

# Converse bounds for private communication over quantum channels

Mark M. Wilde\*

Marco Tomamichel†

Mario Berta‡

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## Abstract

This paper establishes several converse bounds on the private transmission capabilities of a quantum channel. The main conceptual development builds firmly on the notion of a private state, which is a powerful, uniquely quantum method for simplifying the tripartite picture of privacy involving local operations and public classical communication to a bipartite picture of quantum privacy involving local operations and classical communication. This approach has previously led to some of the strongest upper bounds on secret key rates, including the squashed entanglement and the relative entropy of entanglement. Here we use this approach along with a “privacy test” to establish a general meta-converse bound for private communication, which has a number of applications. The meta-converse allows for proving that any quantum channel’s relative entropy of entanglement is a strong converse rate for private communication. For covariant channels, the meta-converse also leads to second-order expansions of relative entropy of entanglement bounds for private communication rates. For such channels, the bounds also apply to the private communication setting in which the sender and receiver are assisted by unlimited public classical communication, and as such, they are relevant for establishing various converse bounds for quantum key distribution protocols conducted over these channels. We find precise characterizations for several channels of interest and apply the methods to establish converse bounds on the private transmission capabilities of all phase-insensitive bosonic channels.

## 1 Introduction

Ever since the discovery of quantum key distribution [BB84], researchers have been interested in exploiting quantum-mechanical effects in order to ensure the secrecy of communication. This has led to a large amount of research in many directions [SBPC<sup>+</sup>09], both experimental and theoretical, and one of the recent challenges has been to connect both of these directions.

On the theoretical side, much progress has been made by generalizing several ideas developed in the context of classical information theory. For example, the wiretap channel is a simple model for private communication, and one can study its capacity for secure data transmission [Wyn75] (see [CK78, TB15, HTW14, Win16] for later progress on refining this capacity). In this model, two

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\*Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Center for Computation and Technology, Louisiana State University, Baton Rouge, Louisiana 70803, USA

†School of Physics, The University of Sydney, Sydney, Australia

‡Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, California 91125, USA

honest parties, usually called Alice (the sender) and Bob (the receiver), are connected by a classical channel. At the same time, there is a classical channel connecting Alice to an eavesdropper or wiretapper, usually called Eve. The goal is to devise a communication scheme such that Alice can communicate to Bob with small error in such a way that Eve gets nearly zero information about the message communicated (with both the probability of error and information leakage vanishing in the limit of many channel uses). One can further generalize the model to allow for public classical communication and study capacities in this context [Mau93, AC93]. However, two major drawbacks of the wiretap model is that the honest parties need to assume that they have fully characterized both 1) their channel and 2) the channel to the eavesdropper, which may not be possible in practice. Nevertheless, techniques developed in the context of the wiretap channel have been foundational to our understanding of information-theoretically secure communication.

Quantum mechanics offers a route around one of the aforementioned problems with the classical model, via the notion of purification. Indeed, for any quantum channel connecting Alice to Bob, there is a purification (or isometric extension) of this channel that is unique up to unitary rotations [Sti55]. All the degrees of freedom that are not accessible to the receiver Bob are accessible to the environment of the channel, and in the spirit of being cautious, as is usually the case in cryptography, we assume that the eavesdropper has full access to the environmental system. For example, communication from Alice to Bob in free space can be modeled by an interaction at a beamsplitter [Sha09], and in the wiretap model, we assume that all of the light that is lost along the way can be collected by the eavesdropper Eve [GSE08]. Thus, in the quantum wiretap model, Alice and Bob can perform parameter estimation in order to characterize their channel, and once they have a complete characterization, they also have a model for the channel to the eavesdropper, circumventing one of the aforementioned problems with the classical model. If we allow for Alice and Bob to make use of public classical communication in addition to the quantum channel (see, e.g., [TGW14b, TGW14a]), then this model is closely related to that which is used in some quantum key distribution protocols. In practice, one drawback of this model is that the channel from Alice to Bob might be changing with time or difficult to characterize, but nevertheless one can study the private capacities of this quantum wiretap channel model in an attempt to gain some understanding of what rates might be achievable in principle.

With this Shannon-theoretic viewpoint, the quantum wiretap model has been studied in much detail. The private capacity of a quantum wiretap channel was defined and characterized in [Dev05, CWY04]. For the class of degradable quantum channels, there is a tractable formula for the private capacity [Smi08]. The same occurs for conjugate degradable [BDHM10], less noisy, and more capable channels [Wat12]. Beyond such channels, little is known and recent evidence suggests that characterizing private capacity effectively could be a very difficult challenge. For example, the formula for private information from [Dev05, CWY04] is now known to be superadditive in general [SRS08, ES15], and the private capacity itself is as well [LWZG09].

More recently there has been progress on characterizing the private capacity when public classical communication is available (for a given channel  $\mathcal{N}$ , let  $P^{\leftrightarrow}(\mathcal{N})$  denote this quantity). Building on the notion of squashed entanglement [CW04] and the fact that this quantity is an upper bound on distillable key [Chr06], the authors of [TGW14b] defined the squashed entanglement of a channel and showed that it is an upper bound on  $P^{\leftrightarrow}(\mathcal{N})$  for any channel  $\mathcal{N}$  (see also [Wil16]). This result thus established a strong limitation for quantum key distribution protocols as discussed in [TGW14a]. Following this development, by building on the notion of relative entropy of entanglement [VP98] and the fact that this quantity is also an upper bound on the distillable key of a

bipartite state [HHHO05, HHHO09], the authors of [PLOB16] defined a channel's relative entropy of entanglement and stated that it is an upper bound on  $P^{\leftrightarrow}(\mathcal{N})$  for any channel  $\mathcal{N}$  that has a “teleportation symmetry” identified in [BDSW96, Section V] and extended in [NFC09, PLOB16]. It is an open question to determine whether the relative entropy of entanglement is an upper bound on the two-way assisted private capacity of a general quantum channel.

Both of the aforementioned upper bounds on  $P^{\leftrightarrow}(\mathcal{N})$  critically rely upon the notion of a private state [HHHO05, HHHO09]. To motivate this notion, consider that the ultimate goal of a  $P^{\leftrightarrow}$  protocol is to generate a secret-key state of the following form:

$$(\mathcal{M}_A \otimes \mathcal{M}_B)(\gamma_{ABE}) = \frac{1}{K} \sum_{i=0}^{K-1} |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E, \quad (1.1)$$

where the  $A$  system is possessed by Alice,  $B$  by Bob,  $E$  by the eavesdropper,  $K$  is the number of key values,  $\gamma_{ABE}$  is some state on systems  $ABE$ ,  $\mathcal{M}(\cdot) = \sum_i |i\rangle\langle i|(\cdot)|i\rangle\langle i|$  is a projective measurement channel with  $\{|i\rangle\}_i$  an orthonormal basis, and  $\sigma_E$  is some state on system  $E$ . The state in (1.1) is such that the systems  $A$  and  $B$  are perfectly correlated (i.e., maximally classically correlated), and the value of the key is uniformly random and independent of Eve's system  $E$ . The main observation of [HHHO05, HHHO09] is that, in principle, every step of a  $P^{\leftrightarrow}$  protocol can be purified, and since these steps are conducted in the laboratories of Alice and Bob, these parties could possess purifying systems of  $\gamma_{ABE}$  (call them  $A'$  and  $B'$ ), such that  $\gamma_{ABA'B'E}$  is a pure state satisfying  $\text{Tr}_{A'B'}\{\gamma_{ABA'B'E}\} = \gamma_{ABE}$ . By employing purification theorems of quantum information theory, the authors of [HHHO05, HHHO09] showed that the reduced state of  $\gamma_{ABA'B'E}$  on the systems  $ABA'B'$  has the following form:

$$\gamma_{ABA'B'} = U_{ABA'B'} (\Phi_{AB} \otimes \theta_{A'B'}) U_{ABA'B'}^\dagger, \quad (1.2)$$

where  $\Phi_{AB}$  is a maximally entangled state,  $U_{ABA'B'}$  is a special kind of unitary called a “twisting,” and  $\theta_{A'B'}$  is an arbitrary state (see Section 2 for more details). Such a state is now known as a bipartite private state and is fully equivalent to the state in (1.1) in the aforementioned sense. This observation thus allows for a perspective change which is helpful for analyzing private communication protocols: one can eliminate the eavesdropper from the analysis, revising the goal of such a protocol to be the production of states of the form in (1.2), and this allows for using the powerful tools of entanglement theory [HHHH09] to analyze secret-key rates.

Not only did the results of [HHHO05, HHHO09] provide a conceptually different method for understanding privacy in the quantum setup, but they also showed how there are fundamental differences between entanglement distillation and secret-key distillation protocols. Indeed, the strongest demonstration of this difference was the realization that there exist quantum channels that have zero capacity to send quantum information and yet can generate private information at a non-zero rate [HHH<sup>+</sup>08a, HHH<sup>+</sup>08b]. This in turn led to the discovery of the superactivation effect [SY08, SSY11]: two quantum channels each having zero quantum capacity can be used together to have a non-zero quantum capacity, by taking advantage of the intricate interplay between privacy and coherence.

In all of the above theoretical analyses, the statements made are asymptotic in nature, applying exclusively to the situation in which a large number of independent and identical channel uses are available. While these works have provided interesting bounds and are conceptually rich, they are somewhat removed from practical situations in which the number of channel uses is limited.

However, some recent works have aimed to bridge this gap for the case of quantum communication [BD10b, MW14, TWW17, BDL16, TBR16], giving more refined bounds on what is possible and impossible for a limited number of channel uses. One goal of the present paper is to bridge the gap for private communication.

Similar to the results from [TGW14a, PLOB16, GEW16, Wil16], the bounds given in this paper can be used to assess the performance of quantum key distribution protocols, as first suggested in [Lüt15]. In particular, one prominent experimental goal has been to build a quantum repeater [SST11, LST09], which is a device that could be inserted between two nodes in a given network to increase the rates of secret key generation. One way to assess the performance of such a repeater is that it should be able to exceed the limitations of the network that hold whenever the repeater is not present [Lüt15], and this has been hailed as one of the main applications of the bounds from [TGW14a, PLOB16, GEW16, Wil16]. However, since these bounds are asymptotic in nature, they have limited applicability to protocols using a channel a finite number of times. On the other hand, the bounds given in this paper can be used to assess the performance of practical, non-asymptotic protocols for certain channels.

**Summary of results.** In this paper, we establish several converse bounds on the private transmission capabilities of a quantum channel. The main conceptual development is a so-called “meta-converse” bound for private communication, which is a general upper bound that can be translated to several regimes of interest (the idea of a “meta-converse” has its roots in the seminal work in [PPV10]). In particular, we can use the meta-converse to establish that a channel’s relative entropy of entanglement is a strong converse rate for private communication, meaning that if the communication rate of a sequence of protocols exceeds this amount, then the probability of a protocol’s failure tends to one exponentially fast in the number of channel uses. The result builds strongly on the approach from [TWW17] (see also [MW14] for progress on a strong converse for private capacity of degradable channels).

We also use the meta-converse bound to establish second-order converse bounds for private communication. In this regime, one fixes the error parameter and asks what is the maximum rate of private communication possible. Here we again find an upper bound in terms of quantities related to the relative entropy of entanglement, but this bound applies only to channels with certain symmetry. For some channels with sufficient symmetry, we establish exact characterizations of the second-order coding rate (and even finer) by combining our upper bounds with the lower bounds from [TBR16]. Finally, we can use the method to establish finite blocklength converse bounds for all single-mode phase-insensitive bosonic channels, and as a consequence, we find that the weak-converse bounds presented in [PLOB16] are in fact strong-converse bounds for two-way assisted private communication. As a special case, we establish that the two-way assisted, unconstrained private and quantum capacities of the pure-loss and quantum-limited amplifier channels satisfy the strong converse property.

The rest of the paper gives details of our results. In the next section (Section 2), we recall many facts from quantum information theory which are needed for the developments, and we establish the notation used in the later parts. Section 2.4 reviews private states [HHHO05, HHHO09], and then Section 3 gives definitions of secret-key transmission protocols and their non-asymptotic achievable rates. Section 4 establishes the general meta-converse bound for any private communication protocol. In Section 5, we use the meta-converse and prior developments in [TWW17] to prove that a channel’s relative entropy of entanglement is a strong converse rate for private communica-

tion. If a channel is “teleportation simulable” (defined later), then the same quantity is a strong converse rate for private communication assisted by public classical communication. In Section 6, we use the meta-converse to establish second-order expansions of relative entropy of entanglement bounds on private communication rates (this is for channels with sufficient symmetry). Section 7 then gives several examples of channels for which we have precise characterizations of their private transmission capabilities, including the qubit dephasing channel, the qubit erasure channel, and any entanglement-breaking channel. Section 8 establishes finite blocklength converse bounds for phase-insensitive bosonic channels. We finally conclude in Section 9 with a summary and some open questions.

## 2 Preliminaries

### 2.1 Quantum information

Much of the background on quantum information theory reviewed here is available in [Wil15]. Let  $\mathcal{L}(\mathcal{H})$  denote the algebra of bounded linear operators acting on a Hilbert space  $\mathcal{H}$ . Let  $\mathcal{L}_+(\mathcal{H})$  denote the subset of positive semi-definite operators. We also write  $X \geq 0$  if  $X \in \mathcal{L}_+(\mathcal{H})$ . An operator  $\rho$  is in the set  $\mathcal{D}(\mathcal{H})$  of density operators (or states) if  $\rho \in \mathcal{L}_+(\mathcal{H})$  and  $\text{Tr}\{\rho\} = 1$ . An operator  $\rho$  is in the set  $\mathcal{D}_{\leq}(\mathcal{H})$  of subnormalized density operators if  $\rho \in \mathcal{L}_+(\mathcal{H})$  and  $\text{Tr}\{\rho\} \leq 1$ . The tensor product of two Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$  is denoted by  $\mathcal{H}_A \otimes \mathcal{H}_B$  or  $\mathcal{H}_{AB}$ . Given a multipartite density operator  $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , we unambiguously write  $\rho_A = \text{Tr}_B\{\rho_{AB}\}$  for the reduced density operator on system  $A$ . We use  $\rho_{AB}$ ,  $\sigma_{AB}$ ,  $\tau_{AB}$ ,  $\omega_{AB}$ , etc. to denote general density operators in  $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , while  $\psi_{AB}$ ,  $\varphi_{AB}$ ,  $\phi_{AB}$ , etc. denote rank-one density operators (pure states) in  $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$  (with it implicit, clear from the context, and the above convention implying that  $\psi_A$ ,  $\varphi_A$ ,  $\phi_A$  may be mixed if  $\psi_{AB}$ ,  $\varphi_{AB}$ ,  $\phi_{AB}$  are pure). A purification  $|\phi^\rho\rangle_{RA} \in \mathcal{H}_R \otimes \mathcal{H}_A$  of a state  $\rho_A \in \mathcal{D}(\mathcal{H}_A)$  is such that  $\rho_A = \text{Tr}_R\{|\phi^\rho\rangle\langle\phi^\rho|_{RA}\}$ . As is conventional, we often say that a unit vector  $|\psi\rangle$  is a pure state or a pure-state vector (while also saying that  $|\psi\rangle\langle\psi|$  is a pure state). An extension of a state  $\rho_A \in \mathcal{S}(\mathcal{H}_A)$  is some state  $\rho_{RA} \in \mathcal{S}(\mathcal{H}_R \otimes \mathcal{H}_A)$  such that  $\text{Tr}_R\{\rho_{RA}\} = \rho_A$ . Often, an identity operator is implicit if we do not write it explicitly (and should be clear from the context). We employ the shorthand  $\text{supp}(A)$  and  $\ker(A)$  to refer to the support and kernel of an operator  $A$ , respectively.

Let  $\{|i\rangle_A\}$  be an orthonormal basis (i.e., the standard basis) associated to a Hilbert space  $\mathcal{H}_A$ , and let  $\{|i\rangle_B\}$  be defined similarly for  $\mathcal{H}_B$ . If these spaces are finite-dimensional and their dimensions are equal ( $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = d$ ), then we define the maximally entangled state vector  $|\Phi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$  as

$$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle_B. \quad (2.1)$$

A state  $\sigma_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$  is separable if it can be written in the following form [Wer89]:

$$\sigma_{AB} = \sum_x p_X(x) |\phi^x\rangle\langle\phi^x|_A \otimes |\varphi^x\rangle\langle\varphi^x|_B, \quad (2.2)$$

where  $p_X$  is a probability distribution and  $\{|\phi^x\rangle_A\}$  and  $\{|\varphi^x\rangle_B\}$  are sets of pure-state vectors. Let  $\mathcal{S}(A:B)$  denote the set of separable states acting on  $\mathcal{H}_A \otimes \mathcal{H}_B$ . Note that

$$\mathcal{S}(A:B) = \text{conv}\{|\phi\rangle\langle\phi|_A \otimes |\varphi\rangle\langle\varphi|_B : |\phi\rangle_A \in \mathcal{H}_A, |\varphi\rangle_B \in \mathcal{H}_B, \|\phi\rangle_A\|_2 = \|\varphi\rangle_B\|_2 = 1\}, \quad (2.3)$$

where  $\text{conv}$  denotes the convex hull.

A linear map  $\mathcal{N}_{A \rightarrow B} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$  is positive if  $\mathcal{N}_{A \rightarrow B}(\sigma_A) \in \mathcal{L}_+(\mathcal{H}_B)$  whenever  $\sigma_A \in \mathcal{L}_+(\mathcal{H}_A)$ . Let  $\text{id}_A$  denote the identity map acting on a system  $A$ . A linear map  $\mathcal{N}_{A \rightarrow B}$  is completely positive if the map  $\text{id}_R \otimes \mathcal{N}_{A \rightarrow B}$  is positive for a reference system  $R$  of arbitrary size. A linear map  $\mathcal{N}_{A \rightarrow B}$  is trace-preserving if  $\text{Tr}\{\mathcal{N}_{A \rightarrow B}(\tau_A)\} = \text{Tr}\{\tau_A\}$  for all input operators  $\tau_A \in \mathcal{L}(\mathcal{H}_A)$ . It is trace non-increasing if  $\text{Tr}\{\mathcal{N}_{A \rightarrow B}(\tau_A)\} \leq \text{Tr}\{\tau_A\}$  for all  $\tau_A \in \mathcal{L}_+(\mathcal{H}_A)$ . A quantum channel is a linear map which is completely positive and trace-preserving (CPTP). Every quantum channel has a Kraus representation as  $\mathcal{N}_{A \rightarrow B}(\tau_A) = \sum_x E^x \tau_A(E^x)^\dagger$  where the Kraus operators  $\{E^x\}$  satisfy  $\sum_x (E^x)^\dagger E^x = I_A$ . A positive operator-valued measure (POVM) is a set  $\{\Lambda^m\}$  of positive semi-definite operators such that  $\sum_m \Lambda^m = I$ . An isometry  $U : \mathcal{H} \rightarrow \mathcal{H}'$  is a linear map such that  $U^\dagger U = I_{\mathcal{H}}$ . An isometric extension  $U_{A \rightarrow BE}^{\mathcal{N}}$  of a quantum channel  $\mathcal{N}_{A \rightarrow B}$  (i.e., its Stinespring dilation [Sti55]) is a linear map that satisfies the following:

$$\text{Tr}_E \left\{ U_{A \rightarrow BE}^{\mathcal{N}} \rho_A (U_{A \rightarrow BE}^{\mathcal{N}})^\dagger \right\} = \mathcal{N}_{A \rightarrow B}(\rho_A), \quad (2.4)$$

$$U_{\mathcal{N}}^\dagger U_{\mathcal{N}} = I_A, \quad (2.5)$$

$$U_{\mathcal{N}} U_{\mathcal{N}}^\dagger = \Pi_{BE}, \quad (2.6)$$

for all states  $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ , where  $\Pi_{BE}$  is a projection onto a subspace of the Hilbert space  $\mathcal{H}_B \otimes \mathcal{H}_E$ . We define an isometric channel  $\mathcal{U}_{A \rightarrow BE}^{\mathcal{N}}$  from the isometry  $U_{A \rightarrow BE}^{\mathcal{N}}$  as

$$\mathcal{U}_{A \rightarrow BE}^{\mathcal{N}}(X_A) = U_{A \rightarrow BE}^{\mathcal{N}} X_A (U_{A \rightarrow BE}^{\mathcal{N}})^\dagger, \quad (2.7)$$

where  $X_A \in \mathcal{L}(\mathcal{H}_A)$ .

