finite cover $\widetilde{X}_K$ as in Proposition 5.1. Suppose that ω is a symplectic form on X_K. Since $[T] \neq 0 \in H_2(X;\mathbb{R}) \cong H_2(X_K;\mathbb{R})$ and $c_1(\mathfrak{k}) \neq 0 \in H^2(X_K;\mathbb{R})$ by Corrollary 6.2, we may perturb and rescale ω so that

$$[\omega]([T]) \neq 0, \quad c_1(\mathfrak{k}) \smile [\omega] \neq 0, \tag{12}$$

and $[\omega] \in H^2(X_K;\mathbb{Z})$. Let Ω be the pull-back of ω on $\widetilde{X}_K$, then Ω is also a symplectic form. Moreover, it follows from (12) that

$$[\Omega]([\widetilde{T}]) \neq 0, \quad c_1(\widetilde{X}_K,\Omega) \smile [\Omega] \neq 0, \tag{13}$$

The inclusion map $S_1 \times N \subset X_K$ induces a map

$$\iota_N^*\colon\ H^2(X_K) \to H^2(S^1 \times N) \cong H^1(S^1) \otimes H^1(N).$$

Let σ be the positive generator of $H^1(S^1)$. Then

$$\iota_N^*([\omega]) = k\sigma \otimes \psi, \text{ for some integer } k \neq 0, \tag{14}$$

by (12).

Let $X_2 \subset \widetilde{X}_K$ be a copy of $S^1 \times \widetilde{N}$, let $X_1 = \widetilde{X}_K \setminus \mathrm{int}(X_2)$, and $M = \partial X_1$. Let $\iota_i^*\colon\ H^2(\widetilde{X}_K) \to H^2(X_i)$, $i = 1, 2$, be the natural maps induced by the inclusion maps.

Let $p^*\colon\ H^2(\widetilde{N}, \partial\widetilde{N}) \to H^2(S^1 \times \widetilde{N}, S^1 \times \partial\widetilde{N})$ be the map induced by the projection. Let q_N be the covering map in (8). If $w \in H^2(\widetilde{N}, \partial\widetilde{N})$, using (14), we have

$$\begin{aligned}
\iota_2^*[\Omega] \smile p^*(w) &= q_N^*(\iota_N^*[\omega]) \smile p^*(w) \\
&= q_N^*(k\sigma \otimes \psi) \smile p^*(w) \\
&= kl\sigma\widetilde{\psi} \smile p^*(w) \\
&= kl\widetilde{\psi} \smile_3 w,
\end{aligned} \tag{15}$$

where $\smile_3$ means the cup product in $(\widetilde{N}, \partial\widetilde{N})$. Here we identify an element $a \cup b \in H^n(Y^n, \partial Y^n)$ with an integer via the isomorphism $H^n(Y^n, \partial Y^n) \cong \mathbb{Z}$.

Let

$$\rho_i\colon\ H^2(X_i, \partial X_i) \to H^2(\widetilde{X}_K), \quad i = 1, 2,$$

be the natural restriction maps, and let

$$\rho = \rho_1 + \rho_2\colon\ H^2(X_1, \partial X_1) \oplus H^2(X_2, \partial X_2) \to H^2(\widetilde{X}_K).$$

Suppose that $z_i \in H^2(X_i, \partial X_i)$, $i = 1, 2$, then it is elementary to check

$$\rho(z_1, z_2) \smile [\Omega] = z_1 \smile \iota_1^*[\Omega] + z_2 \smile \iota_2^*[\Omega]. \tag{16}$$

Suppose that $n = c_1(\widetilde{X}, \Omega) \smile [\Omega]$. Then $n \neq 0$ by (13). Using Theorem 3.3 and (16), we have

$$\begin{aligned} & \sum_{z \in H^2(\widetilde{X}_K), z \smile [\Omega] = n} SW_{\widetilde{X}_K}(z) \\ = & \sum_{\substack{z_1 \in H^2(X_1, \partial X_1) \\ z_2 \in H^2(X_2, \partial X_2) \\ z_1 \smile \iota_1^*[\Omega] + z_2 \smile \iota_2^*[\Omega] = n}} SW_{X_1}(z_1) SW_{X_2}(z_2) \\ = & \sum_{z_1 \in H^2(X_1, \partial X_1)} SW_{X_1}(z_1) \cdot \left(\sum_{\substack{z_2 \in H^2(X_2, \partial X_2) \\ z_2 \smile \iota_2^*[\Omega] = n - z_1 \smile \iota_1^*[\Omega]}} SW_{X_2}(z_2) \right). \end{aligned} \tag{17}$$

It follows from (15) and Theorem 3.4 that the inner sum in (17) is a coefficient in $\widetilde{\psi}_*(\Delta_{\widetilde{N}})$, which is zero by (11). Hence the right hand side of (17) is zero. This contradicts Theorem 4.3 and the fact that $n \neq 0$. □

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