The trace distance between two quantum states  $\rho, \sigma \in \mathcal{D}(\mathcal{H})$  is equal to  $\|\rho - \sigma\|_1$ , where  $\|C\|_1 \equiv \text{Tr}\{\sqrt{C^\dagger C}\}$  for any operator  $C$ . It has a direct operational interpretation in terms of the distinguishability of these states. That is, if  $\rho$  or  $\sigma$  are prepared with equal probability and the task is to distinguish them via some quantum measurement, then the optimal success probability in doing so is equal to  $(1 + \|\rho - \sigma\|_1 / 2) / 2$ . The fidelity is defined as  $F(\rho, \sigma) \equiv \|\sqrt{\rho} \sqrt{\sigma}\|_1^2$  [Uhl76], and more generally we can use the same formula to define  $F(P, Q)$  if  $P, Q \in \mathcal{L}_+(\mathcal{H})$ . Uhlmann's theorem states that [Uhl76]

$$F(\rho_A, \sigma_A) = \max_U |\langle \phi^\sigma |_{RA} U_R \otimes I_A | \phi^\rho \rangle_{RA}|^2, \quad (2.8)$$

where  $|\phi^\rho\rangle_{RA}$  and  $|\phi^\sigma\rangle_{RA}$  are fixed purifications of  $\rho_A$  and  $\sigma_A$ , respectively, and the optimization is with respect to all unitaries  $U_R$ . The same statement holds more generally for  $P, Q \in \mathcal{L}_+(\mathcal{H})$ . The fidelity is invariant with respect to isometries and monotone non-decreasing with respect to channels. The sine distance or  $C$ -distance between two quantum states  $\rho, \sigma \in \mathcal{D}(\mathcal{H})$  was defined as

$$C(\rho, \sigma) \equiv \sqrt{1 - F(\rho, \sigma)} \quad (2.9)$$

and proven to be a metric in [Ras02, Ras03, GLN05, Ras06]. It was later [TCR09] (under the name “purified distance”) shown to be a metric on subnormalized states  $\rho, \sigma \in \mathcal{D}_{\leq}(\mathcal{H})$  via the embedding

$$P(\rho, \sigma) \equiv C(\rho \oplus [1 - \text{Tr}\{\rho\}], \sigma \oplus [1 - \text{Tr}\{\sigma\}]). \quad (2.10)$$

## 2.2 Quantum channels with symmetries

Let  $G$  be a finite group, and for every  $g \in G$ , let  $g \rightarrow U_A(g)$  and  $g \rightarrow V_B(g)$  be unitary representations acting on the input and output spaces of a quantum channel  $\mathcal{N}_{A \rightarrow B}$ , respectively. Then a quantum channel  $\mathcal{N}_{A \rightarrow B}$  is covariant with respect to these representations if the following relation holds for all input density operators  $\rho_A \in \mathcal{D}(A)$  and group elements  $g \in G$  [Hol02]:

$$\mathcal{N}_{A \rightarrow B} \left( U_A(g) \rho_A U_A^\dagger(g) \right) = V_B(g) \mathcal{N}_{A \rightarrow B}(\rho_A) V_B^\dagger(g). \quad (2.11)$$

**Definition 1 (Covariant channel)** A quantum channel is covariant if it is covariant with respect to a group which has a representation  $U(g)$  on  $\mathcal{H}_A$  that is a unitary one-design, the latter meaning that the channel  $\frac{1}{|G|} \sum_{g \in G} U(g)(\cdot)U(g)^\dagger$  always outputs the maximally mixed state.

The teleportation protocol is a basic primitive in quantum information [BBC<sup>+</sup>93]. We say that a channel is “teleportation-simulable” with associated state  $\omega_{AB}$  if it can be realized by the action of the teleportation protocol on one share of a bipartite state  $\omega_{AB}$  [BDSW96, Section V]. That is, a channel  $\mathcal{N}_{A' \rightarrow B}$  is teleportation-simulable with associated state  $\omega_{AB}$  if there exists a state  $\omega_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ , with  $\mathcal{H}_A \simeq \mathcal{H}_{A'}$ , such that for all  $\rho_{A'} \in \mathcal{D}(\mathcal{H}_{A'})$

$$\mathcal{N}_{A' \rightarrow B}(\rho_{A'}) = \mathcal{T}_{AA'B \rightarrow B}(\rho_{A'} \otimes \omega_{AB}), \quad (2.12)$$

where  $\mathcal{T}_{AA'B \rightarrow B}$  is a channel corresponding to a general teleportation protocol [Wer01] (note here that the correction operations might need to be adapted for the output space of the channel  $\mathcal{N}_{A' \rightarrow B}$ , which could be different from the input space). The advantage of channels possessing this symmetry is that any protocol involving  $\mathcal{N}_{A' \rightarrow B}$  can be replaced by one which involves Alice and Bob sharing  $\omega_{AB}$  and performing quantum teleportation to simulate  $\mathcal{N}_{A' \rightarrow B}$ .

The following proposition establishes that every covariant channel is teleportation-simulable. Its proof is given in Appendix A and extends earlier developments in [Wer01] and [LM15, Eqs. (53)–(56)].

**Proposition 2** If a quantum channel  $\mathcal{N}_{A \rightarrow B}$  is covariant (as given in Definition 1), then it is teleportation simulable with associated state  $\omega_{AB} \equiv \mathcal{N}_{A' \rightarrow B}(\Phi_{AA'})$ .

## 2.3 Local operations and (public) classical communication

A quantum instrument is a quantum channel that accepts a quantum system as input and outputs two systems: a classical one and a quantum one [DL70, Dav76, Oza84]. More formally, a quantum instrument is a collection  $\{\mathcal{N}^x\}$  of completely positive trace non-increasing maps, such that the sum map  $\sum_x \mathcal{N}^x$  is a quantum channel. We can write the action of a quantum instrument on an input density operator  $\rho \in \mathcal{D}(\mathcal{H})$  as the following quantum channel:

$$\rho \rightarrow \sum_x \mathcal{N}^x(\rho) \otimes |x\rangle\langle x|, \quad (2.13)$$

where  $\{|x\rangle\}_x$  is an orthonormal basis labeling the classical output of the instrument.

It is common in quantum communication theory to consider the framework of local operations and classical communication (LOCC) [BDSW96, CLM<sup>+</sup>14], which consists of particular interactions between two parties usually called Alice and Bob. A round of LOCC (or LOCC channel) consists of a finite number of compositions of the following:

1. Alice performs a quantum instrument, which has both a quantum and classical output. She forwards the classical output to Bob, who then performs a quantum channel conditioned on the classical data received. This sequence of actions corresponds to the following channel:

$$\sum_x \mathcal{E}_A^x \otimes \mathcal{F}_B^x, \quad (2.14)$$

where  $\{\mathcal{E}_A^x\}$  is a collection of completely positive maps such that  $\sum_x \mathcal{E}_A^x$  is a quantum channel and  $\{\mathcal{F}_B^x\}$  is a collection of quantum channels.

2. The situation is reversed, with Bob performing the initial instrument, who forwards the classical data to Alice, who then performs a quantum channel conditioned on the classical data. This sequence of actions corresponds to a channel of the form in (2.14), with the  $A$  and  $B$  labels switched.

The framework of local operations and public communication (LOPC) is essentially the same as LOCC, except that the terminology implies that there is a third party Eve (an eavesdropper) who receives a copy of all of the classical data exchanged between Alice and Bob.

A channel is separable [Rai08] if it has Kraus operators of the form  $C_A^x \otimes D_B^x$ , where  $\sum_x (C_A^x)^\dagger C_A^x \otimes (D_B^x)^\dagger D_B^x = I_{AB}$ . Every LOCC channel is separable, but the opposite is not always true [BDF<sup>+</sup>99]. A channel is separability preserving [HN03, BP10] if it preserves the set of separable states. The swap operator is an example of a separability-preserving channel that is not a separable channel.

## 2.4 Private states

Private states are an essential notion for our analysis [HHHO05, HHHO09], and we review their basics here.

**Definition 3** A tripartite key state  $\gamma_{ABE} \in \mathcal{D}(\mathcal{H}_{ABE})$  contains  $\log K$  bits of secret key if there exists a state  $\sigma_E \in \mathcal{D}(\mathcal{H}_E)$  and a projective measurement channel  $\mathcal{M}(\cdot) = \sum_i |i\rangle\langle i|(\cdot)|i\rangle\langle i|$ , where  $\{|i\rangle\}_i$  is an orthonormal basis, such that

$$(\mathcal{M}_A \otimes \mathcal{M}_B)(\gamma_{ABE}) = \frac{1}{K} \sum_{i=0}^{K-1} |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E. \quad (2.15)$$

That is, we see that the systems  $A$  and  $B$  are maximally classically correlated, and the key value is uniformly random and independent of the  $E$  system. Physically, we can think of the  $A$  system as being in Alice's laboratory,  $B$  in Bob's, and  $E$  in Eve's. We also think of Alice and Bob as two honest parties and Eve as a malicious eavesdropper whose system should ideally be independent of the key systems possessed by Alice and Bob.

Purifying such a state  $\gamma_{ABE}$  with two systems  $A'$  and  $B'$ , thinking of  $A'$  as being available to Alice and  $B'$  as being available to Bob (or alternatively simply as not being available to Eve), and tracing out the  $E$  system then leads to the notion of a bipartite private state  $\gamma_{ABA'B'}$  [HHHO05, HHHO09]. As shown in [HHHO05, HHHO09], any such state  $\gamma_{ABA'B'} \in \mathcal{D}(\mathcal{H}_{ABA'B'})$  takes a canonical form:

**Definition 4** A bipartite private state  $\gamma_{A''B''} \in \mathcal{D}(\mathcal{H}_{A''B''})$  contains  $\log K$  bits of secret key if  $\mathcal{H}_{A''} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$  and  $\mathcal{H}_{B''} = \mathcal{H}_B \otimes \mathcal{H}_{B'}$  such that  $\gamma_{ABA'B'} \in \mathcal{D}(\mathcal{H}_{ABA'B'})$  has the following form:

$$\gamma_{ABA'B'} = U_{ABA'B'} (\Phi_{AB} \otimes \theta_{A'B'}) U_{ABA'B'}^\dagger, \quad (2.16)$$

where  $\Phi_{AB}$  is a maximally entangled state of Schmidt rank  $K$ ,  $U_{ABA'B'}$  is a “twisting” unitary of the form

$$U_{ABA'B'} = \sum_{i,j=0}^{K-1} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U_{A'B'}^{ij}, \quad (2.17)$$

with each  $U_{A'B'}^{ij}$  a unitary, and  $\theta_{A'B'} \in \mathcal{D}(\mathcal{H}_{A'B'})$ .

The systems  $A'$  and  $B'$  are called the “shield” systems because they, along with the twisting unitary, can help to protect the key in systems  $A$  and  $B$  from any party possessing a purification of  $\gamma_{ABA'B'}$ . Such bipartite private states are in one-to-one correspondence with the tripartite key states given in (2.15) [HHHO05, HHHO09]. That is, for every state  $\gamma_{ABE}$  of the form in (2.15), we can find a state of the form in (2.16) and vice versa. We summarize this as the following proposition:

**Proposition 5 ([HHHO05, HHHO09])** Bipartite private states and tripartite key states are equivalent. That is, for  $\gamma_{ABA'B'}$  a bipartite private state,  $\gamma_{ABE}$  is a tripartite key state for any purification  $\gamma_{ABA'B'E}$  of  $\gamma_{ABA'B'}$ . Conversely, for any tripartite key state  $\gamma_{ABE}$  and any purification  $\gamma_{ABA'B'E}$  of it,  $\gamma_{ABA'B'}$  is a bipartite private state.

This correspondence takes on a more physical form (reviewed in Section 3), which is that any tripartite protocol whose aim it is to extract tripartite key states of the form in (2.15) is in one-to-one correspondence with a bipartite protocol whose aim it is to extract bipartite private states of the form in (2.16) [HHHO05, HHHO09].

**Definition 6** A state  $\rho_{ABE} \in \mathcal{D}(\mathcal{H}_{ABE})$  is an  $\varepsilon$ -approximate tripartite key state if there exists a tripartite key state  $\gamma_{ABE}$  of the form in (2.15) such that

$$F(\rho_{ABE}, \gamma_{ABE}) \geq 1 - \varepsilon, \quad (2.18)$$

where  $\varepsilon \in [0, 1]$ . Similarly, a state  $\rho_{ABA'B'} \in \mathcal{D}(\mathcal{H}_{ABA'B'})$  is an  $\varepsilon$ -approximate bipartite private state if there exists a bipartite private state  $\gamma_{ABA'B'} \in \mathcal{D}(\mathcal{H}_{ABA'B'})$  of the form in (2.15) such that

$$F(\rho_{ABA'B'}, \gamma_{ABA'B'}) \geq 1 - \varepsilon. \quad (2.19)$$

Approximate tripartite key states are in one-to-one correspondence with approximate bipartite private states [HHHO09, Theorem 5], as summarized below:

**Proposition 7 ([HHHO05, HHHO09])** If  $\rho_{ABA'B'}$  is an  $\varepsilon$ -approximate bipartite key state with  $K$  key values, then Alice and Bob hold an  $\varepsilon$ -approximate tripartite key state with  $K$  key values. The converse statement is true as well.

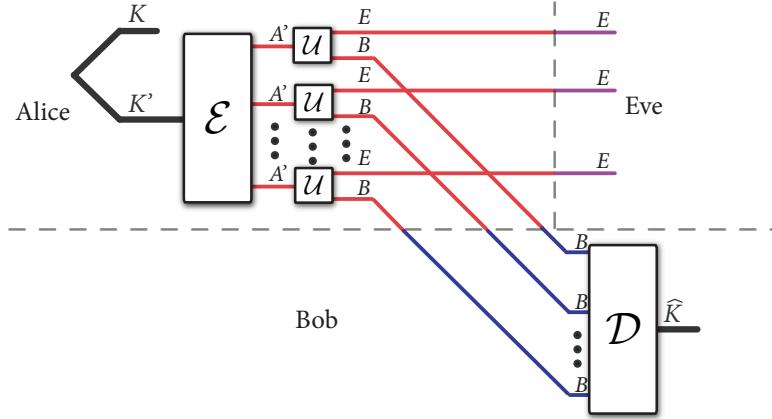


Figure 1: A protocol for secret-key transmission over a quantum channel.

### 3 Secret key transmission and generation over quantum channels

In this section, we define secret-key transmission and generation codes and corresponding measures of their performance. We also review the identification from [HHHO05, HHHO09], which shows how a tripartite key distillation protocol is in one-to-one correspondence with a bipartite private state distillation protocol.

#### 3.1 Secret-key transmission codes

Given is a quantum channel  $\mathcal{N}_{A' \rightarrow B}$ . Let  $\mathcal{N}_{A' \rightarrow B}^{\otimes n}$  denote the tensor-product channel,  $U_{A' \rightarrow BE}^{\mathcal{N}}$  an isometric extension of  $\mathcal{N}_{A' \rightarrow B}$ , and  $\mathcal{U}_{A' \rightarrow BE}^{\mathcal{N}}$  the associated isometric channel. A secret-key transmission protocol for  $n$  channel uses consists of a triple  $\{|K|, \mathcal{E}, \mathcal{D}\}$ , where  $|K|$  is the size of the secret key to be generated,  $\mathcal{E}_{K' \rightarrow A'^n}$  is the encoder (a CPTP map), and  $\mathcal{D}_{B^n \rightarrow \hat{K}}$  is the decoder (another CPTP map). The protocol begins with a third party preparing a maximally classically correlated state  $\bar{\Phi}_{KK'}$  of the following form:

$$\bar{\Phi}_{KK'} \equiv \frac{1}{|K|} \sum_{i=0}^{|K|-1} |i\rangle\langle i|_K \otimes |i\rangle\langle i|_{K'}, \quad (3.1)$$

and then sending the  $K'$  system to Alice. Alice then inputs the  $K'$  system to an encoder  $\mathcal{E}_{K' \rightarrow A'^n}$ , transmits the  $A'^n$  systems through the tensor-power channel  $(\mathcal{U}_{A' \rightarrow BE}^{\mathcal{N}})^{\otimes n}$ , and the receiver Bob applies the decoder  $\mathcal{D}_{B^n \rightarrow \hat{K}}$  to the systems  $B^n$ . The state at the end of the protocol is as follows:

$$\rho_{K\hat{K}E^n} \equiv (\mathcal{D}_{B^n \rightarrow \hat{K}} \circ (\mathcal{U}_{A' \rightarrow BE}^{\mathcal{N}})^{\otimes n} \circ \mathcal{E}_{K' \rightarrow A'^n})(\bar{\Phi}_{KK'}). \quad (3.2)$$

Figure 1 depicts such a protocol.

A triple  $(n, P, \varepsilon)$  consists of the number  $n$  of channel uses, the rate  $P$  of secret-key transmission, and the error  $\varepsilon \in [0, 1]$ . Such a triple is achievable on  $\mathcal{N}_{A' \rightarrow B}$  if there exists a secret-key transmission protocol  $\{|K|, \mathcal{E}, \mathcal{D}\}$  and some state  $\omega_{E^n} \in \mathcal{D}(\mathcal{H}_{E^n})$  such that  $\frac{1}{n} \log |K| \geq P$  and

$$F(\bar{\Phi}_{K\hat{K}} \otimes \omega_{E^n}, \rho_{K\hat{K}E^n}) \geq 1 - \varepsilon. \quad (3.3)$$

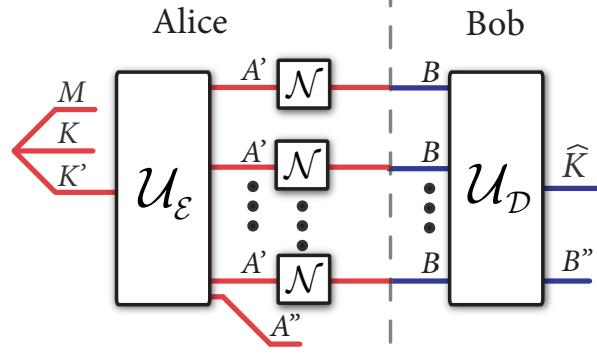


Figure 2: A purified version of a secret-key transmission protocol.

Thus, the goal of such a secret-key transmission protocol is to realize an  $\varepsilon$ -approximate tripartite secret-key state as defined in (2.18).

Note that the above definition of secret-key transmission combines the error probability and the security parameter into a single parameter  $\varepsilon$ , in contrast to the definitions from [Dev05, CWY04, MW14, HTW14, Win16]. Doing so is consistent with the definition of private capacity or distillable key from [HHHO05, HHHO09, Chr06] and turns out to be beneficial for the developments in this paper. Furthermore, we argue in Appendix B how a converse bound according to the above definition of privacy gives a converse bound according to quantum generalizations of the privacy definition from [HTW14].

As mentioned before Definition 6, it is possible to purify a secret-key transmission protocol [HHHO05, HHHO09], such that every step is performed coherently and the ultimate goal is to realize a private bipartite state  $\gamma_{K_A K_B S_A S_B}$ , where we now denote the key systems by  $K$  and the shield systems by  $S$ . In the class of protocols discussed above, this consists of replacing each step with the following:

1. A third party preparing a purification of the state  $\bar{\Phi}_{KK'}$ , which is a ‘‘GHZ state’’ that we denote by  $|\Phi^{\text{GHZ}}\rangle_{KK'M} \equiv |K|^{-1/2} \sum_i |i\rangle_K \otimes |i\rangle_{K'} \otimes |i\rangle_M$ , and giving the  $K'$  system to Alice,
2. Alice performing an isometric extension of the encoder  $\mathcal{E}_{K' \rightarrow A'^n}$ , denoted by  $\mathcal{U}_{K' \rightarrow A'^n A''}^{\mathcal{E}}$ ,
3. Bob performing an isometric extension of the decoder  $\mathcal{D}_{B^n \rightarrow \hat{K}}$ , denoted by  $\mathcal{U}_{B^n \rightarrow \hat{K} B''}^{\mathcal{D}}$ .

Figure 2 depicts such a purified version of a secret-key transmission protocol. By employing [HHHO09, Theorem 5], we find that (3.3) implies that

$$F(\gamma_{K_A K_B S_A S_B}, \rho_{K \hat{K} M A'' B''}) \geq 1 - \varepsilon, \quad (3.4)$$

for some private state  $\gamma_{K_A K_B S_A S_B}$ , where we make the identifications  $K_A \equiv K$ ,  $K_B \equiv \hat{K}$ ,  $S_A \equiv M A''$ , and  $S_B \equiv B''$ , and

$$\rho_{K \hat{K} M A'' B'' E^n} \equiv (\mathcal{U}_{B^n \rightarrow \hat{K} B''}^{\mathcal{D}} \circ (\mathcal{U}_{A' \rightarrow BE}^N)^{\otimes n} \circ \mathcal{U}_{K' \rightarrow A'^n A''}^{\mathcal{E}})(\Phi_{KK'M}^{\text{GHZ}}). \quad (3.5)$$

### 3.2 Secret-key generation codes

A secret-key generation protocol is defined similarly as above, with some key differences however. The main difference is that the objective is secret key *generation*. As such, a secret-key generation protocol for  $n$  channel uses consists of a triple  $\{|K|, \varrho_{KA'^n}, \mathcal{D}\}$ , where  $|K|$  is the size of the secret key to be generated,  $\varrho_{KA'^n}$  is the initial state, and  $\mathcal{D}_{B^n \rightarrow \hat{K}}$  is the decoder. Such a protocol begins with Alice preparing the state  $\varrho_{KA'^n}$ , sending the  $A'^n$  systems through the tensor-power channel  $(\mathcal{U}_{A' \rightarrow BE}^N)^{\otimes n}$ , and the receiver Bob applies the decoder  $\mathcal{D}_{B^n \rightarrow \hat{K}}$  to the systems  $B^n$ . The state at the end of the protocol is as follows:

$$\varrho_{K\hat{K}E^n} \equiv (\mathcal{D}_{B^n \rightarrow \hat{K}} \circ (\mathcal{U}_{A' \rightarrow BE}^N)^{\otimes n})(\varrho_{KA'^n}). \quad (3.6)$$

A triple  $(n, P, \varepsilon)$  for secret-key generation consists of the number  $n$  of channel uses, the rate  $P$  of secret-key generation, and the error  $\varepsilon \in [0, 1]$ . Such a triple is achievable on  $\mathcal{N}_{A' \rightarrow B}$  for secret-key generation if there exists a secret-key generation protocol  $\{|K|, \varrho_{KA'^n}, \mathcal{D}\}$  and some state  $\omega_{E^n} \in \mathcal{D}(\mathcal{H}_{E^n})$  such that  $\frac{1}{n} \log |K| \geq P$  and

$$F(\bar{\Phi}_{K\hat{K}} \otimes \omega_{E^n}, \varrho_{K\hat{K}E^n}) \geq 1 - \varepsilon. \quad (3.7)$$

Thus, the goal of such a secret-key generation protocol is to generate an  $\varepsilon$ -approximate tripartite secret-key state as defined in (2.18).

Any secret-key generation protocol can be purified as discussed in the previous section, such that the goal is to generate an  $\varepsilon$ -approximate bipartite private state.

Finally, note that the encoder and decoder for an  $(n, P, \varepsilon)$  secret-key transmission protocol can be used directly to realize an  $(n, P, \varepsilon)$  secret-key generation protocol, simply by setting the initial state  $\varrho_{KA'^n}$  for the secret-key generation protocol equal to  $\mathcal{E}_{K' \rightarrow A'^n}(\bar{\Phi}_{KK'})$ . The converse realization is not possible without the assistance of another resource.

### 3.3 Non-asymptotic achievable regions

**Unassisted protocols.** The non-asymptotic private achievable region of a quantum channel is the union of all triples  $(n, P, \varepsilon)$  for secret-key transmission, and we are interested in understanding two different boundaries of this region, defined as

$$\hat{P}_{\mathcal{N}}(n, \varepsilon) \equiv \max \{P : (n, P, \varepsilon) \text{ is achievable for } \mathcal{N}\}, \quad (3.8)$$

$$\hat{\varepsilon}_{\mathcal{N}}(n, P) \equiv \min \{\varepsilon : (n, P, \varepsilon) \text{ is achievable for } \mathcal{N}\}. \quad (3.9)$$

In this paper, we investigate both of these boundaries. The first boundary  $\hat{P}_{\mathcal{N}}(n, \varepsilon)$  identifies how the rate can change as a function of  $n$  for fixed error  $\varepsilon$ , and second-order coding rates can characterize this boundary for sufficiently large  $n$ . The second boundary  $\hat{\varepsilon}_{\mathcal{N}}(n, P)$  identifies how the error can change as a function of  $n$  for fixed rate  $P$ , and error exponents and strong converse exponents characterize this boundary (in this paper we focus exclusively on bounds on strong converse exponents).

**LOPC/LOCC-assisted protocols.** We can extend all of the above definitions to the case in which Alice and Bob employ classical communication to aid in their goal of establishing a secret key. We call such a protocol a secret-key-agreement protocol. The most general such protocol in the tripartite picture consists of rounds of local operations and public communication (LOPC)

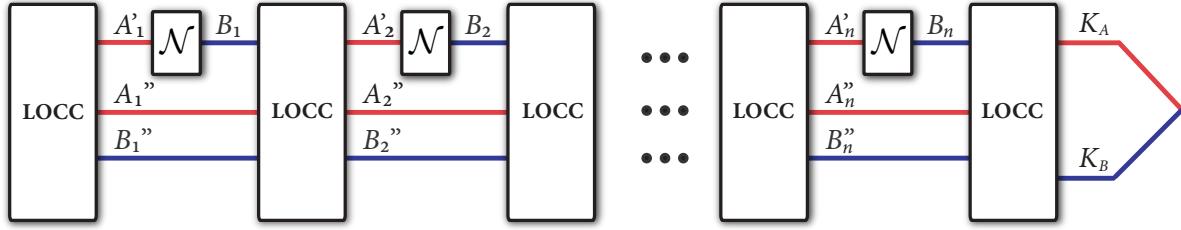


Figure 3: An LOCC-assisted secret-key agreement protocol. For  $i \in \{1, \dots, n\}$ , the  $A''_i$  and  $B''_i$  systems are scratch registers of arbitrary size that Alice and Bob can use in between every channel use. The output of the protocol are systems  $K_A$  and  $K_B$  which contain the key realized by the protocol.

interleaved between every channel use. By purifying every operation and the classical data communicated in such a protocol, we can describe such a protocol in the bipartite picture, which consists of local operations and classical communication (LOCC) [HHHO05, HHHO09]. Figure 3 depicts such an LOCC-assisted protocol. The output of such a protocol is compared via the fidelity with a private state, and by [HHHO09, Theorem 5], it meets the same fidelity requirement as the original tripartite formulation. We define a triple  $(n, P, \varepsilon)$  to be achievable if there exists a secret-key agreement protocol of the above form that generates an  $\varepsilon$ -approximate tripartite secret key state in the tripartite picture or, equivalently, an  $\varepsilon$ -approximate bipartite private state in the bipartite picture. We define the achievable rate region as before as the union of all achievable rate triples, and we are interested in the boundaries, defined as

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon) \equiv \max \{P : (n, P, \varepsilon) \text{ is achievable for } \mathcal{N} \text{ using } \leftrightarrow\}, \quad (3.10)$$

$$\hat{\varepsilon}_{\mathcal{N}}^{\leftrightarrow}(n, P) \equiv \min \{\varepsilon : (n, P, \varepsilon) \text{ is achievable for } \mathcal{N} \text{ using } \leftrightarrow\}, \quad (3.11)$$

where  $\leftrightarrow$  indicates that the protocol is LOCC-assisted.

There is no difference in the performance of secret-key transmission and secret-key generation protocols whenever classical communication is available for free. To see this, consider that any  $(n, P, \varepsilon)$  LOCC-assisted secret-key transmission protocol realizes an  $(n, P, \varepsilon)$  LOCC-assisted secret-key generation protocol, for reasons similar to those that we discussed previously. When classical communication is available for free, any  $(n, P, \varepsilon)$  LOCC-assisted secret-key generation protocol is an instance of an  $(n, P, \varepsilon)$  LOCC-assisted secret-key transmission protocol. This follows by an application of the well known one-time pad protocol. That is, suppose that the secret-key generation protocol produces an  $\varepsilon$ -approximate key state of size  $|K|$  shared between Alice and Bob. Then if the third party had given the system  $K'$  of  $\bar{\Phi}_{KK'}$  to Alice, she and Bob could employ a one-time pad protocol, using their generated  $\varepsilon$ -approximate key as a resource, in order for Alice to transmit the  $K'$  system. The effect is to realize an  $\varepsilon$ -approximate key state shared between the third party and Bob (due to the monotonicity of fidelity with respect to any quantum channel; in this case, the one-time pad protocol is a particular quantum channel). The resulting protocol is then an  $(n, P, \varepsilon)$  LOCC-assisted secret-key transmission protocol.

**CPPP-assisted protocols.** Similarly, we can define a class of protocols that consist of a single

round of LOPC/LOCC, using the channel  $n$  times, and a final round of LOPC/LOCC. We call these classical pre- and post-processing (CPPP) protocols and define the corresponding boundaries as

$$\hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \equiv \max \{P : (n, P, \varepsilon) \text{ is achievable for } \mathcal{N} \text{ using cппp}\}, \quad (3.12)$$

$$\hat{\varepsilon}_{\mathcal{N}}^{\text{CPPP}}(n, P) \equiv \min \{\varepsilon : (n, P, \varepsilon) \text{ is achievable for } \mathcal{N} \text{ using cппp}\}. \quad (3.13)$$

For similar reasons as discussed directly above, there is no difference in the performance of secret-key transmission and secret-key generation protocols that are CPPP-assisted.

From the definitions, we see that the following inequalities hold:

$$\hat{P}_{\mathcal{N}}(n, \varepsilon) \leq \hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \leq \hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon), \quad (3.14)$$

$$\hat{\varepsilon}_{\mathcal{N}}(n, P) \geq \hat{\varepsilon}_{\mathcal{N}}^{\text{CPPP}}(n, P) \geq \hat{\varepsilon}_{\mathcal{N}}^{\leftrightarrow}(n, P), \quad (3.15)$$

because adding an extra resource can only help to increase the rate or reduce the error.

The structure of protocols involving adaptive LOCC can simplify immensely for channels that are teleportation-simulable. This was realized in [BDSW96, Section V] for discrete-variable channels and extended in [NFC09, PLOB16] to continuous-variable bosonic channels (a general review of the idea is available in [PLOB16]). Since secret-key agreement protocols in the tripartite picture can be recast as LOCC protocols in the bipartite picture, this reduction by teleportation applies to them as well. For such channels, a secret-key agreement protocol can be simulated by one in which every channel use is replaced by Alice and Bob sharing the state  $\omega_{AB}$  from (2.12) and then performing the teleportation protocol. This allows for Alice to take “full control” of the channel, delaying or advancing its use at will. What this translates to for an adaptive secret-key agreement protocol is that all of the adaptive rounds of LOCC can be delayed until the very end of the protocol, such that the resulting protocol is a special kind of CPPP-assisted protocol. Thus, for a teleportation-simulable channel  $\mathcal{N}_{A' \rightarrow B}^{\text{TP}}$  with associated state  $\omega_{AB}$ , any secret-key agreement protocol can be simulated by one which leads to a state of the following form:

$$\Lambda_{A^n B^n \rightarrow K_A K_B}(\omega_{AB}^{\otimes n}), \quad (3.16)$$

where  $\Lambda_{A^n B^n \rightarrow K_A K_B}$  is an LOCC channel with  $K_A$  and  $K_B$  the key systems generated for Alice and Bob, respectively. For several teleportation-simulable channels of interest, it suffices to take  $\omega_{AB} = \mathcal{N}_{A' \rightarrow B}^{\text{TP}}(\Phi_{AA'})$ , in which Alice prepares a maximally entangled state and sends one share of it through the channel. These observations were applied in [BDSW96, Section V] to the case of quantum communication protocols and in [PLOB16] to private communication after definitions of private capacity were settled in [Dev05, CWY04].

### 3.4 Relation to entanglement transmission

Entanglement transmission achieves the task of secret key transmission. That is, an entanglement transmission code of rate  $R$  and fidelity  $\geq 1 - \varepsilon$  can serve as a secret-key transmission code with the same rate and fidelity. We formalize this relation now by defining entanglement transmission codes, which are strongly related to quantum communication codes [BKN00]. An entanglement transmission code is a triple  $\{|M|\, \mathcal{E}, \mathcal{D}\}$ , which consists of an encoding  $\mathcal{E}_{M' \rightarrow A'^n}$  and a decoding  $\mathcal{D}_{B^n \rightarrow \hat{M}}$ . We say that a triple  $(n, R, \varepsilon)$  is achievable if  $\frac{1}{n} \log |M| \geq R$  and

$$F(\Phi_{M\hat{M}}, (\mathcal{D}_{B^n \rightarrow \hat{M}} \circ \mathcal{N}_{A' \rightarrow B}^{\otimes n} \circ \mathcal{E}_{M' \rightarrow A'^n})(\Phi_{MM'})) \geq 1 - \varepsilon. \quad (3.17)$$

Since  $\Phi_{M\hat{M}}$  is a particular kind of private state with trivial twisting unitary and trivial shield systems (called ‘‘abelian twisting’’ in [HHHO09, Section VI-A]), such a code is achievable for secret-key transmission in the bipartite picture of privacy. Due to the relation between the bipartite picture of privacy and the tripartite picture [HHHO09, Theorem 5], the triple  $(n, R, \varepsilon)$  is achievable in the tripartite picture as well. As a consequence, we can always give lower bounds on secret-key transmission rates in terms of entanglement transmission rates.

We can extend all of the various definitions given previously for secret-key transmission to the case of entanglement transmission (see [TBR16] for details), and this gives us all of the quantities from the previous section, with  $P$  replaced by  $Q$ . The relation of the non-asymptotic  $Q$  quantities to privacy is summarized as follows:

$$\hat{Q}_{\mathcal{N}}(n, \varepsilon) \leq \hat{P}_{\mathcal{N}}(n, \varepsilon), \quad Q_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \leq \hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon), \quad \hat{Q}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon) \leq \hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon). \quad (3.18)$$

## 4 General (meta-converse) bounds

### 4.1 Information measures for the general (meta-converse) bounds

The general meta-converse bound in Section 4.3 is given in terms of the following quantity, defined for  $\rho \in \mathcal{D}(\mathcal{H})$ ,  $\sigma \in \mathcal{L}_+(\mathcal{H})$ , and  $\varepsilon \in [0, 1]$  as

$$D_H^\varepsilon(\rho\|\sigma) \equiv -\log [\inf\{\text{Tr}\{\Lambda\sigma\} : 0 \leq \Lambda \leq I \wedge \text{Tr}\{\Lambda\rho\} \geq 1 - \varepsilon\}]. \quad (4.1)$$

If  $\sigma$  is a quantum state,  $D_H^\varepsilon(\rho\|\sigma)$  has an interpretation as the optimal exponent of the Type II error in a hypothesis test to distinguish  $\rho$  from  $\sigma$ , given the constraint that the Type I error should not exceed  $\varepsilon$  [HP91]. This quantity was used effectively in one-shot quantum information theory [BD10b, BD11, WR12] and given the name ‘‘hypothesis testing relative entropy’’ in [WR12]. The hypothesis testing relative entropy is monotone non-increasing with respect to quantum channels as well (see, e.g., [WR12]). By inspecting the definition, one can see that the following statement holds for  $\varepsilon \in [0, 1)$

$$\rho = \sigma \quad \Rightarrow \quad D_H^\varepsilon(\rho\|\sigma) = -\log(1 - \varepsilon). \quad (4.2)$$

That is, the conditions  $\rho = \sigma$  and  $\text{Tr}\{\Lambda\rho\} \geq 1 - \varepsilon$  imply that  $\text{Tr}\{\Lambda\sigma\} \geq 1 - \varepsilon$  for all  $\Lambda$ , and we can take  $\Lambda = (1 - \varepsilon)I$  to achieve this bound.

From this quantity follows an information measure [BD11, Definition 4] closely related to the relative entropy of entanglement [VP98]:

$$E_R^\varepsilon(A; B)_\rho \equiv \inf_{\sigma_{AB} \in \mathcal{S}(A:B)} D_H^\varepsilon(\rho_{AB}\|\sigma_{AB}). \quad (4.3)$$

This quantity is an LOCC monotone, meaning that

$$E_R^\varepsilon(A; B)_\rho \geq E_R^\varepsilon(A'; B')_\omega, \quad (4.4)$$

for  $\omega_{A'B'} \equiv \Lambda_{AB \rightarrow A'B'}(\rho_{AB})$ , with  $\Lambda_{AB \rightarrow A'B'}$  an LOCC channel. This follows because the underlying quantity  $D_H^\varepsilon$  is monotone non-increasing with respect to quantum channels and the set of separable states is closed under LOCC channels (see [BD11, Lemma 1] for an explicit proof). More generally,  $E_R^\varepsilon(A; B)_\rho$  is monotone non-increasing with respect to separability-preserving channels for the same reasons. We can extend the definition in (4.3) to be a function of a quantum channel  $\mathcal{N}_{A' \rightarrow B}$ :

$$E_R^\varepsilon(\mathcal{N}) \equiv \sup_{|\psi\rangle_{AA'} \in \mathcal{H}_{AA'}} E_R^\varepsilon(A; B)_\rho, \quad (4.5)$$

where  $\rho_{AB} \equiv \mathcal{N}_{A' \rightarrow B}(\psi_{AA'})$ . Note that it suffices to perform the optimization with respect to pure states due to the fact that  $D_H^\varepsilon$  satisfies the data processing inequality. The quantity  $E_R^\varepsilon(\mathcal{N})$  (and later related ones) will play an important role in establishing upper bounds on the private transmission capabilities of a quantum channel.

## 4.2 Privacy test

Here we define a “privacy test” as a method for testing whether a given bipartite state is private. It forms an essential component of the general meta-converse bound given in Section 4.3. In some sense, this notion is already implicit in the developments of [HHHO09, Eqns. (282)–(284)] and is stated even more explicitly in [HHH<sup>+</sup>08b, HHH<sup>+</sup>08a]. We state the notion here concretely for completeness.

**Definition 8 (Privacy test)** Let  $\gamma_{ABA'B'} \in \mathcal{D}(\mathcal{H}_{ABA'B'})$  be a bipartite private state as given in Definition 4. A privacy test corresponding to  $\gamma_{ABA'B'}$  (a  $\gamma$ -privacy test) is defined as the following dichotomic measurement:

$$\{\Pi_{ABA'B'}, I_{ABA'B'} - \Pi_{ABA'B'}\}, \quad (4.6)$$

where  $\Pi_{ABA'B'} \equiv U_{ABA'B'} (\Phi_{AB} \otimes I_{A'B'}) U_{ABA'B'}^\dagger$  and  $U_{ABA'B'}$  is the unitary specified in (2.17).

If one has access to the systems  $ABA'B'$  of a bipartite state  $\rho_{ABA'B'}$  and has a description of  $\gamma_{ABA'B'}$  satisfying (2.19), then the  $\gamma$ -privacy test decides whether  $\rho_{ABA'B'}$  is a private state with respect to  $\gamma_{ABA'B'}$ . The first outcome corresponds to the decision “yes, it is a  $\gamma$ -private state,” and the second outcome corresponds to “no.” Physically, this test is just untwisting the purported private state and projecting onto a maximally entangled state. The following lemma states that the probability for an  $\varepsilon$ -approximate bipartite private state to pass the  $\gamma$ -privacy test is high:

**Lemma 9** Let  $\varepsilon \in [0, 1]$  and let  $\rho_{ABA'B'} \in \mathcal{D}(\mathcal{H}_{ABA'B'})$  be an  $\varepsilon$ -approximate private state as given in Definition 6, with  $\gamma_{ABA'B'}$  satisfying (2.19). The probability for  $\rho_{ABA'B'}$  to pass the  $\gamma$ -privacy test is never smaller than  $1 - \varepsilon$ :

$$\text{Tr}\{\Pi_{ABA'B'} \rho_{ABA'B'}\} \geq 1 - \varepsilon, \quad (4.7)$$

where  $\Pi_{ABA'B'}$  is defined as above.

**Proof.** One can see this bound explicitly by inspecting the following steps:

$$\text{Tr}\{\Pi_{ABA'B'} \rho_{ABA'B'}\} = \langle \Phi |_{AB} \text{Tr}_{A'B'} \{U_{ABA'B'}^\dagger \rho_{ABA'B'} U_{ABA'B'}\} | \Phi \rangle_{AB} \quad (4.8)$$

$$= F(\Phi_{AB}, \text{Tr}_{A'B'} \{U_{ABA'B'}^\dagger \rho_{ABA'B'} U_{ABA'B'}\}) \quad (4.9)$$

$$\geq F(\Phi_{AB} \otimes \theta_{A'B'}, U_{ABA'B'}^\dagger \rho_{ABA'B'} U_{ABA'B'}) \quad (4.10)$$

$$= F(U_{ABA'B'} (\Phi_{AB} \otimes \theta_{A'B'}) U_{ABA'B'}^\dagger, \rho_{ABA'B'}) \quad (4.11)$$

$$= F(\gamma_{ABA'B'}, \rho_{ABA'B'}) \geq 1 - \varepsilon. \quad (4.12)$$

The steps follow as a consequence of several properties of the fidelity recalled in Section 2. ■

For completeness, we think it is worthwhile to recall the brief proof of (4.13) below from [HHHO09, Eqns. (282)–(284)].

**Lemma 10 ([HHHO09, Eqn. (281)])** *For a separable state  $\sigma_{ABA'B'} \in \mathcal{S}(AA':BB')$ , the probability of passing any  $\gamma$ -privacy test is never larger than  $1/K$ :*

$$\mathrm{Tr}\{\Pi_{ABA'B'}\sigma_{ABA'B'}\} \leq \frac{1}{K}, \quad (4.13)$$

where  $K$  is the number of values that the secret key can take (i.e.,  $K = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$ ).

**Proof.** The idea is to begin by establishing the bound for any pure product state  $|\phi\rangle_{AA'} \otimes |\varphi\rangle_{BB'}$ . We can expand these states with respect to the standard bases of  $A$  and  $B$  as follows:

$$|\phi\rangle_{AA'} \otimes |\varphi\rangle_{BB'} = \left[ \sum_{i=0}^{K-1} \alpha_i |i\rangle_A \otimes |\phi_i\rangle_{A'} \right] \otimes \left[ \sum_{j=0}^{K-1} \beta_j |j\rangle_B \otimes |\varphi_j\rangle_{B'} \right], \quad (4.14)$$

where  $\sum_{i=0}^{K-1} |\alpha_i|^2 = \sum_{j=0}^{K-1} |\beta_j|^2 = 1$ . A few steps of calculation then lead to the following equalities:

$$\begin{aligned} & \mathrm{Tr}\{\Pi_{ABA'B'}|\phi\rangle\langle\phi|_{AA'} \otimes |\varphi\rangle\langle\varphi|_{BB'}\} \\ &= \mathrm{Tr}\{U_{ABA'B'} (\Phi_{AB} \otimes I_{A'B'}) U_{ABA'B'}^\dagger |\phi\rangle\langle\phi|_{AA'} \otimes |\varphi\rangle\langle\varphi|_{BB'}\} \end{aligned} \quad (4.15)$$

$$= \frac{1}{K} \sum_{i,j=0}^{K-1} \alpha_i \beta_i \alpha_j^* \beta_j^* \langle \xi_j | \xi_i \rangle_{A'B'}, \quad (4.16)$$

where  $|\xi_i\rangle_{A'B'} \equiv (U_{A'B'}^{ii})^\dagger |\phi_i\rangle_{A'} |\varphi_i\rangle_{B'}$  is a quantum state. The desired bound is then equivalent to

$$\sum_{i,j=0}^{K-1} \alpha_i \beta_i \alpha_j^* \beta_j^* \langle \xi_j | \xi_i \rangle_{A'B'} \leq 1. \quad (4.17)$$

Setting  $\alpha_i = \sqrt{p_i} e^{i\theta_i}$  and  $\beta_i = \sqrt{q_i} e^{i\eta_i}$ , we find that

$$\sum_{i,j=0}^{K-1} \alpha_i \beta_i \alpha_j^* \beta_j^* \langle \xi_j | \xi_i \rangle_{A'B'} = \left| \sum_{i,j=0}^{K-1} \sqrt{p_i q_i p_j q_j} e^{i(\theta_i + \eta_i - \theta_j - \eta_j)} \langle \xi_j | \xi_i \rangle_{A'B'} \right| \quad (4.18)$$

$$\leq \sum_{i,j=0}^{K-1} \sqrt{p_i q_i p_j q_j} |\langle \xi_j | \xi_i \rangle_{A'B'}| \leq \sum_{i,j=0}^{K-1} \sqrt{p_i q_i p_j q_j} \quad (4.19)$$

$$= \left[ \sum_{i=0}^{K-1} \sqrt{p_i q_i} \right]^2 \leq 1, \quad (4.20)$$

where the last inequality holds for all probability distributions (this is just the statement that the classical fidelity cannot exceed one). The above reasoning thus establishes (4.13) for pure product states, and the bound for general separable states follows because every such state can be written as a convex combination of pure product states. ■

The bounds in (4.7) and (4.13) are the core ones underlying all of our converse bounds in this paper.

### 4.3 Statements of general (meta-converse) bounds

We now establish some general bounds on the achievable regions discussed in Section 3.3.

**Theorem 11** *Let  $\mathcal{N}_{A' \rightarrow B}$  be a quantum channel. Then for any fixed  $\varepsilon \in (0, 1)$ , the achievable region with CPPP assistance satisfies*

$$\hat{P}_{\mathcal{N}}^{\text{CPPP}}(1, \varepsilon) \leq E_R^\varepsilon(\mathcal{N}). \quad (4.21)$$

**Proof.** Consider any CPPP-assisted protocol that achieves a rate  $\hat{P}_{\mathcal{N}}^{\text{CPPP}}(1, \varepsilon) \equiv \hat{P}$ , formulated in the bipartite picture as discussed in the previous section. Let  $\omega_{A_0 A' B_0}$  denote the state generated by the first round of LOCC. Note that  $\omega_{A_0 A' B_0}$  is a separable state:  $\omega_{A_0 A' B_0} \in \mathcal{S}(A_0 A' : B_0)$ . The  $A'$  system of this state gets sent through the channel  $\mathcal{N}_{A' \rightarrow B}$ , leading to the state

$$\theta_{A_0 B B_0} \equiv \mathcal{N}_{A' \rightarrow B}(\omega_{A_0 A' B_0}). \quad (4.22)$$

Alice and Bob apply an LOCC decoder  $\mathcal{D}_{A_0 B B_0 \rightarrow K_A K_B S_A S_B}$ , which consists of a round of LOCC, leading to the state

$$\omega_{K_A K_B S_A S_B} \equiv \mathcal{D}_{A_0 B B_0 \rightarrow K_A K_B S_A S_B}(\theta_{A_0 B B_0}). \quad (4.23)$$

By assumption we have that

$$F(\gamma_{K_A K_B S_A S_B}, \omega_{K_A K_B S_A S_B}) \geq 1 - \varepsilon, \quad (4.24)$$

for some private state  $\gamma_{K_A K_B S_A S_B}$ . By Lemma 9, there is a projector  $\Pi_{K_A K_B S_A S_B}$  corresponding to a  $\gamma$ -privacy test of the form in Definition 8, such that

$$\text{Tr}\{\Pi_{K_A K_B S_A S_B} \omega_{K_A K_B S_A S_B}\} \geq 1 - \varepsilon. \quad (4.25)$$

From Lemma 10, we have that

$$\text{Tr}\{\Pi_{K_A K_B S_A S_B} \sigma_{K_A K_B S_A S_B}\} \leq 2^{-\hat{P}}, \quad (4.26)$$

for any separable state  $\sigma_{K_A K_B S_A S_B} \in \mathcal{S}(K_A S_A : K_B S_B)$ . Thus, this test is feasible for  $D_H^\varepsilon(\omega \| \sigma)$  and we find that

$$\hat{P} \leq D_H^\varepsilon(\omega_{K_A K_B S_A S_B} \| \sigma_{K_A K_B S_A S_B}) \quad (4.27)$$

for any separable state  $\sigma_{K_A K_B S_A S_B} \in \mathcal{S}(K_A S_A : K_B S_B)$ . Let  $\tau_{A_0 B} \in \mathcal{S}(A_0 : B)$ . From the quasi-convexity of  $D_H^\varepsilon$  we find that there exist pure states  $\psi_{A_0 A'}$  and  $\varphi_{B_0}$  such that

$$D_H^\varepsilon(\mathcal{N}_{A' \rightarrow B}(\psi_{A_0 A'}) \| \tau_{A_0 B}) = D_H^\varepsilon(\mathcal{N}_{A' \rightarrow B}(\psi_{A_0 A'}) \otimes \varphi_{B_0} \| \tau_{A_0 B} \otimes \varphi_{B_0}) \quad (4.28)$$

$$\geq D_H^\varepsilon(\mathcal{N}_{A' \rightarrow B}(\omega_{A_0 A' B_0}) \| \tau_{A_0 B} \otimes \varphi_{B_0}) \quad (4.29)$$

$$\geq D_H^\varepsilon(\omega_{K_A K_B S_A S_B} \| \sigma_{K_A K_B S_A S_B}) \quad (4.30)$$

$$\geq \hat{P}, \quad (4.31)$$

where we take  $\sigma_{K_A K_B S_A S_B} = \mathcal{D}_{A_0 B B_0 \rightarrow K_A K_B S_A S_B}(\tau_{A_0 B} \otimes \varphi_{B_0})$ . The first equality follows because  $D_H^\varepsilon$  is invariant with respect to tensoring in the same state on an extra system (doing so does not change the constrained Type II error in a quantum hypothesis test). The first inequality follows from quasi-convexity of  $D_H^\varepsilon$ . The second inequality follows from the monotonicity of  $D_H^\varepsilon$  with

respect to quantum channels. Since the decoder  $\mathcal{D}_{A_0BB_0 \rightarrow K_AK_BS_AS_B}$  is an LOCC channel, we can conclude that  $\sigma_{K_AK_BS_AS_B} \in \mathcal{S}(K_A S_A : K_B S_B)$ . The final inequality follows from (4.27). Since the inequality holds for any choice  $\tau_{A_0B} \in \mathcal{S}(A_0 : B)$ , we can conclude that

$$E_R^\varepsilon(\mathcal{N}_{A' \rightarrow B}(\psi_{A_0A'})) \geq \hat{P}. \quad (4.32)$$

Optimizing over all input states  $\psi_{A_0A'}$ , we can conclude the statement of the proposition. ■

The above theorem immediately leads to the following bound for any quantum channel  $\mathcal{N}$ :

$$\hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \leq \frac{1}{n} E_R^\varepsilon(\mathcal{N}^{\otimes n}). \quad (4.33)$$

For a teleportation-simulable channel  $\mathcal{N}_{A' \rightarrow B}^{\text{TP}}$  as defined in (2.12) with associated state  $\omega_{AB}$ , combining (3.16) and a proof similar to that for Theorem 11 leads to the following bound:

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon) \leq \frac{1}{n} E_R^\varepsilon(A^n; B^n)_{\omega^{\otimes n}}. \quad (4.34)$$

Note that this latter bound will lead us to a complete proof of the main result presented in [PLOB16], in addition to other more refined statements. In contrast to the approach presented in [PLOB16], our result has no dependence on the dimension of the shield systems. This is in particular beneficial for the treatment of quantum Gaussian channels (discussed in Section 8).

## 5 Relative entropy of entanglement as a strong converse rate

In this section, we prove two strong converse theorems for private communication. Before doing so, we review various definitions of private capacities and strong converse rates for private communication, and we also review several Rényi entropic measures that play a role in establishing the strong converse theorems.

### 5.1 Definitions of private capacities and strong converse rates

A rate  $r$  is achievable for secret-key transmission over the channel  $\mathcal{N}$  if there exists a sequence of secret-key transmission protocols  $\{(n, P_n, \varepsilon_n)\}_{n \in \mathbb{N}}$ , such that

$$\liminf_{n \rightarrow \infty} P_n \geq r \quad \text{and} \quad \lim_{n \rightarrow \infty} \varepsilon_n = 0. \quad (5.1)$$

The private capacity of  $\mathcal{N}$ , denoted  $P(\mathcal{N})$ , is equal to the supremum of all achievable rates [Dev05, CWY04]. Equivalently, we have that

$$P(\mathcal{N}) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \hat{P}_{\mathcal{N}}(n, \varepsilon). \quad (5.2)$$

Analogously, the CPPP-assisted private capacity of  $\mathcal{N}$ , denoted  $P_{\text{CPPP}}(\mathcal{N})$ , is equal to the supremum of all CPPP-assisted achievable rates, and we have a similar definition for  $P_{\leftrightarrow}(\mathcal{N})$ . Similarly,

$$P_{\text{CPPP}}(\mathcal{N}) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon), \quad (5.3)$$

$$P_{\leftrightarrow}(\mathcal{N}) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon). \quad (5.4)$$

On the other hand,  $r$  is a strong converse rate for secret-key transmission if for every sequence of secret-key transmission protocols  $\{(n, P_n, \varepsilon_n)\}_{n \in \mathbb{N}}$  as above, we have

$$\liminf_{n \rightarrow \infty} P_n > r \implies \lim_{n \rightarrow \infty} \varepsilon_n = 1. \quad (5.5)$$

The strong converse private capacity, denoted  $P^\dagger(\mathcal{N})$ , is equal to the infimum of all strong converse rates. Analogously, the CPPP-assisted strong converse private capacity of  $\mathcal{N}$ , denoted  $P_{\text{cppp}}^\dagger(\mathcal{N})$ , is equal to the infimum of all CPPP-assisted strong converse rates (and similarly for  $P_{\leftrightarrow}^\dagger(\mathcal{N})$ ). The following inequalities hold by definition:

$$P(\mathcal{N}) \leq P_{\text{cppp}}(\mathcal{N}) \leq P_{\leftrightarrow}(\mathcal{N}), \quad (5.6)$$

$$P^\dagger(\mathcal{N}) \leq P_{\text{cppp}}^\dagger(\mathcal{N}) \leq P_{\leftrightarrow}^\dagger(\mathcal{N}), \quad (5.7)$$

$$P(\mathcal{N}) \leq P^\dagger(\mathcal{N}), \quad (5.8)$$

$$P_{\text{cppp}}(\mathcal{N}) \leq P_{\text{cppp}}^\dagger(\mathcal{N}), \quad (5.9)$$

$$P_{\leftrightarrow}(\mathcal{N}) \leq P_{\leftrightarrow}^\dagger(\mathcal{N}). \quad (5.10)$$

We argue in Appendix B how a strong converse rate according to the above definitions of privacy is a strong converse rate according to quantum generalizations of the definitions from [HTW14] (see Appendix B for specifics).

Finally, we say that a channel  $\mathcal{N}$  satisfies the strong converse property for private communication if  $P(\mathcal{N}) = P^\dagger(\mathcal{N})$ . Similarly, we say that a channel  $\mathcal{N}$  satisfies the strong converse property for CPPP-assisted private communication if  $P_{\text{cppp}}(\mathcal{N}) = P_{\text{cppp}}^\dagger(\mathcal{N})$ , and a similar statement if  $P_{\leftrightarrow}(\mathcal{N}) = P_{\leftrightarrow}^\dagger(\mathcal{N})$ .

For the capacities, we find that

$$Q(\mathcal{N}) \leq P(\mathcal{N}), \quad Q_{\text{cppp}}(\mathcal{N}) \leq P_{\text{cppp}}(\mathcal{N}), \quad Q_{\leftrightarrow}(\mathcal{N}) \leq P_{\leftrightarrow}(\mathcal{N}), \quad (5.11)$$

$$Q^\dagger(\mathcal{N}) \leq P^\dagger(\mathcal{N}), \quad Q_{\text{cppp}}^\dagger(\mathcal{N}) \leq P_{\text{cppp}}^\dagger(\mathcal{N}), \quad Q_{\leftrightarrow}^\dagger(\mathcal{N}) \leq P_{\leftrightarrow}^\dagger(\mathcal{N}), \quad (5.12)$$

$$Q(\mathcal{N}) \leq Q^\dagger(\mathcal{N}), \quad Q_{\text{cppp}}(\mathcal{N}) \leq Q_{\text{cppp}}^\dagger(\mathcal{N}), \quad Q_{\leftrightarrow}(\mathcal{N}) \leq Q_{\leftrightarrow}^\dagger(\mathcal{N}), \quad (5.13)$$

where the quantum capacity  $Q$  and the strong converse quantum capacity  $Q^\dagger$  are defined analogously to  $P$  and  $P^\dagger$ . In summary, any lower bound on a rate of entanglement transmission for a given scenario is a lower bound for secret-key transmission in the same scenario.

## 5.2 Rényi relative entropies and related measures

Let  $\rho \in \mathcal{D}(\mathcal{H})$  and  $\sigma \in \mathcal{L}_+(\mathcal{H})$ . The quantum relative entropy  $D(\rho\|\sigma)$  is defined as [Ume62]

$$D(\rho\|\sigma) \equiv \begin{cases} \text{Tr}\{\rho(\log \rho - \log \sigma)\} & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty & \text{else} \end{cases}. \quad (5.14)$$

Throughout we take the logarithm (denoted by  $\log$ ) to be base two unless stated otherwise. The relative entropy  $D(\rho\|\sigma)$  is monotone with respect to quantum channels [Lin75, Uhl77], in the sense that

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad (5.15)$$

for  $\mathcal{N}$  a quantum channel.

The sandwiched Rényi relative entropy is defined for  $\alpha \in (0, 1) \cup (1, \infty)$  as [MLDS<sup>+</sup>13, WWY14]:

$$\tilde{D}_\alpha(\rho\|\sigma) \equiv \begin{cases} \frac{2\alpha}{\alpha-1} \log \|\sigma^{(1-\alpha)/2\alpha} \rho^{1/2}\|_{2\alpha} & \text{if } (\alpha \in (0, 1) \wedge \text{supp}(\rho) \not\subseteq \text{supp}(\sigma)) \\ +\infty & \vee (\text{supp}(\rho) \subseteq \text{supp}(\sigma)) \\ & \text{else} \end{cases}, \quad (5.16)$$

where  $\|A\|_p \equiv [\text{Tr}\{|A|^p\}]^{1/p}$  is the  $p$ -norm of an operator  $A$  for  $p \geq 1$  and  $|A| \equiv \sqrt{A^\dagger A}$  (note that we define  $\|A\|_p$  as above even for  $p \in [0, 1)$  when it is not a norm). Both the quantum and sandwiched relative entropies are additive in the following sense:

$$D(\rho_0 \otimes \rho_1\|\sigma_0 \otimes \sigma_1) = D(\rho_0\|\sigma_0) + D(\rho_1\|\sigma_1), \quad (5.17)$$

$$\tilde{D}_\alpha(\rho_0 \otimes \rho_1\|\sigma_0 \otimes \sigma_1) = \tilde{D}_\alpha(\rho_0\|\sigma_0) + \tilde{D}_\alpha(\rho_1\|\sigma_1), \quad (5.18)$$

where  $\rho_i \in \mathcal{D}(\mathcal{H}_i)$  and  $\sigma_i \in \mathcal{L}_+(\mathcal{H}_i)$  for  $i \in \{0, 1\}$ . The following limits hold [MLDS<sup>+</sup>13, WWY14]

$$\lim_{\alpha \rightarrow 1^-} \tilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma), \quad \lim_{\alpha \rightarrow \infty} \tilde{D}_\alpha(\rho\|\sigma) = D_{\max}(\rho\|\sigma), \quad (5.19)$$

where  $D_{\max}(\rho\|\sigma) \equiv 2 \log \|\sigma^{-1/2} \rho^{1/2}\|_\infty$  [Dat09]. The quantity  $\tilde{D}_\alpha$  is monotone with respect to quantum channels [FL13], in the sense that

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad (5.20)$$

for  $\mathcal{N}$  a quantum channel and  $\alpha \in [1/2, 1) \cup (1, \infty]$ . The quantity  $\tilde{D}_\alpha$  is also monotone with respect to the Rényi parameter [MLDS<sup>+</sup>13, Bei13]: for  $1 < \alpha < \beta$ , the following inequality holds

$$D(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) \leq \tilde{D}_\beta(\rho\|\sigma). \quad (5.21)$$

The following inequality relates  $D_H^\varepsilon(\rho\|\sigma)$  to  $\tilde{D}_\alpha(\rho\|\sigma)$  for  $\alpha \in (1, \infty)$  and  $\varepsilon \in (0, 1)$ :

$$D_H^\varepsilon(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1} \log (1/(1-\varepsilon)). \quad (5.22)$$

This inequality is implicit in the literature [HP91, Nag01, ON00] (an explicit proof for the interested reader is available as [CMW16, Lemma 5]).

The relative entropy of entanglement of a state  $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$  is defined as [VP98]

$$E_R(A; B)_\rho \equiv \min_{\sigma_{AB} \in \mathcal{S}(A:B)} D(\rho_{AB}\|\sigma_{AB}), \quad (5.23)$$

and we define a related Rényi relative entropy of entanglement as well:

$$\tilde{E}_{R,\alpha}(A; B)_\rho \equiv \inf_{\sigma_{AB} \in \mathcal{S}(A:B)} \tilde{D}_\alpha(\rho_{AB}\|\sigma_{AB}). \quad (5.24)$$

Note that an alternative definition of Rényi relative entropy of entanglement has already been given in [Sha14], in terms of the Rényi relative entropy defined in [Pet86]. The relative entropies of entanglement are LOCC monotones and more generally separability-preserving monotones, as defined and justified previously in Section 4.1. The following subadditivity relations hold:

$$E_R(A_0 A_1; B_0 B_1)_{\rho^0 \otimes \rho^1} \leq E_R(A_0; B_0)_{\rho^0} + E_R(A_1; B_1)_{\rho^1}, \quad (5.25)$$

$$\tilde{E}_{R,\alpha}(A_0 A_1; B_0 B_1)_{\rho^0 \otimes \rho^1} \leq \tilde{E}_{R,\alpha}(A_0; B_0)_{\rho^0} + \tilde{E}_{R,\alpha}(A_1; B_1)_{\rho^1}, \quad (5.26)$$

where  $\rho_{A_i B_i}^i \in \mathcal{D}(\mathcal{H}_{A_i B_i})$  for  $i \in \{0, 1\}$ . These follow from the additivity relations in (5.17)–(5.18) and because the separable states in  $\mathcal{S}(A_0 A_1 : B_0 B_1)$  considered for the infima on the left-hand side need not be a tensor product. We extend these definitions to be functions of a quantum channel  $\mathcal{N}_{A' \rightarrow B}$ , which we call the channel's relative entropy of entanglement:

$$E_R(\mathcal{N}) \equiv \sup_{|\psi\rangle_{AA'} \in \mathcal{H}_{AA'}} E_R(A; B)_\rho, \quad (5.27)$$

$$\tilde{E}_{R,\alpha}(\mathcal{N}) \equiv \sup_{|\psi\rangle_{AA'} \in \mathcal{H}_{AA'}} \tilde{E}_{R,\alpha}(A; B)_\rho, \quad (5.28)$$

where  $\rho_{AB} \equiv \mathcal{N}_{A' \rightarrow B}(\psi_{AA'})$ .

By a standard continuity argument (see, e.g., [MH11, TWW17, CMW16, MO14, DW15]), the following limits hold

$$\lim_{\alpha \rightarrow 1} \tilde{E}_{R,\alpha}(A; B)_\rho = E_R(A; B)_\rho, \quad (5.29)$$

$$\lim_{\alpha \rightarrow 1} \tilde{E}_{R,\alpha}(\mathcal{N}) = E_R(\mathcal{N}). \quad (5.30)$$

Monotonicity of these quantities with respect to the Rényi parameter follows from (5.21): for  $1 < \alpha < \beta$ , the following inequalities hold

$$E_R(A; B)_\rho \leq \tilde{E}_{R,\alpha}(A; B)_\rho \leq \tilde{E}_{R,\beta}(A; B)_\rho, \quad (5.31)$$

$$E_R(\mathcal{N}) \leq \tilde{E}_{R,\alpha}(\mathcal{N}) \leq \tilde{E}_{R,\beta}(\mathcal{N}). \quad (5.32)$$

The inequality in (5.22) allows us to relate  $\tilde{E}_{R,\alpha}$  to  $E_R^\varepsilon$  for  $\alpha \in (1, \infty)$  and  $\varepsilon \in (0, 1)$ :

$$E_R^\varepsilon(A; B)_\rho \leq \tilde{E}_{R,\alpha}(A; B)_\rho + \frac{\alpha}{\alpha - 1} \log(1/(1 - \varepsilon)), \quad (5.33)$$

$$E_R^\varepsilon(\mathcal{N}) \leq \tilde{E}_{R,\alpha}(\mathcal{N}) + \frac{\alpha}{\alpha - 1} \log(1/(1 - \varepsilon)). \quad (5.34)$$

These latter two inequalities are helpful for obtaining the strong converse theorems given below.

### 5.3 Statements of strong converse results

We begin by establishing the following strong converse theorem for teleportation-simulable channels:

**Theorem 12** *If a channel  $\mathcal{N}^{\text{TP}}$  is teleportation-simulable as defined in (2.12) (with associated state  $\omega_{AB}$ ), then  $E_R(A; B)_\omega$  is a strong converse rate for two-way assisted private communication:*

$$P_{\leftrightarrow}^{\dagger}(\mathcal{N}^{\text{TP}}) \leq E_R(A; B)_\omega. \quad (5.35)$$

**Proof.** Let  $\alpha \in (1, \infty)$ . A consequence of (4.34) and a rewriting of (5.33) is the following bound on the optimal fidelity of any two-way assisted protocol for a teleportation-simulable channel  $\mathcal{N}^{\text{TP}}$ :

$$1 - \hat{\varepsilon}_{\mathcal{N}}^{\leftrightarrow}(n, P^{\leftrightarrow}) \leq 2^{-n(\frac{\alpha-1}{\alpha})(P^{\leftrightarrow} - \frac{1}{n}\tilde{E}_{R,\alpha}(A^n; B^n)_{\omega \otimes n})} \quad (5.36)$$

$$\leq 2^{-n(\frac{\alpha-1}{\alpha})(P^{\leftrightarrow} - \tilde{E}_{R,\alpha}(A; B)_\omega)}, \quad (5.37)$$

where the second inequality follows from (5.26), i.e, the subadditivity of  $\tilde{E}_{R,\alpha}$  with respect to tensor-product states. Thus, if  $P^{\leftrightarrow} > E_R(A;B)_\omega$ , then by (5.31) and (5.29), there exists  $\alpha > 1$  such that  $P^{\leftrightarrow} > \tilde{E}_{R,\alpha}(A;B)_\omega$  and so the optimal error  $\hat{\varepsilon}_N^{\leftrightarrow}(n, P^{\leftrightarrow})$  increases exponentially fast to one with exponent  $(\frac{\alpha-1}{\alpha})(P^{\leftrightarrow} - \tilde{E}_{R,\alpha}(A;B)_\omega)$ . ■

Next we establish that a channel's relative entropy of entanglement from (5.27) is a strong converse rate for CPPP-assisted private communication (and thus for unassisted private communication as well by (5.7)).

**Theorem 13** *For any channel  $N$ , its relative entropy of entanglement is a strong converse rate for CPPP-assisted private communication:*

$$P_{\text{CPPP}}^\dagger(N) \leq E_R(N). \quad (5.38)$$

We do not give a detailed proof of the theorem above, because it follows from several results already available in [TWW17]. Here we merely collect the needed statements and give a proof sketch. A proof for Proposition 14 below follows by the same proof given for [TWW17, Proposition 2]:

**Proposition 14** *Let  $N_{A' \rightarrow B}$  be a quantum channel that is covariant with respect to a group  $G$  (as defined in Section 2) and let  $\rho_{A'} \in \mathcal{D}(\mathcal{H}_{A'})$ ,  $\phi_{AA'}^\rho \in \mathcal{H}_A \otimes \mathcal{H}_{A'}$  be a purification of  $\rho_{A'}$ , and  $\rho_{AB} = N_{A' \rightarrow B}(\phi_{AA'}^\rho)$ . Let  $\bar{\rho}_{A'}$  denote the group expectation of  $\rho_{A'}$ , i.e.,*

$$\bar{\rho}_{A'} = \frac{1}{|G|} \sum_g U_{A'}(g) \rho_{A'} U_{A'}^\dagger(g), \quad (5.39)$$

and let  $\phi_{AA'}^{\bar{\rho}}$  be a purification of  $\bar{\rho}_{A'}$  and  $\bar{\rho}_{AB} = N_{A' \rightarrow B}(\phi_{AA'}^{\bar{\rho}})$ . Then, for  $\alpha \in (1, \infty)$  and  $\varepsilon \in (0, 1)$ , the following inequalities hold

$$E_R(A;B)_{\bar{\rho}} \geq E_R(A;B)_\rho, \quad (5.40)$$

$$E_R^\varepsilon(A;B)_{\bar{\rho}} \geq E_R^\varepsilon(A;B)_\rho, \quad (5.41)$$

$$\tilde{E}_{R,\alpha}(A;B)_{\bar{\rho}} \geq \tilde{E}_{R,\alpha}(A;B)_\rho. \quad (5.42)$$

The following theorem is a consequence of Proposition 14, the permutation covariance of any i.i.d. channel, and an application of the quantum de Finetti theorem, in the form of the postselection technique [CKR09] (see [TWW17, Theorem 6] for an explicit proof of the theorem below).

**Theorem 15** *Let  $N_{A' \rightarrow B}$  be a quantum channel. For all  $\alpha > 1$  and  $n \in \mathbb{N}$ , we have*

$$\tilde{E}_{R,\alpha}(N^{\otimes n}) \leq n \tilde{E}_{R,\alpha}(N) + \frac{\alpha |A'|^2}{\alpha - 1} \log n. \quad (5.43)$$

To arrive at the statement in Theorem 13, we note that (4.33) and (5.34) lead to the following bound on the optimal fidelity of any CPPP-assisted protocol:

$$1 - \hat{\varepsilon}_N^{\text{CPPP}}(n, P) \leq 2^{-\left(\frac{\alpha-1}{\alpha}\right)(nP - \tilde{E}_{R,\alpha}(N^{\otimes n}))} \quad (5.44)$$

$$\leq 2^{-\left(\frac{\alpha-1}{\alpha}\right)\left(nP - n\tilde{E}_{R,\alpha}(N) - \frac{\alpha|A'|^2}{\alpha-1} \log n\right)} \quad (5.45)$$

$$= n^{|A'|^2} 2^{-n\left(\frac{\alpha-1}{\alpha}\right)(P - \tilde{E}_{R,\alpha}(N))}, \quad (5.46)$$

where the second inequality follows from Theorem 15. Thus, if  $P > E_R(\mathcal{N})$ , then by (5.32) and (5.30), there exists  $\alpha > 1$  such that  $P > \tilde{E}_{R,\alpha}(\mathcal{N})$  and so the optimal error  $\hat{\varepsilon}_{\mathcal{N}}^{\text{cppp}}(n, P)$  increases exponentially fast to one with exponent  $(\frac{\alpha-1}{\alpha})(P - \tilde{E}_{R,\alpha}(\mathcal{N}))$  (the polynomial prefactor  $n^{|A'|^2}$  does not contribute to the exponent). Thus Theorem 13 follows.

**Remark 16** *The following regularized versions of the bounds in Theorems 12 and 13 hold, by applying the same argument as given in [TWW17, Theorem 8]:*

$$P_{\leftrightarrow}^{\dagger}(\mathcal{N}^{\text{TP}}) \leq \inf_{\ell \geq 1} \frac{1}{\ell} E_R(A^{\ell}; B^{\ell})_{\omega \otimes \ell}, \quad (5.47)$$

$$P_{\text{cppp}}^{\dagger}(\mathcal{N}) \leq \inf_{\ell \geq 1} \frac{1}{\ell} E_R(\mathcal{N}^{\otimes \ell}), \quad (5.48)$$

where  $\mathcal{N}^{\text{TP}}$  is a teleportation-simulable channel with associated state  $\omega_{AB}$  and  $\mathcal{N}$  is an arbitrary channel.

#### 5.4 Strong converses for particular channels

Two particular channels of interest for which we can establish the strong converse property for their private capacities are generalized dephasing channels and quantum erasure channels. A generalized dephasing channel is any channel with an isometric extension of the form

$$U_{A \rightarrow BE}^{\mathcal{N}} \equiv \sum_{x=0}^{d-1} |x\rangle_B \langle x|_A \otimes |\psi_x\rangle_E, \quad (5.49)$$

where the states  $|\psi_x\rangle$  are arbitrary (not necessarily orthonormal). Hence, the specification for such a channel is as follows:

$$\mathcal{N}(\rho) = \sum_{x,y=0}^{d-1} \langle x|_A \rho |y\rangle_A \langle \psi_y | \psi_x \rangle |x\rangle \langle y|_B. \quad (5.50)$$

**Proposition 17** *Let  $\mathcal{N}$  be a generalized dephasing channel of the form (5.49). Then*

$$I_c(\mathcal{N}) = P(\mathcal{N}) = P^{\dagger}(\mathcal{N}) = P_{\text{cppp}}(\mathcal{N}) = P_{\text{cppp}}^{\dagger}(\mathcal{N}) = E_R(\mathcal{N}), \quad (5.51)$$

where  $I_c(\mathcal{N})$  is the coherent information of the channel, defined in (6.9).

A proof for the above proposition proceeds exactly as in the proof of [TWW17, Proposition 10].

A quantum erasure channel is defined as follows:

$$\mathcal{E}_{A' \rightarrow B}^p : \rho_{A'} \mapsto (1-p)\rho_B + p|e\rangle\langle e|_B, \quad (5.52)$$

where  $p \in [0, 1]$  is the erasure probability,  $\rho_B$  is an isometric embedding of  $\rho_{A'}$  into  $\mathcal{H}_B$ , and  $|e\rangle$  is a quantum state orthogonal to  $\rho_B$ . The ccppp- and two-way assisted private capacity of this channel was presented in [GEW16, Section IV] and [PLOB16] to be equal to  $(1-p) \log |A|$ . This channel is teleportation-simulable, and the associated state  $\omega_{AB}$  in this case can be taken as  $\omega_{AB} = \mathcal{E}_{A' \rightarrow B}^p(\Phi_{AA'})$ . Applying Theorem 12 and the method of proof from [TWW17, Proposition 11], we obtain the following:

**Proposition 18** *Let  $\mathcal{E}^p$  be an erasure channel of the form (5.52). Then*

$$P_{\text{cppp}}(\mathcal{E}^p) = P_{\text{cppp}}^{\dagger}(\mathcal{E}^p) = P_{\leftrightarrow}(\mathcal{E}^p) = P_{\leftrightarrow}^{\dagger}(\mathcal{E}^p) = (1-p) \log |A|. \quad (5.53)$$

## 6 Second-order expansions for private communication

A recent goal of research in quantum information theory has been to determine second-order characterizations of various quantum communication tasks [TH13, Li14, TT15, DTW14, BDL16, TBR16]. In this program, the goal is to determine the highest rate of communication achievable for a given task when constrained to meet a fixed (constant) error probability and with a sufficiently large number of channel uses available. What one finds here is called the “Gaussian approximation,” which often serves as a good guideline for rates that are achievable at a fixed error and finite blocklength. Thus, this research goal is especially important nowadays given that experimentalists have limited control over quantum systems, only being able to manipulate small numbers of qubits coherently.

Two of the main tools which are consistently used in a second-order analysis are the quantum relative entropy variance and a second-order expansion of  $D_H^\varepsilon$ . The quantum relative entropy variance  $V(\rho\|\sigma)$  is defined as [Li14, TH13]

$$V(\rho\|\sigma) \equiv \text{Tr}\{\rho [\log \rho - \log \sigma - D(\rho\|\sigma)]^2\}. \quad (6.1)$$

whenever  $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ . The following second order expansion holds for  $n \propto 1/\varepsilon^2$  and  $\mathcal{H}$  a finite-dimensional Hilbert space [Li14, TH13]:

$$D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n). \quad (6.2)$$

In the above, we have used the cumulative distribution function for a standard normal random variable:

$$\Phi(a) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a dx \exp(-x^2/2), \quad (6.3)$$

and its inverse, defined as  $\Phi^{-1}(\varepsilon) \equiv \sup \{a \in \mathbb{R} \mid \Phi(a) \leq \varepsilon\}$ . It should be clear from the context whether  $\Phi$  refers to the maximally entangled state or (6.3). A recent alternative proof of (6.2) is available in [DPR16].

### 6.1 Converse (upper) bounds

If a quantum channel is teleportation-simulable, then it is possible to give a second-order expansion of the upper bounds from (4.33) and (4.34), by employing (6.2). (Recall from Proposition 2 that every covariant channel is teleportation-simulable.) Before doing so, we define the following variance quantity:

$$V_{E_R}^\varepsilon(A;B)_\rho \equiv \begin{cases} \sup_{\sigma_{AB} \in \Pi_S} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon < 1/2 \\ \inf_{\sigma_{AB} \in \Pi_S} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon \geq 1/2 \end{cases}, \quad (6.4)$$

where  $\Pi_S \subseteq \mathcal{S}(A:B)$  is the set of separable states achieving the minimum in  $E_R(A;B)_\rho$ . This leads us to the following theorem, as an immediate consequence of the above reasoning:

**Theorem 19** *If a quantum channel  $\mathcal{N}_{A' \rightarrow B}$  is teleportation-simulable with associated state  $\omega_{AB}$ , then*

$$\hat{P}_{\mathcal{N}}(n, \varepsilon) \leq \hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \leq \hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon) \leq E_R(A;B)_\omega + \sqrt{\frac{V_{E_R}^\varepsilon(A;B)_\omega}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right). \quad (6.5)$$

## 6.2 Achievable rates and lower bounds

As discussed in Section 3.4, entanglement transmission achieves the task of secret-key transmission, and this relationship allows for giving lower bounds on achievable secret-key transmission rates by employing known lower bounds on achievable entanglement transmission rates. For some channels of interest, this approach leads to a tight second-order characterization of their private transmission capabilities. We now briefly review some known lower bounds on achievable entanglement transmission rates [BDL16, TBR16]. Two quantities which arise in such a setting are the conditional quantum entropy and conditional entropy variance [TH13], defined for  $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$  as

$$H(A|B)_\rho \equiv -D(\rho_{AB}\|I_A \otimes \rho_B), \quad (6.6)$$

$$V(A|B)_\rho \equiv V(\rho_{AB}\|I_A \otimes \rho_B). \quad (6.7)$$

The coherent information is defined as  $I(A\rangle B)_\rho \equiv -H(A|B)_\rho$  [SN96] and its corresponding variance is  $V(A\rangle B)_\rho \equiv V(A|B)_\rho$ . Using these quantities, the maximum entanglement transmission rate  $\hat{Q}_{\mathcal{N}}(n, \varepsilon)$  possible has the following general lower bound for  $\varepsilon \in (0, 1)$  [BDL16, TBR16]:

$$\hat{Q}_{\mathcal{N}}(n, \varepsilon) \geq \hat{Q}_{\text{lower}, \mathcal{N}}(n, \varepsilon) \equiv I_c(\mathcal{N}) + \sqrt{\frac{V_c^\varepsilon(\mathcal{N})}{n}} \Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right), \quad (6.8)$$

where  $I_c(\mathcal{N})$  is the channel's coherent information:

$$I_c(\mathcal{N}) \equiv \max_{|\psi\rangle_{AA'} \in \mathcal{H}_{AA'}} I(A\rangle B)_\theta, \quad (6.9)$$

$\theta_{AB} \equiv \mathcal{N}_{A' \rightarrow B}(\psi_{AA'})$ , and  $V_c^\varepsilon(\mathcal{N})$  is the channel's conditional entropy variance:

$$V_c^\varepsilon(\mathcal{N}) \equiv \begin{cases} \min_{\psi_{AA'} \in \Pi} V(A|B)_\theta & \text{for } \varepsilon < 1/2 \\ \max_{\psi_{AA'} \in \Pi} V(A|B)_\theta & \text{for } \varepsilon \geq 1/2 \end{cases}. \quad (6.10)$$

The set  $\Pi \subseteq \mathcal{D}(\mathcal{H}_{AA'})$  is the set of all states achieving the maximum in (6.9). For channels with sufficient symmetry, such as covariant generalized dephasing channels, the characterization in (6.8) is tight, in the sense that  $\hat{Q}_{\mathcal{N}}(n, \varepsilon) = \hat{Q}_{\text{lower}, \mathcal{N}}(n, \varepsilon)$  for sufficiently large  $n$  [TBR16]. Due to (3.14) and (3.18), we can also conclude that the maximum rates possible for secret-key transmission have the same lower bound for  $\varepsilon \in (0, 1)$ :

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon) \geq \hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \geq \hat{P}_{\mathcal{N}}(n, \varepsilon) \geq \hat{Q}_{\text{lower}, \mathcal{N}}(n, \varepsilon). \quad (6.11)$$

The protocol for achieving the lower bound in (6.8) does not utilize forward or backward classical communication in any way. Proposition 21 below gives a lower bound on the one-shot distillable entanglement of a bipartite state  $\rho_{AB}$ . Such protocols allow for classical communication assistance in one direction. Note that lower bounds on one-shot distillable entanglement have previously appeared in the literature [Ber08, BD10a], but the bound given below allows for a tighter characterization when we later consider the i.i.d. case and second-order expansions. We give a proof of Proposition 21 in Appendix C.

**Definition 20** *The smooth conditional max-entropy  $H_{\max}^\xi(A|B)_\rho$  of a bipartite state  $\rho_{AB}$  is defined for  $\xi \in [0, 1]$  as*

$$H_{\max}^\xi(A|B)_\rho \equiv \inf_{\tilde{\rho}_{AB} \in \mathcal{B}^\xi(\rho_{AB})} \sup_{\sigma_B \in \mathcal{D}(\mathcal{H}_B)} \log F(\tilde{\rho}_{AB}, I_A \otimes \sigma_B), \quad (6.12)$$

where

$$\mathcal{B}^\xi(\rho_{AB}) \equiv \left\{ \rho'_{AB} \in \mathcal{D}_{\leq}(\mathcal{H}_{AB}) : P(\rho_{AB}, \rho'_{AB}) \leq \xi \right\}, \quad (6.13)$$

with  $P$  denoting the purified distance in (2.10).

**Proposition 21** Let  $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ ,  $\varepsilon \in [0, 1]$ , and  $\eta \in [0, \sqrt{\varepsilon}]$ . Then there exists a one-way entanglement distillation protocol  $\Lambda_{AB \rightarrow A'B'}$ , utilizing classical communication from Alice to Bob, such that

$$F(\Phi_{A'B'}, \Lambda_{AB \rightarrow A'B'}(\rho_{AB})) \geq 1 - \varepsilon, \quad (6.14)$$

where  $\Phi_{A'B'}$  is a maximally entangled state of Schmidt rank  $d$  and

$$\log d = -H_{\max}^{\sqrt{\varepsilon}-\eta}(A|B)_\rho - 4 \log\left(\frac{1}{\eta}\right). \quad (6.15)$$

One strategy for generating entanglement or secret key by means of a quantum channel is for

1. Alice to prepare  $n$  copies of a given state  $\psi_{AA'}$ ,
2. Alice to send the  $n$  systems labeled by  $A'$  through the memoryless channel  $\mathcal{N}_{A' \rightarrow B}^{\otimes n}$ , which leads to  $n$  copies of a bipartite state  $\theta_{AB} \equiv \mathcal{N}_{A' \rightarrow B}(\psi_{AA'})$ , and then for
3. Alice and Bob to perform entanglement distillation by means of backward classical communication from Bob to Alice.

Since the classical communication is now from Bob to Alice, the number of  $\varepsilon$ -approximate ebits that they can generate using this method is equal to  $-H_{\max}^{\sqrt{\varepsilon}-\eta}(B^n|A^n)_{\theta^{\otimes n}} - 4 \log\left(\frac{1}{\eta}\right)$ , by applying Proposition 21.

Steps 1-3 above realize an entanglement generation protocol. If the goal is entanglement transmission, Alice and Bob could subsequently perform quantum teleportation [BBC<sup>+</sup>93], using forward classical communication from Alice to Bob, to transmit any system Alice possesses to Bob. By the monotonicity of the fidelity with respect to quantum channels (and the teleportation protocol realizing a channel), the fidelity of entanglement transmission is  $\geq 1 - \varepsilon$  if the fidelity of entanglement generation is  $\geq 1 - \varepsilon$ .

Using standard methods for second-order expansions [TH13], we find that

$$\hat{P}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \geq \hat{Q}_{\mathcal{N}}^{\text{CPPP}}(n, \varepsilon) \geq I_{\text{rev}}(\mathcal{N}) + \sqrt{\frac{V_{\text{rev}}^\varepsilon(\mathcal{N})}{n}} \Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right) \quad (6.16)$$

where  $I_{\text{rev}}(\mathcal{N})$  is the channel's reverse coherent information (see [DJKR06, Section 5.3] and [GPPLS09]):

$$I_{\text{rev}}(\mathcal{N}) \equiv \max_{|\psi\rangle_{AA'} \in \mathcal{H}_{AA'}} I(B\rangle A)_\theta, \quad (6.17)$$

$\theta_{AB} \equiv \mathcal{N}_{A' \rightarrow B}(\psi_{AA'})$ , and  $V_{\text{rev}}^\varepsilon(\mathcal{N})$  is the channel's reverse conditional entropy variance:

$$V_{\text{rev}}^\varepsilon(\mathcal{N}) \equiv \begin{cases} \min_{\psi_{AA'} \in \Pi_{\text{rev}}} V(B\rangle A)_\theta & \text{for } \varepsilon < 1/2 \\ \max_{\psi_{AA'} \in \Pi_{\text{rev}}} V(B\rangle A)_\theta & \text{for } \varepsilon \geq 1/2 \end{cases}. \quad (6.18)$$

The set  $\Pi_{\text{rev}} \subseteq \mathcal{D}(\mathcal{H}_{AA'})$  is the set of all states achieving the maximum in (6.17).

Covariant dephasing channels and quantum erasure channels are two classes of channels for which we have tight second-order characterizations of their private transmission capabilities, due to their symmetries. In the next section, we provide characterizations of the private transmission capabilities of qubit versions of these channels that are tight to even the third order.

## 7 Channels with higher-order characterizations

This section details several examples of channels for which we can provide precise characterizations of their private communication abilities. Some of the results rely heavily on those from [TBR16], which in turn rely upon recent results from classical information theory (see the references given in [TBR16]). For this reason, in these cases we keep the exposition brief and point to [TBR16] for details.

### 7.1 Qubit dephasing channel

The qubit dephasing channel performs the following transformation on an input qubit density operator:

$$\mathcal{Z}^\gamma : \rho \longmapsto (1 - \gamma) \rho + \gamma Z \rho Z, \quad (7.1)$$

where  $\gamma \in (0, 1)$  is the dephasing parameter and  $Z$  is shorthand for the Pauli  $\sigma_Z$  operator. This channel is teleportation-simulable because it arises from the action of the teleportation protocol on the state  $\mathcal{Z}_{A' \rightarrow B}^\gamma(\Phi_{AA'}^+)$ , where  $|\Phi^+\rangle_{AA'} \equiv (|00\rangle_{AA'} + |11\rangle_{AA'})/\sqrt{2}$ . As a consequence, the upper bound from (4.34) applies. By using the same method given in [TBR16], which relates to results for the binary symmetric channel from classical information theory, we can establish a third-order upper bound. We can also follow the achievability strategy from [TBR16] to establish a matching lower bound. This leads to the following result:

**Proposition 22** *For the qubit dephasing channel  $\mathcal{Z}^\gamma$  with  $\gamma \in (0, 1)$ , the boundary  $\hat{P}(n; \varepsilon)$  satisfies*

$$\hat{P}(n, \varepsilon) = \hat{P}^{\text{CPPP}}(n, \varepsilon) = \hat{P}^{\leftrightarrow}(n, \varepsilon) = 1 - h(\gamma) + \sqrt{\frac{v(\gamma)}{n}} \Phi^{-1}(\varepsilon) + \frac{\log n}{2n} + O\left(\frac{1}{n}\right), \quad (7.2)$$

where  $\Phi$  is the cumulative standard Gaussian distribution,  $h(\gamma)$  denotes the binary entropy and  $v(\gamma)$  the corresponding variance, defined as

$$h(\gamma) \equiv -\gamma \log \gamma - (1 - \gamma) \log(1 - \gamma), \quad (7.3)$$

$$v(\gamma) \equiv \gamma(\log \gamma + h(\gamma))^2 + (1 - \gamma)(\log(1 - \gamma) + h(\gamma))^2. \quad (7.4)$$

Thus, for this channel, there is no difference between its private and quantum transmission capabilities.

### 7.2 Qubit erasure channel

The qubit erasure channel, defined for  $|A| = 2$  in (5.52), is another example of a channel for which we can obtain a precise characterization. This channel is teleportation-simulable because it arises from the action of the teleportation protocol on  $\mathcal{E}_{A' \rightarrow B}^p(\Phi_{AA'}^+)$ . Thus, we can apply the upper bound from (4.34) and the same reasoning from [TBR16] to establish a precise upper bound on the rates of private communication achievable with LOCC-assistance. Also, the same achievability protocol from [TBR16] gives a lower bound that matches this upper bound, giving us the following:

**Proposition 23** *For the qubit erasure channel  $\mathcal{E}^p$  with  $p \in (0, 1)$ , the boundary  $\hat{P}_{\mathcal{E}^p}^{\leftrightarrow}(n, \varepsilon)$  satisfies*

$$\varepsilon = \sum_{l=n-k+1}^n \binom{n}{l} p^l (1-p)^{n-l} \left(1 - 2^{n(1-\hat{P}_{\mathcal{E}^p}^{\leftrightarrow}(n, \varepsilon))-l}\right). \quad (7.5)$$

The same equation holds for  $\hat{P}_{\mathcal{E}^p}^{\text{CPPP}}(n, \varepsilon)$ . Moreover, the following expansion holds

$$\hat{P}_{\mathcal{E}^p}^{\text{CPPP}}(n, \varepsilon) = \hat{P}_{\mathcal{E}^p}^{\leftrightarrow}(n, \varepsilon) = 1 - p + \sqrt{\frac{p(1-p)}{n}} \Phi^{-1}(\varepsilon) + O\left(\frac{1}{n}\right). \quad (7.6)$$

Thus, again for this channel, there is no difference between its private and quantum transmission capabilities. The statement given above provides a strong refinement of the recent results in [GEW16] for the qubit erasure channel, which established  $1 - p$  as the two-way assisted private capacity of the qubit erasure channel.

### 7.3 Entanglement-breaking channels

Entanglement-breaking channels have played an important role in the development of quantum information theory [HSR03], in some sense being the class of channels which are most similar to a classical channel. A formal definition is that a channel  $\mathcal{N}_{A' \rightarrow B}$  is entanglement breaking if the state  $\mathcal{N}_{A' \rightarrow B}(\rho_{AA'})$  is separable regardless of the input state  $\rho_{AA'}$ . As shown in [HSR03], it suffices to check this condition for a single input: the maximally entangled state  $\Phi_{AA'}$ .

Entanglement-breaking channels can be simulated by local operations and classical communication: every such channel can be simulated by a measurement followed by a preparation of a state conditioned on the measurement outcome [HSR03, Hol08]. As a consequence, any  $P^{\leftrightarrow}$  protocol using an entanglement-breaking channel  $n$  times can only generate a separable state at the end of the protocol. Applying the same method of proof as in Theorem 11 and the observation from (4.2), we find that the following bound holds for any entanglement-breaking channel  $\mathcal{N}$ :

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon) \leq -\frac{1}{n} \log(1 - \varepsilon). \quad (7.7)$$

Thus, for these channels, the first, second, and third order terms all vanish, implying that such channels have essentially no capability to transmit private information. Given (3.18), the same upper bound holds for  $\hat{Q}_{\mathcal{N}}^{\leftrightarrow}(n, \varepsilon)$ , a result already obtained by the transposition bound method from [MHRW16, Section IV-C].

## 8 Quantum Gaussian channels

Quantum Gaussian channels are an important model for communication in realistic settings, such as free space and fiber-optic communication. A relevant subclass are the phase-insensitive channels, which add noise equally to the position and momentum quadrature of a given bosonic mode. Several results are now known for the various capacities of these channels (see the reviews in [Sha09, WPGP<sup>+</sup>12, HG12] and see [GGPCH14, PLOB16] for more recent developments).

In this section, we are interested in establishing bounds on the private and quantum communication capabilities of three kinds of phase-insensitive bosonic channels: the thermalizing channels, the amplifier channels, and the additive noise channels. Each of these are defined respectively by the following Heisenberg input-output relations:

$$\hat{b} = \sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{e}, \quad (8.1)$$

$$\hat{b} = \sqrt{G} \hat{a} + \sqrt{G-1} \hat{e}^\dagger, \quad (8.2)$$

$$\hat{b} = \hat{a} + (x + ip) / \sqrt{2}, \quad (8.3)$$

where  $\hat{a}$ ,  $\hat{b}$ , and  $\hat{e}$  are the field-mode annihilation operators for the sender's input, the receiver's output, and the environment's input of these channels, respectively.

The channel in (8.1) is a thermalizing channel, in which the environmental mode is prepared in a thermal state  $\theta(N_B)$  of mean photon number  $N_B \geq 0$ , defined as

$$\theta(N_B) \equiv \frac{1}{N_B + 1} \sum_{n=0}^{\infty} \left( \frac{N_B}{N_B + 1} \right)^n |n\rangle\langle n|, \quad (8.4)$$

where  $\{|n\rangle\}_{n=0}^{\infty}$  is the orthonormal, photonic number-state basis. When  $N_B = 0$ ,  $\theta(N_B)$  reduces to the vacuum state, in which case the resulting channel in (8.1) is called the pure-loss channel—it is said to be quantum-limited in this case because the environment is injecting the minimum amount of noise allowed by quantum mechanics. The parameter  $\eta \in [0, 1]$  is the transmissivity of the channel, representing the average fraction of photons making it from the input to the output of the channel. The channel in (8.1) is entanglement-breaking when  $(1 - \eta) N_B \geq \eta$  [Hol08]. Let  $\mathcal{L}_{\eta, N_B}$  denote this channel, and we make the further abbreviation  $\mathcal{L}_\eta \equiv \mathcal{L}_{\eta, N_B=0}$  when it is the pure-loss channel.

The channel in (8.2) is an amplifier channel, and the parameter  $G \geq 1$  is its gain. For this channel, the environment is prepared in the thermal state  $\theta(N_B)$ . If  $N_B = 0$ , the amplifier channel is said to be quantum-limited for a similar reason as stated above. The channel in (8.2) is entanglement-breaking when  $(G - 1) N_B \geq 1$  [Hol08]. Let  $\mathcal{A}_{G, N_B}$  denote this channel, and we make the further abbreviation  $\mathcal{A}_G \equiv \mathcal{A}_{G, N_B=0}$  when it is the quantum-limited amplifier channel.

Finally, the channel in (8.3) is an additive noise channel, representing a quantum generalization of the classical additive white Gaussian noise channel. In (8.3),  $x$  and  $p$  are zero-mean Gaussian random variables each having variance  $\xi \geq 0$ . The channel in (8.3) is entanglement-breaking when  $\xi \geq 1$  [Hol08]. Let  $\mathcal{W}_\xi$  denote this channel. Kraus representations for the channels in (8.1)–(8.3) are available in [ISS11].

For our purposes, it suffices to consider the three kinds of channels given above. All other phase-insensitive Gaussian channels are entanglement-breaking, and they thus have their private and quantum communication abilities severely limited as discussed in Section 7.3. Figure 1 in [GGPCH14] is helpful for understanding the various phase-insensitive quantum Gaussian channels (the channels given in (8.1)–(8.3) all fall within the white-shaded area in that figure).

Recently, the authors of [PLOB16] presented several upper bounds on the two-way assisted private capacities of these channels:

$$P_{\leftrightarrow}(\mathcal{L}_{\eta, N_B}) \leq -\log((1 - \eta) \eta^{N_B}) - g(N_B), \quad (8.5)$$

$$P_{\leftrightarrow}(\mathcal{A}_{G, N_B}) \leq \log\left(\frac{G^{N_B+1}}{G - 1}\right) - g(N_B), \quad (8.6)$$

$$P_{\leftrightarrow}(\mathcal{W}_\xi) \leq \frac{\xi - 1}{\ln 2} - \log \xi, \quad (8.7)$$

where

$$g(x) \equiv (x + 1) \log(x + 1) - x \log x \quad (8.8)$$

is the quantum entropy of a thermal state with mean photon number  $x \geq 0$ . If the channels are entanglement breaking (specific parameter values discussed above), then the upper bound can be taken as zero.

When considering capacities of communication, it is a common convention to impose an energy constraint on the modes being input to the channel uses. This constraint acknowledges the practical reality that a communication scheme cannot consume an arbitrary amount of energy in any given protocol. However, it is known that the quantum and private capacities of quantum Gaussian channels are bounded even if an infinite amount of energy is available [HW01, TGW14b] (this is in contrast to the classical capacity [HW01]). Thus, we can consider both the unconstrained capacity (with no constraint on energy consumption) and the constrained capacity in these scenarios. Note that the bounds in (8.5)–(8.7) apply to both the constrained and unconstrained capacities.

Regarding lower bounds on the capacities, an achievable rate for both quantum and private data transmission is given by the reverse coherent information (as discussed in Section 6.2):

$$I(B\rangle A)_\omega \equiv H(A)_\omega - H(AB)_\omega, \quad (8.9)$$

where  $\omega_{AB} \equiv \mathcal{N}_{A' \rightarrow B}(\psi_{AA'})$ . Alternatively, the coherent information  $I(A\rangle B)_\omega$  is an achievable rate as discussed in Section 6.2. Evaluating these quantities for the channels of interest in (8.1)–(8.3) leads to the following lower bounds on the unconstrained two-way assisted quantum capacities [PGPBL09]:

$$-\log(1 - \eta) - g(N_B) \leq Q_{\leftrightarrow}(\mathcal{L}_{\eta, N_B}), \quad (8.10)$$

$$\log\left(\frac{G}{G - 1}\right) - g(N_B) \leq Q_{\leftrightarrow}(\mathcal{A}_{G, N_B}), \quad (8.11)$$

$$-1/\ln 2 - \log \xi \leq Q_{\leftrightarrow}(\mathcal{W}_\xi). \quad (8.12)$$

For the pure-loss and quantum-limited amplifier channels, we thus have an exact characterization of their unconstrained capacities [PLOB16]:

$$Q_{\leftrightarrow}(\mathcal{L}_\eta) = P_{\leftrightarrow}(\mathcal{L}_\eta) = -\log(1 - \eta), \quad (8.13)$$

$$Q_{\leftrightarrow}(\mathcal{A}_G) = P_{\leftrightarrow}(\mathcal{A}_G) = \log\left(\frac{G}{G - 1}\right). \quad (8.14)$$

The following theorem refines the upper bounds in (8.5)–(8.7), which in turn establishes the rates as strong converse rates and solidifies the claims of [PLOB16]:

**Theorem 24** *Let  $V_{\mathcal{L}_{\eta, N_B}}$ ,  $V_{\mathcal{A}_{G, N_B}}$ , and  $V_{\mathcal{W}_\xi}$  be the unconstrained relative entropy variances of the thermalizing, amplifier, and additive-noise channels, respectively:*

$$V_{\mathcal{L}_{\eta, N_B}} \equiv N_B(N_B + 1) \log^2(\eta [N_B + 1]/N_B), \quad (8.15)$$

$$V_{\mathcal{A}_{G, N_B}} \equiv N_B(N_B + 1) \log^2(G^{-1} [N_B + 1]/N_B), \quad (8.16)$$

$$V_{\mathcal{W}_\xi} \equiv (1 - \xi)^2 / \ln^2 2. \quad (8.17)$$

The following converse bounds hold for all  $\varepsilon \in (0, 1)$ ,  $n \geq 1$ , and  $N_B > 0$ :

$$\hat{P}_{\mathcal{L}_{\eta, N_B}}^{\leftrightarrow}(n, \varepsilon) \leq -\log((1 - \eta) \eta^{N_B}) - g(N_B) + \sqrt{\frac{2V_{\mathcal{L}_{\eta, N_B}}}{n(1 - \varepsilon)}} + \frac{C(\varepsilon)}{n}, \quad (8.18)$$

$$\hat{P}_{\mathcal{A}_{G, N_B}}^{\leftrightarrow}(n, \varepsilon) \leq \log\left(\frac{G^{N_B+1}}{G - 1}\right) - g(N_B) + \sqrt{\frac{2V_{\mathcal{A}_{G, N_B}}}{n(1 - \varepsilon)}} + \frac{C(\varepsilon)}{n}, \quad (8.19)$$

$$\hat{P}_{\mathcal{W}_\xi}^{\leftrightarrow}(n, \varepsilon) \leq \frac{\xi - 1}{\ln 2} - \log \xi + \sqrt{\frac{2V_{\mathcal{W}_\xi}}{n(1 - \varepsilon)}} + \frac{C(\varepsilon)}{n}, \quad (8.20)$$

where  $C(\varepsilon) \equiv \log 6 + 2 \log ([1 + \varepsilon] / [1 - \varepsilon])$ . For the pure-loss channel  $\mathcal{L}_\eta$  and quantum-limited amplifier channel  $\mathcal{A}_G$ , the following bounds hold

$$\hat{P}_{\mathcal{L}_\eta}^{\leftrightarrow}(n, \varepsilon) \leq -\log(1 - \eta) + \frac{C(\varepsilon)}{n}, \quad (8.21)$$

$$\hat{P}_{\mathcal{A}_G}^{\leftrightarrow}(n, \varepsilon) \leq \log\left(\frac{G}{G - 1}\right) + \frac{C(\varepsilon)}{n}. \quad (8.22)$$

**Proof.** We will argue a proof of the bound (8.18) for the thermalizing channel, then for the pure-loss channel, and finally the other bounds will follow from similar reasoning. First, consider an arbitrary  $(n, P^{\leftrightarrow}, \varepsilon)$  protocol for the thermalizing channel  $\mathcal{L}_{\eta, N_B}$ . It consists of using the channel  $n$  times and interleaving rounds of LOCC between every channel use. Let  $\zeta_{\hat{A}\hat{B}}^n$  denote the final state of Alice and Bob at the end of this protocol.

By the teleportation reduction procedure from [BDSW96, Section V] and [NFC09] (see also the review in [PLOB16]), such a protocol can be simulated by preparing  $n$  two-mode squeezed vacuum (TMSV) states each having energy  $\mu - 1/2$  (where we think of  $\mu \geq 1/2$  as a very large positive real), sending one mode of each TMSV through each channel use, and then performing continuous-variable quantum teleportation [BK98] to delay all of the LOCC operations until the end of the protocol. Let  $\rho_{\eta, N_B}^\mu$  denote the state resulting from sending one share of the TMSV through the thermalizing channel, and let  $\zeta'_{\hat{A}\hat{B}}(n, \mu)$  denote the state at the end of the simulation. Let  $\varepsilon_{\text{TP}}(n, \mu)$  denote the “infidelity” of the simulation:

$$\varepsilon_{\text{TP}}(n, \mu) \equiv 1 - F(\zeta_{\hat{A}\hat{B}}^n, \zeta'_{\hat{A}\hat{B}}(n, \mu)). \quad (8.23)$$

Due to the fact that continuous-variable teleportation induces a perfect quantum channel when infinite energy is available [BK98], the following limit holds for every  $n$ :

$$\limsup_{\mu \rightarrow \infty} \varepsilon_{\text{TP}}(n, \mu) = 0. \quad (8.24)$$

Note also that  $\varepsilon_{\text{TP}}(n, \mu)$  is a monotone non-increasing function of  $\mu$ , because the fidelity of continuous-variable teleportation increases with increasing energy [BK98]. By using that  $\sqrt{1 - F(\rho, \sigma)}$  is a distance measure for states  $\rho$  and  $\sigma$  (and thus obeys a triangle inequality) [GLN05], the simulation leads to an  $(n, P^{\leftrightarrow}, \varepsilon(n, \mu))$  protocol for the thermalizing channel, where

$$\varepsilon(n, \mu) \equiv \min\left\{1, \left[\sqrt{\varepsilon} + \sqrt{\varepsilon_{\text{TP}}(n, \mu)}\right]^2\right\}. \quad (8.25)$$

Observe that  $\limsup_{\mu \rightarrow \infty} \varepsilon(n, \mu) = \varepsilon$ , so that the simulated protocol has equivalent performance to the original protocol in the infinite-energy limit. However, the simulated protocol has a simpler form than the original one, and since this procedure can be conducted for any  $(n, P^{\leftrightarrow}, \varepsilon)$  protocol for the channel  $\mathcal{L}_{\eta, N_B}$ , we find that the following bound applies by invoking reasoning similar to that needed to arrive at (4.34):

$$\hat{P}_{\mathcal{L}_{\eta, N_B}}^{\leftrightarrow}(n, \varepsilon) \leq \hat{P}_{\rho_{\eta, N_B}^\mu}^{\leftrightarrow}(n, \varepsilon(n, \mu)) \quad (8.26)$$

$$\leq \frac{1}{n} E_R^{\varepsilon(n, \mu)}(A^n; B^n)_{(\rho_{\eta, N_B}^\mu)^{\otimes n}} \quad (8.27)$$

$$\leq \frac{1}{n} D_H^{\varepsilon(n, \mu)}((\rho_{\eta, N_B}^\mu)^{\otimes n} \| (\sigma_{\eta, N_B}^\mu)^{\otimes n}), \quad (8.28)$$

where  $\hat{P}_{\rho_{\eta,N_B}^\mu}^{\leftrightarrow}(n, \varepsilon(n, \mu))$  denotes the distillable key using  $n$  copies of  $\rho_{\eta,N_B}^\mu$  up to error  $\varepsilon(n, \mu)$  and the state  $\sigma_{\eta,N_B}^\mu$  is a particular separable state chosen as in (D.6) (in Appendix D). By applying the Chebyshev-like bound given in Appendix E, we find that

$$\frac{1}{n} D_H^{\varepsilon(n, \mu)}((\rho_{\eta,N_B}^\mu)^{\otimes n} \| (\sigma_{\eta,N_B}^\mu)^{\otimes n}) \leq D(\rho_{\eta,N_B}^\mu \| \sigma_{\eta,N_B}^\mu) + \sqrt{\frac{2V(\rho_{\eta,N_B}^\mu \| \sigma_{\eta,N_B}^\mu)}{n(1 - \varepsilon(n, \mu))}} + C(\varepsilon(n, \mu))/n. \quad (8.29)$$

Thus, the following bound holds for every  $\mu$  sufficiently large (so that  $\varepsilon(n, \mu) \in (0, 1)$ ):

$$\hat{P}_{\mathcal{L}_{\eta,N_B}}^{\leftrightarrow}(n, \varepsilon) \leq D(\rho_{\eta,N_B}^\mu \| \sigma_{\eta,N_B}^\mu) + \sqrt{\frac{2V(\rho_{\eta,N_B}^\mu \| \sigma_{\eta,N_B}^\mu)}{n(1 - \varepsilon(n, \mu))}} + C(\varepsilon(n, \mu))/n. \quad (8.30)$$

From the developments in [PLOB16] and Appendix D, we have the following expansions about  $\mu = \infty$ :

$$D(\rho_{\eta,N_B}^\mu \| \sigma_{\eta,N_B}^\mu) = -\log((1 - \eta)\eta^{N_B}) - g(N_B) + O(\mu^{-1}), \quad (8.31)$$

$$V(\rho_{\eta,N_B}^\mu \| \sigma_{\eta,N_B}^\mu) = V_{\mathcal{L}_{\eta,N_B}} + O(\mu^{-1}). \quad (8.32)$$

We can then take the limit in (8.30) as  $\mu \rightarrow \infty$  to conclude the bound stated in (8.18).

To recover the other bounds in (8.19) and (8.20), we apply the same reasoning as above but instead use the infinite-energy expansions of the relative entropy and the relative entropy variance given in [PLOB16] and Appendix D, respectively.

The bound in (8.21) for the pure-loss channel follows similarly but requires a careful argument with appropriate limits because the propositions in Appendix E are stated to hold only for faithful states (i.e., positive-definite states). The main idea of the proof below is to apply the following statement given in [DTW14]: for  $\rho', \rho, \sigma \in \mathcal{D}(\mathcal{H})$ ,  $\varepsilon \in (0, 1)$ , and  $\delta \in (0, 1 - \varepsilon)$ ,

$$\frac{1}{2} \|\rho' - \rho\|_1 \leq \delta \quad \Rightarrow \quad D_H^\varepsilon(\rho \| \sigma) \leq D_H^{\varepsilon+\delta}(\rho' \| \sigma). \quad (8.33)$$

To this end, we can now repeat the reasoning from above for the pure-loss channel  $\mathcal{L}_\eta$ . Let  $\rho_\eta^\mu$  be the state arising from sending one mode of a TMSV through the pure-loss channel  $\mathcal{L}_\eta$ . This state is  $f(\delta_n)$ -close in trace distance to a faithful state  $\rho_{\eta, \delta_n}^\mu$  that would result from sending one share of a TMSV through a thermalizing channel  $\mathcal{L}_{\eta, \delta_n}$ , where  $\delta_n > 0$  is a tunable parameter that we will eventually take to zero at the end of the argument and the function  $f$  is such that  $\lim_{\delta_n \rightarrow 0} f(\delta_n) = 0$ . We can choose  $\delta_n$  small enough and the energy  $\mu$  large enough such that  $\varepsilon(n, \mu) + f(\delta_n) \in (0, 1)$  for a given  $n$ . Proceeding as in (8.28), we pick  $\sigma_{\eta, \delta_n}^\mu$  to be a separable state “tuned” for  $\rho_{\eta, \delta_n}^\mu$ , chosen as in (D.6) (in Appendix D), and we find the following for all  $n \geq 1$ , for all  $\mu$  sufficiently large, and  $\delta_n$  small enough:

$$\hat{P}_{\mathcal{L}_\eta}^{\leftrightarrow}(n, \varepsilon) \leq \hat{P}_{\rho_\eta^\mu}^{\leftrightarrow}(n, \varepsilon(n, \mu)) \quad (8.34)$$

$$\leq \frac{1}{n} E_R^{\varepsilon(n, \mu)}(A^n; B^n)_{(\rho_\eta^\mu)^{\otimes n}} \quad (8.35)$$

$$\leq \frac{1}{n} D_H^{\varepsilon(n, \mu)}((\rho_\eta^\mu)^{\otimes n} \| (\sigma_{\eta, \delta_n}^\mu)^{\otimes n}) \quad (8.36)$$

$$\leq \frac{1}{n} D_H^{\varepsilon(n, \mu) + f(\delta_n)}((\rho_{\eta, \delta_n}^\mu)^{\otimes n} \| (\sigma_{\eta, \delta_n}^\mu)^{\otimes n}) \quad (8.37)$$

$$\leq D(\rho_{\eta, \delta_n}^\mu \| \sigma_{\eta, \delta_n}^\mu) + \sqrt{\frac{2V(\rho_{\eta, \delta_n}^\mu \| \sigma_{\eta, \delta_n}^\mu)}{n(1 - \varepsilon(n, \mu) - \delta_n)}} + C(\varepsilon(n, \mu) + \delta_n)/n. \quad (8.38)$$

The first inequality follows from the teleportation simulation argument. The second follows by invoking reasoning similar to that needed to arrive at (4.34). The third inequality follows by picking the separable state in  $E_R^{\varepsilon(n,\mu)}$  to be  $(\sigma_{\eta,\delta_n}^\mu)^{\otimes n}$ . The fourth inequality follows from (8.33). The last inequality follows by applying the Chebyshev-like bound given in Appendix E. We can now take the limit as  $\mu \rightarrow \infty$ , applying (8.31)–(8.32), and find that

$$\hat{P}_{\mathcal{L}_\eta}^{\leftrightarrow}(n, \varepsilon) \leq -\log((1-\eta)\eta^{\delta_n}) - g(\delta_n) + \sqrt{\frac{2V_{\mathcal{L}_\eta, \delta_n}}{n(1-\varepsilon-\delta_n)}} + C(\varepsilon+\delta_n)/n. \quad (8.39)$$

Since the above bound holds for all sufficiently small  $\delta_n$ , we can now take the limit as  $\delta_n \rightarrow 0$ , and find the following bound:

$$\hat{P}_{\mathcal{L}_\eta}^{\leftrightarrow}(n, \varepsilon) \leq -\log(1-\eta) + C(\varepsilon)/n, \quad (8.40)$$

which follows because

$$\lim_{\delta_n \rightarrow 0} \left[ -\log((1-\eta)\eta^{\delta_n}) - g(\delta_n) \right] = -\log(1-\eta), \quad (8.41)$$

$$\lim_{\delta_n \rightarrow 0} V_{\mathcal{L}_\eta, \delta_n} = 0. \quad (8.42)$$

Similar reasoning applies to get the bound in (8.22) for the quantum-limited amplifier channel. ■

**Corollary 25** *As a consequence of Theorem 24, the following bounds hold*

$$P_{\leftrightarrow}^{\dagger}(\mathcal{L}_{\eta, N_B}) \leq -\log((1-\eta)\eta^{N_B}) - g(N_B), \quad (8.43)$$

$$P_{\leftrightarrow}^{\dagger}(\mathcal{A}_{G, N_B}) \leq \log\left(\frac{G^{N_B+1}}{G-1}\right) - g(N_B), \quad (8.44)$$

$$P_{\leftrightarrow}^{\dagger}(\mathcal{W}_\xi) \leq \frac{\xi-1}{\ln 2} - \log \xi, \quad (8.45)$$

establishing the upper bounds in (8.5)–(8.7) as strong converse rates for the constrained and unconstrained private and quantum capacities of these channels. Furthermore, by taking the same limit in (8.21)–(8.22) and by combining with the statements in (8.13)–(8.14), we can conclude that the unconstrained, two-way assisted private and quantum capacities of the pure-loss and quantum-limited amplifier channels satisfy the strong converse property:

$$Q_{\leftrightarrow}(\mathcal{L}_\eta) = Q_{\leftrightarrow}^{\dagger}(\mathcal{L}_\eta) = P_{\leftrightarrow}(\mathcal{L}_\eta) = P_{\leftrightarrow}^{\dagger}(\mathcal{L}_\eta) = -\log(1-\eta), \quad (8.46)$$

$$Q_{\leftrightarrow}(\mathcal{A}_G) = Q_{\leftrightarrow}^{\dagger}(\mathcal{A}_G) = P_{\leftrightarrow}(\mathcal{A}_G) = P_{\leftrightarrow}^{\dagger}(\mathcal{A}_G) = \log\left(\frac{G}{G-1}\right). \quad (8.47)$$

We note that the corollary above improves upon the upper bound on  $Q_{\leftrightarrow}^{\dagger}(\mathcal{L}_\eta)$  and  $Q_{\leftrightarrow}^{\dagger}(\mathcal{A}_G)$  that one gets by applying the transposition bound [HW01, MHRW16] (in fact there cannot be any further improvements of the result stated above due to the equalities).

## 9 Conclusion

We have outlined a general approach for obtaining converse bounds for the private transmission capabilities of a quantum channel, which builds strongly on the notion of a private state [HHHO05,

HHHO09] and the relative entropy of entanglement bound therein. We first obtained a general meta-converse bound and then applied it to obtain strong converse and second-order bounds for private communication, building upon the methods of [TWW17] and [TBR16]. For several channels of interest, we can go a step further and give precise characterizations, as was done in [TBR16]. Notable examples include the phase-insensitive bosonic channels as well. In particular, we have established the strong converse property for the unconstrained private and quantum capacities of the pure-loss and quantum-limited amplifier channels, in addition to some converse bounds for more general phase-insensitive bosonic channels. Several of these bounds are relevant for understanding the limitations of quantum key distribution protocols performed over such channels.

Going forward from here, it is desirable to obtain a second-order expansion of the achievability results from [Dev05, CWY04]. Progress in the classical case is available in [Tan12, YSP16], but the problem seems generally open there as well. It is also known that there are quantum channels which have zero quantum capacity but non-zero private capacity [HHH<sup>+</sup>08a, HHH<sup>+</sup>08b], and a second-order analysis might further elucidate this phenomenon. It would be interesting as well to prove that the squashed entanglement of a quantum channel  $\mathcal{N}$  is an upper bound on  $Q_{\leftrightarrow}^{\dagger}(\mathcal{N})$  and  $P_{\leftrightarrow}^{\dagger}(\mathcal{N})$  (this question has remained open since [TGW14b]).

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## A Covariant channels are teleportation simulable

In this appendix, we give a proof of Proposition 2: any covariant channel, as defined in Definition 1, is teleportation simulable.

Let  $\mathcal{N} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$  be a quantum channel, and let  $G$  be a group with unitary representations  $U_A^g$  and  $V_B^g$  for  $g \in G$ , such that

$$\frac{1}{|G|} \sum_g U_A^g X_A (U_A^g)^\dagger = \text{Tr}\{X_A\} \pi_A, \quad (\text{A.1})$$

$$\mathcal{N}_{A \rightarrow B}(U_A^g X_A (U_A^g)^\dagger) = V_B^g \mathcal{N}_{A \rightarrow B}(X_A) (V_B^g)^\dagger, \quad (\text{A.2})$$

where  $X_A \in \mathcal{L}(\mathcal{H}_A)$  and  $\pi$  denotes the maximally mixed state. Consider that

$$\frac{1}{|G|} \sum_g U_{A'}^g \Phi_{A'A} (U_{A'}^g)^\dagger = \pi_{A'} \otimes \pi_A, \quad (\text{A.3})$$

where  $\Phi$  denotes a maximally entangled state and  $A'$  is a system isomorphic to  $A$ . Note that in order for  $\{U_A^g\}$  to satisfy (A.1), it is necessary that  $|A|^2 \leq |G|$  [AMTdW00]. Consider the POVM  $\{E_{A'A}^g\}_g$ , with  $A'$  a system isomorphic to  $A$  and each element  $E_{A'A}^g$  defined as

$$E_{A'A}^g \equiv \frac{|A|^2}{|G|} U_{A'}^g \Phi_{A'A} (U_{A'}^g)^\dagger. \quad (\text{A.4})$$

It follows from the fact that  $|A|^2 \leq |G|$  and (A.3) that  $\{E_{A'A}^g\}_g$  is a valid POVM.

The simulation of the channel  $\mathcal{N}_{A \rightarrow B}$  via teleportation begins with a state  $\rho_{A'}$  and a shared resource  $\omega_{AB} \equiv \mathcal{N}_{A'' \rightarrow B}(\Phi_{AA''})$ . The desired outcome is for Bob to receive the state  $\mathcal{N}_{A \rightarrow B}(\rho_A)$  and for the protocol to work independently of the input state  $\rho_A$ . The first step is for Alice to perform the measurement  $\{E_{A'A}^g\}_g$  on systems  $A'A$  and then send the outcome  $g$  to Bob. Based on the outcome  $g$ , Bob then performs  $V_B^g$ . The following analysis demonstrates that this protocol works, by simplifying the form of the post-measurement state:

$$|G| \text{Tr}_{AA'} \{E_{A'A}^g (\rho_{A'} \otimes \omega_{AB})\} = |A|^2 \text{Tr}_{AA'} \{U_{A'}^g |\Phi\rangle_{A'A} \langle \Phi|_{A'A} (U_{A'}^g)^\dagger (\rho_{A'} \otimes \omega_{AB})\} \quad (\text{A.5})$$

$$= |A|^2 \langle \Phi|_{A'A} (U_{A'}^g)^\dagger (\rho_{A'} \otimes \omega_{AB}) U_{A'}^g |\Phi\rangle_{A'A} \quad (\text{A.6})$$

$$= |A|^2 \langle \Phi|_{A'A} (U_{A'}^g)^\dagger \rho_{A'} U_{A'}^g \otimes \mathcal{N}_{A'' \rightarrow B}(\Phi_{AA''}) |\Phi\rangle_{A'A} \quad (\text{A.7})$$

$$= |A|^2 \langle \Phi|_{A'A} \left[ (U_A^g)^\dagger \rho_A U_A^g \right]^* \mathcal{N}_{A'' \rightarrow B}(\Phi_{AA''}) |\Phi\rangle_{A'A}. \quad (\text{A.8})$$

The first three equalities follow by substitution and some rewriting. The fourth equality follows from the fact that

$$\langle \Phi|_{A'A} M_{A'} = \langle \Phi|_{A'A} M_A^* \quad (\text{A.9})$$

for any operator  $M$  and where  $*$  denotes the complex conjugate, taken with respect to the basis in which  $|\Phi\rangle_{A'A}$  is defined. Continuing, we have that

$$(A.8) = |A| \text{Tr}_A \left\{ \left[ (U_A^g)^\dagger \rho_A U_A^g \right]^* \mathcal{N}_{A'' \rightarrow B}(\Phi_{AA''}) \right\} \quad (\text{A.10})$$

$$= |A| \text{Tr}_A \left\{ \mathcal{N}_{A'' \rightarrow B} \left( \left[ (U_{A''}^g)^\dagger \rho_{A''} U_{A''}^g \right]^\dagger \Phi_{AA''} \right) \right\} \quad (\text{A.11})$$

$$= \mathcal{N}_{A'' \rightarrow B} \left( \left[ (U_{A''}^g)^\dagger \rho_{A''} U_{A''}^g \right]^\dagger \right) \quad (\text{A.12})$$

$$= \mathcal{N}_{A'' \rightarrow B} \left( (U_{A''}^g)^\dagger \rho_{A''} U_{A''}^g \right) \quad (\text{A.13})$$

$$= (V_B^g)^\dagger \mathcal{N}_{A'' \rightarrow B}(\rho_{A''}) V_B^g. \quad (\text{A.14})$$

The first equality follows because  $|A| \langle \Phi|_{A'A} (I_{A'} \otimes M_{AB}) |\Phi\rangle_{A'A} = \text{Tr}_A \{M_{AB}\}$  for any operator  $M_{AB}$ . The second equality follows by applying the conjugate transpose of (A.9). The final equality follows from the covariance property of the channel.

Thus, if Bob finally performs the unitary  $V_B^g$  upon receiving  $g$  via a classical channel from Alice, then the output of the protocol is  $\mathcal{N}_{A'' \rightarrow B}(\rho_{A''})$ , so that this protocol simulates the action of the channel  $\mathcal{N}$  on the state  $\rho$ .

## B Definitions of privacy and converse bounds

One of the main results of this appendix is to show that a converse bound with the definition of privacy from (3.3) implies a converse bound for a quantum generalization of the definition of privacy from [HTW14]. Note that we need to modify the definition in (3.3) slightly as given in Definition 27 below, but all of our converse bounds apply for this modified definition of privacy.

We begin by recalling the two notions of privacy. Consider a tripartite state  $\rho_{KLE}$  of the following form:

$$\rho_{KLE} = \frac{1}{|K|} \sum_{k,l} p(l|k) |k\rangle\langle k|_K \otimes |l\rangle\langle l|_L \otimes \rho_E^{k,l}, \quad (\text{B.1})$$

which is the kind of state that gets generated at the end of a secret-key transmission protocol. Specifically, the classical variable in system  $K$  is generated uniformly at random, and at the end of the protocol the receiver decodes it in the system  $L$ , which is intended to be one share of a secret key correlated with system  $K$ . The system  $E$  represents the eavesdropper's system, which can be correlated with systems  $K$  and  $L$ .

**Definition 26** *The state  $\rho_{KLE}$  is an  $(\varepsilon, \delta)$  Type I secret-key state for  $\varepsilon, \delta \in [0, 1]$  if*

$$\Pr\{K \neq L\} \leq \varepsilon, \quad (\text{B.2})$$

$$\frac{1}{2} \|\rho_{KE} - \pi_K \otimes \rho_E\|_1 \leq \delta, \quad (\text{B.3})$$

where  $\Pr\{K \neq L\} = \frac{1}{|K|} \sum_{l \neq k} p(l|k)$  and  $\pi_K = I_K / |K|$  is the maximally mixed state.

**Definition 27** *The state  $\rho_{KLE}$  is an  $\eta$  Type II secret-key state for  $\eta \in [0, 1]$  if*

$$F(\rho_{KLE}, \bar{\Phi}_{KL} \otimes \rho_E) = 1 - \eta, \quad (\text{B.4})$$

where  $\bar{\Phi}_{KL}$  is the maximally classically correlated state, defined as

$$\bar{\Phi}_{KL} \equiv \frac{1}{|K|} \sum_k |k\rangle\langle k|_K \otimes |k\rangle\langle k|_L. \quad (\text{B.5})$$

Observe that the main difference between Definitions 3 and 27 is that in the former, we allow for the ideal state of the eavesdropper's system to be arbitrary, whereas in the latter, we demand that the ideal state of the eavesdropper's system is equal to the marginal  $\rho_E$  of  $\rho_{KLE}$ . Note that this constraint does not affect any of our converse bounds in the main text, but here we show how it allows us to connect to other notions of privacy in the literature.

**Proposition 28** *For the definitions given above, the following bound holds*

$$1 - \sqrt{F(\rho_{KLE}, \bar{\Phi}_{KL} \otimes \rho_E)} \leq \Pr\{K \neq L\} + \frac{1}{2} \|\rho_{KE} - \pi_K \otimes \rho_E\|_1. \quad (\text{B.6})$$

Thus, an  $\eta$  Type II secret-key state with  $\eta \rightarrow 1$  is an  $(\varepsilon, \delta)$  Type I secret-key state with  $\varepsilon + \delta \rightarrow \xi \geq 1$ .

**Proof.** We follow the proof of [PR14, Theorem 4.1] closely. Let  $\gamma_{KLE}$  denote the following state obtained by discarding the  $L$  system of  $\rho_{KLE}$  and copying the contents of the  $K$  system to the  $L$  system:

$$\gamma_{KLE} \equiv \frac{1}{|K|} \sum_{k,l} p(l|k) |k\rangle\langle k|_K \otimes |k\rangle\langle k|_L \otimes \rho_E^{k,l}. \quad (\text{B.7})$$

Then by the triangle inequality consider that

$$\|\rho_{KLE} - \bar{\Phi}_{KL} \otimes \rho_E\|_1 \leq \|\rho_{KLE} - \gamma_{KLE}\|_1 + \|\gamma_{KLE} - \bar{\Phi}_{KL} \otimes \rho_E\|_1. \quad (\text{B.8})$$

The following holds

$$\|\gamma_{KLE} - \bar{\Phi}_{KL} \otimes \rho_E\|_1 = \|\gamma_{KE} \otimes |0\rangle\langle 0|_L - \pi_K \otimes |0\rangle\langle 0|_L \otimes \rho_E\|_1 \quad (\text{B.9})$$

$$= \|\gamma_{KE} - \pi_K \otimes \rho_E\|_1, \quad (\text{B.10})$$

because we can perform an invertible controlled-subtraction from system  $K$  to system  $L$ , giving the first equality, and then we can discard the system  $L$  because it does not change the trace distance. Now consider that

$$\|\rho_{KLE} - \gamma_{KLE}\|_1 = \left\| \frac{1}{|K|} \sum_{k,l} p(l|k) |k\rangle\langle k|_K \otimes (|l\rangle\langle l|_L - |k\rangle\langle k|_L) \otimes \rho_E^{k,l} \right\|_1 \quad (\text{B.11})$$

$$\leq \frac{1}{|K|} \sum_{k,l} p(l|k) \left\| |k\rangle\langle k|_K \otimes (|l\rangle\langle l|_L - |k\rangle\langle k|_L) \otimes \rho_E^{k,l} \right\|_1 \quad (\text{B.12})$$

$$= \frac{1}{|K|} \sum_{k,l} p(l|k) \| |l\rangle\langle l|_L - |k\rangle\langle k|_L \|_1 \quad (\text{B.13})$$

$$= \frac{2}{|K|} \sum_{k \neq l} p(l|k) = 2 \Pr\{K \neq L\}, \quad (\text{B.14})$$

where the inequality follows from convexity of the trace norm. Using a well known relation between fidelity and trace distance [FvdG98] and combining with the above, we find that

$$1 - \sqrt{F(\rho_{KLE}, \bar{\Phi}_{KL} \otimes \rho_E)} \leq \frac{1}{2} \|\rho_{KLE} - \bar{\Phi}_{KL} \otimes \rho_E\|_1 \quad (\text{B.15})$$

$$\leq \Pr\{K \neq L\} + \frac{1}{2} \|\rho_{KE} - \pi_K \otimes \rho_E\|_1, \quad (\text{B.16})$$

concluding the proof. ■

As a consequence of the above proposition, if there is a sequence of private communication protocols such that  $\eta \rightarrow 1$ , then by the bound in (B.6), we necessarily have that  $\varepsilon + \delta \rightarrow \xi \geq 1$ . So our approach in Section 5 gets strong converse rates for all  $\varepsilon$  and  $\delta$  such that  $\varepsilon + \delta < 1$ , which is the same regime for which the authors of [HTW14] were able to establish strong converse rates for the classical wiretap channel.

We can also show the following alternate relation between the two notions of privacy:

**Proposition 29** *For the definitions given above, the following bounds hold*

$$\Pr\{K \neq L\} \leq \sqrt{1 - F(\rho_{KLE}, \bar{\Phi}_{KL} \otimes \rho_E)}, \quad (\text{B.17})$$

$$\frac{1}{2} \|\rho_{KE} - \pi_K \otimes \rho_E\|_1 \leq \sqrt{1 - F(\rho_{KLE}, \bar{\Phi}_{KL} \otimes \rho_E)}. \quad (\text{B.18})$$

Thus, an  $\eta$  Type II secret-key state is an  $(\sqrt{\eta}, \sqrt{\eta})$  Type I secret-key state.

**Proof.** The inequality in (B.17) follows because

$$\Pr\{K \neq L\} = \frac{1}{2} \|\rho_{KL} - \bar{\Phi}_{KL}\|_1 \quad (\text{B.19})$$

$$\leq \sqrt{1 - F(\rho_{KL}, \bar{\Phi}_{KL})} \quad (\text{B.20})$$

$$\leq \sqrt{1 - F(\rho_{KLE}, \bar{\Phi}_{KL} \otimes \rho_E)}. \quad (\text{B.21})$$

In the above, the first equality is well known and straightforward to verify. The first inequality is a consequence of a well known relation between fidelity and trace distance [FvdG98]. The second inequality follows from the monotonicity of fidelity. The inequality in (B.18) follows for similar reasons, because

$$\frac{1}{2} \|\rho_{KE} - \pi_K \otimes \rho_E\|_1 \leq \sqrt{1 - F(\rho_{KE}, \bar{\Phi}_K \otimes \rho_E)} \quad (\text{B.22})$$

$$\leq \sqrt{1 - F(\rho_{KLE}, \bar{\Phi}_{KL} \otimes \rho_E)}, \quad (\text{B.23})$$

which concludes the proof. ■

## C One-shot distillable entanglement lower bound

Here we give a proof of Proposition 21, regarding a lower bound on one-shot distillable entanglement. Let  $\rho_{ABE} \in \mathcal{D}(\mathcal{H}_{ABE})$  purify the state  $\rho_{AE}$ . Let  $\mathcal{T}_{A \rightarrow A_1 X_A}$  be the following quantum channel corresponding to a quantum instrument:

$$\mathcal{T}_{A \rightarrow A_1 X_A}(\cdot) \equiv \sum_x \mathcal{V}_{A \rightarrow A_1}^x (P_A^x(\cdot) P_A^x) \otimes |x\rangle\langle x|_{X_A}, \quad (\text{C.1})$$

where  $\{P_A^x\}$  is a set of projectors such that  $\sum_x P_A^x = I_A$  and each  $\mathcal{V}_{A \rightarrow A_1}^x$  is an isometric channel that isometrically embeds the subspace onto which  $P_A^x$  projects into  $\mathcal{H}_{A_1}$ . For simplicity, we assume here and in what follows that  $|A_1|$  divides  $|A|$ .

In this paragraph, we prove a “non-smooth” bound and later convert this to a smooth-entropy bound. The non-smooth decoupling theorem of [DBWR14, Theorem 3.3] then states that

$$\int_{\mathbb{U}(A)} dU \left\| \mathcal{T}_{A \rightarrow A_1 X_A}(U_A \rho_{AE} U_A^\dagger) - \tau_{A_1 X_A} \otimes \rho_E \right\|_1 \leq 2^{-\frac{1}{2} H_2(A|E)_\rho} 2^{-\frac{1}{2} H_2(A'|A_1 X_A)_\tau}, \quad (\text{C.2})$$

where  $H_2(C|D)_\sigma$  denotes the conditional collision entropy of a bipartite state  $\sigma_{CD} \in \mathcal{D}(\mathcal{H}_{CD})$ , defined as

$$H_2(C|D)_\sigma \equiv \sup_{\omega_D \in \mathcal{D}(\mathcal{H}_D)} -\log \text{Tr} \left\{ \left( \omega_D^{-1/4} \sigma_{CD} \omega_D^{-1/4} \right)^2 \right\}. \quad (\text{C.3})$$

In the above,  $\rho_{AE}$  is the reduction of  $\rho_{ABE}$ , and  $\tau_{A_1 X_A}$  is the reduction of the following state:

$$\mathcal{T}_{A \rightarrow A_1 X_A}(\Phi_{AA'}) \equiv \tau_{A' A_1 X_A}. \quad (\text{C.4})$$

Note under our assumption that  $|A_1|$  divides  $|A|$ , we have that

$$H_2(A'|A_1X_A)_\tau = -\log |A_1|, \quad (\text{C.5})$$

$$\tau_{A_1X_A} = \frac{I_{A_1}}{|A_1|} \otimes \frac{I_{X_A}}{|X_A|}. \quad (\text{C.6})$$

If we choose  $|A_1|$  as

$$\log |A_1| = H_2(A|E)_\rho - 2 \log \left( \frac{1}{\varepsilon} \right), \quad (\text{C.7})$$

then it follows from the above that

$$\int_{\mathbb{U}(A)} dU \left\| \mathcal{T}_{A \rightarrow A_1X_A}(U_A \rho_{AE} U_A^\dagger) - \tau_{A_1X_A} \otimes \rho_E \right\|_1 \leq \varepsilon. \quad (\text{C.8})$$

This implies that there exists a unitary  $U_A$  such that

$$P \left( \mathcal{T}_{A \rightarrow A_1X_A}(U_A \rho_{AE} U_A^\dagger), \frac{I_{A_1}}{|A_1|} \otimes \frac{I_{X_A}}{|X_A|} \otimes \rho_E \right) \leq \sqrt{\varepsilon}, \quad (\text{C.9})$$

for

$$\log |A_1| = -H_{\max}(A|B)_\rho - 2 \log \left( \frac{1}{\varepsilon} \right), \quad (\text{C.10})$$

due to the fact that

$$H_2(A|E)_\rho \geq H_{\min}(A|E)_\rho = -H_{\max}(A|B)_\rho, \quad (\text{C.11})$$

where

$$H_{\min}(A|E)_\rho \equiv \sup_{\omega_E \in \mathcal{D}(\mathcal{H}_E)} \sup \left\{ \lambda \in \mathbb{R} : \rho_{AE} \leq 2^{-\lambda} I_A \otimes \omega_E \right\}, \quad (\text{C.12})$$

and the equality  $H_{\min}(A|E)_\rho = -H_{\max}(A|B)_\rho$  follows from the duality result in [KRS09]. Let

$$\sigma_{A_1EX_A} \equiv \mathcal{T}_{A \rightarrow A_1X_A}(U_A \rho_{AE} U_A^\dagger) \quad (\text{C.13})$$

$$\equiv \sum_x p_X(x) |x\rangle\langle x|_{X_A} \otimes \sigma_{A_1E}^x. \quad (\text{C.14})$$

By rewriting (C.9) as

$$\left[ \sum_x \sqrt{p_X(x) \frac{1}{|X_A|}} \sqrt{F} \left( \sigma_{A_1E}^x, \frac{I_{A_1}}{|A_1|} \otimes \rho_E \right) \right]^2 \geq 1 - \varepsilon, \quad (\text{C.15})$$

and applying Uhlmann's theorem, we find that there exists a set  $\{\mathcal{U}_{B \rightarrow B_1\bar{B}}^x\}$  of isometric channels such that

$$\left[ \sum_x \sqrt{p_X(x) \frac{1}{|X_A|}} \sqrt{F} \left( \mathcal{U}_{B \rightarrow B_1\bar{B}}^x(\sigma_{A_1BE}^x), \Phi_{A_1B_1} \otimes \psi_{\bar{B}E} \right) \right]^2 \geq 1 - \varepsilon, \quad (\text{C.16})$$

where  $\sigma_{A_1BE}^x$  is a conditional state arising from  $\mathcal{T}_{A \rightarrow A_1X_A}(U_A \rho_{ABE} U_A^\dagger)$  and  $\psi_{\bar{B}E}$  purifies  $\rho_E$ . Using the monotonicity of fidelity under partial trace and rewriting (C.16) in terms of purified distance,

we can conclude that there exists a channel  $\Lambda_{BX_B \rightarrow B_1}$ , where  $X_B$  is a classical copy of  $X_A$  sent over a classical channel to Bob, such that

$$P(\Lambda_{BX_B \rightarrow B_1}(\sigma_{A_1 BX_B}), \Phi_{A_1 B_1}) \leq \sqrt{\varepsilon}. \quad (\text{C.17})$$

Now pick  $\bar{\rho}_{AB} \in \mathcal{B}^{\sqrt{\varepsilon}-\eta}(\rho_{AB})$  such that  $H_{\max}^{\sqrt{\varepsilon}-\eta}(A|B)_{\rho} = H_{\max}(A|B)_{\bar{\rho}}$  (the ball  $\mathcal{B}^{\sqrt{\varepsilon}-\eta}(\rho_{AB})$  of states around  $\rho_{AB}$  is with respect to purified distance). Let  $\bar{\rho}_{ABE}$  purify  $\bar{\rho}_{AB}$ . Then by the non-smooth bound above, we find that

$$P\left(\bar{\sigma}_{A_1 EX_A}, \frac{I_{A_1}}{|A_1|} \otimes \bar{\omega}_{X_A E}\right) \leq \eta, \quad (\text{C.18})$$

for

$$\log |A_1| = -H_{\max}^{\sqrt{\varepsilon}-\eta}(A|B)_{\rho} - 4 \log\left(\frac{1}{\eta}\right), \quad (\text{C.19})$$

and where

$$\bar{\sigma}_{A_1 EX_A} \equiv \mathcal{T}_{A \rightarrow A_1 X_A}(U_A \bar{\rho}_{AE} U_A^\dagger), \quad (\text{C.20})$$

$$\bar{\omega}_{X_A E} \equiv \frac{I_{X_A}}{|X_A|} \otimes \bar{\rho}_E. \quad (\text{C.21})$$

Then

$$\begin{aligned} P\left(\sigma_{A_1 EX_A}, \frac{I_{A_1}}{|A_1|} \otimes \bar{\omega}_{X_A E}\right) &\leq P\left(\mathcal{T}_{A \rightarrow A_1 X_A}(U_A \rho_{AE} U_A^\dagger), \mathcal{T}_{A \rightarrow A_1 X_A}(U_A \bar{\rho}_{AE} U_A^\dagger)\right) \\ &\quad + P\left(\mathcal{T}_{A \rightarrow A_1 X_A}(U_A \bar{\rho}_{AE} U_A^\dagger), \frac{I_{A_1}}{|A_1|} \otimes \bar{\omega}_{X_A E}\right) \end{aligned} \quad (\text{C.22})$$

$$\leq P(\rho_{AE}, \bar{\rho}_{AE}) + \eta \quad (\text{C.23})$$

$$\leq (\sqrt{\varepsilon} - \eta) + \eta = \sqrt{\varepsilon}. \quad (\text{C.24})$$

Applying Uhlmann's theorem once again as done above and converting purified distance to fidelity concludes the proof.

## D Variance of the relative entropy of entanglement for phase-insensitive Gaussian channels

In this appendix, we detail the calculation of the variance of the relative entropy of entanglement for the phase-insensitive Gaussian channels given in (8.1)–(8.3). In particular, we establish the formulas given in (8.15)–(8.17). To begin with and as reviewed in [PLOB16], given a two-mode state with covariance matrix in standard form as in [WTLB16, Eq. (E1)], it is a separable state if

$$c \leq c_{\text{sep}} \equiv \sqrt{(a - 1/2)(b - 1/2)}. \quad (\text{D.1})$$

For any given  $a$  and  $b$ , the two-mode Gaussian state in standard form and having maximal correlations between the two modes has  $c$  set to  $c_{\text{sep}}$  (the amount of correlations is quantified by quantum discord as done in [PSB<sup>+</sup>14]). This choice turns out to be a guiding principle for selecting a separable state closest in “relative entropy distance” to the state at the output of a given channel.

We begin by reviewing the various pairs of states from [PLOB16] for comparison. The state that we consider at the input of any of the channels is the two-mode squeezed vacuum, which has zero mean and standard-form covariance matrix

$$V_{\text{in}}^\mu \equiv \begin{bmatrix} \mu & c \\ c & \mu \end{bmatrix} \oplus \begin{bmatrix} \mu & -c \\ -c & \mu \end{bmatrix}, \quad (\text{D.2})$$

where  $\mu \geq 1/2$  is the energy of the reduced state on a single mode (directly related to the amount of entanglement in the state) and  $c \equiv \sqrt{\mu^2 - 1/4}$ . Sending one mode of this state through the three channels of interest (thermal, amplifier, additive-noise) leads to two-mode states  $\rho_{\eta,\omega}^\mu$ ,  $\rho_{G,\omega}^\mu$ , and  $\rho_\xi^\mu$  with the following respective covariance matrices:

$$V_{\eta,\omega}^\mu \equiv \begin{bmatrix} \mu & \sqrt{\eta}c \\ \sqrt{\eta}c & \eta\mu + (1-\eta)\omega \end{bmatrix} \oplus \begin{bmatrix} \mu & -\sqrt{\eta}c \\ -\sqrt{\eta}c & \eta\mu + (1-\eta)\omega \end{bmatrix}, \quad (\text{D.3})$$

$$V_{G,\omega}^\mu \equiv \begin{bmatrix} \mu & \sqrt{G}c \\ \sqrt{G}c & G\mu + (G-1)\omega \end{bmatrix} \oplus \begin{bmatrix} \mu & -\sqrt{G}c \\ -\sqrt{G}c & G\mu + (G-1)\omega \end{bmatrix}, \quad (\text{D.4})$$

$$V_\xi^\mu \equiv \begin{bmatrix} \mu & c \\ c & \mu + \xi \end{bmatrix} \oplus \begin{bmatrix} \mu & -c \\ -c & \mu + \xi \end{bmatrix}, \quad (\text{D.5})$$

where  $\omega \equiv N_B + 1/2 \geq 1/2$ . Using the aforementioned guiding principle, the resulting separable states  $\sigma_{\eta,\omega}^\mu$ ,  $\sigma_{G,\omega}^\mu$ , and  $\sigma_\xi^\mu$  for evaluating the bounds have zero mean and the following respective covariance matrices:

$$V_{\eta,\omega}^{\mu,\text{sep}} \equiv \begin{bmatrix} \mu & \sqrt{\eta}c_1 \\ \sqrt{\eta}c_1 & \eta\mu + (1-\eta)\omega \end{bmatrix} \oplus \begin{bmatrix} \mu & -\sqrt{\eta}c_1 \\ -\sqrt{\eta}c_1 & \eta\mu + (1-\eta)\omega \end{bmatrix}, \quad (\text{D.6})$$

$$V_{G,\omega}^{\mu,\text{sep}} \equiv \begin{bmatrix} \mu & \sqrt{G}c_2 \\ \sqrt{G}c_2 & G\mu + (G-1)\omega \end{bmatrix} \oplus \begin{bmatrix} \mu & -\sqrt{G}c_2 \\ -\sqrt{G}c_2 & G\mu + (G-1)\omega \end{bmatrix}, \quad (\text{D.7})$$

$$V_\xi^{\mu,\text{sep}} \equiv \begin{bmatrix} \mu & c_3 \\ c_3 & \mu + \xi \end{bmatrix} \oplus \begin{bmatrix} \mu & -c_3 \\ -c_3 & \mu + \xi \end{bmatrix}, \quad (\text{D.8})$$

where

$$c_1 \equiv \sqrt{(\mu - 1/2)(\eta\mu + (1-\eta)\omega - 1/2)}, \quad (\text{D.9})$$

$$c_2 \equiv \sqrt{(\mu - 1/2)(G\mu + (G-1)\omega - 1/2)}, \quad (\text{D.10})$$

$$c_3 \equiv \sqrt{(\mu - 1/2)(\mu + \xi - 1/2)}. \quad (\text{D.11})$$

Using the formula from [WTLB16, Lemma 3] and relying on a computer algebra package to handle tedious algebraic manipulations,<sup>1</sup> we find the following expansions of the various relative entropy variances about  $\mu = \infty$ :

$$V(\rho_{\eta,\omega}^\mu \| \sigma_{\eta,\omega}^\mu) = (\omega^2 - 1/4) \ln^2 \left( \eta \frac{2\omega + 1}{2\omega - 1} \right) + O(\mu^{-1}), \quad (\text{D.12})$$

$$V(\rho_{G,\omega}^\mu \| \sigma_{G,\omega}^\mu) = (\omega^2 - 1/4) \ln^2 \left( G^{-1} \frac{2\omega + 1}{2\omega - 1} \right) + O(\mu^{-1}), \quad (\text{D.13})$$

$$V(\rho_\xi^\mu \| \sigma_\xi^\mu) = (1 - \xi)^2 + O(\mu^{-1}). \quad (\text{D.14})$$

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<sup>1</sup>Mathematica source files are included in the arXiv posting of this paper.

Note that the values  $\eta \frac{2\omega+1}{2\omega-1} > 1$ ,  $G^{-1} \frac{2\omega+1}{2\omega-1} > 1$ , and  $\xi < 1$  correspond to values for which the channels are not entanglement-breaking. Also, when  $\omega = 1/2$ , the first two channels become the pure-loss channel and the quantum-limited amplifier channel, and we find that  $V(\rho_{\eta,\omega=0}^\mu \| \sigma_{\eta,\omega=0}^\mu) = V(\rho_{G,\omega=0}^\mu \| \sigma_{G,\omega=0}^\mu) = O(\mu^{-1})$ , so that in these cases the relative entropy variance vanishes in the infinite-energy limit  $\mu \rightarrow \infty$ . Using the equality  $\omega = N_B + 1/2$ , we find that

$$(\omega^2 - 1/4) \ln^2 \left( \eta \frac{2\omega+1}{2\omega-1} \right) = N_B(N_B + 1) \ln^2 \left( \eta \frac{N_B + 1}{N_B} \right), \quad (\text{D.15})$$

$$(\omega^2 - 1/4) \ln^2 \left( G^{-1} \frac{2\omega+1}{2\omega-1} \right) = N_B(N_B + 1) \ln^2 \left( G^{-1} \frac{N_B + 1}{N_B} \right). \quad (\text{D.16})$$

## E Upper bound for the hypothesis testing relative entropy

This appendix provides some necessary details for the bounds given in the proof of Theorem 24. As these bounds follow directly from the developments in [JOPS12, Eq. (6.5)], [DPR16, Section 3], and [Pol10, Lemma 15], we point to these works for the necessary background. Below we begin with an important proposition from [JOPS12, Eq. (6.5)] and then show how it leads to the desired bound in (E.2). For convenience, in this appendix we take  $D(\rho\|\sigma)$ ,  $V(\rho\|\sigma)$ , and  $D_H^\varepsilon(\rho\|\sigma)$  to be defined with respect to the natural logarithm.

The following proposition is available as [JOPS12, Eq. (6.5)] and restated as [DPR16, Corollary 2]:

**Proposition 30 ([JOPS12, Eq. (6.5)])** *Let  $\rho$  and  $\sigma$  be faithful states acting on a separable Hilbert space  $\mathcal{H}$ , let  $T$  be a measurement operator acting on  $\mathcal{H}$  and such that  $0 \leq T \leq I$ , and let  $v, \theta \in \mathbb{R}$ . Then*

$$e^{-\theta} \text{Tr}\{(I - T)\rho\} + \text{Tr}\{T\sigma\} \geq \frac{e^{-\theta}}{1 + e^{v-\theta}} \Pr\{X \leq v\}, \quad (\text{E.1})$$

where  $X$  is a random variable with mean  $D(\rho\|\sigma)$  and variance  $V(\rho\|\sigma)$ .

The following proposition follows from combining Proposition 30 with the reasoning used to establish [Pol10, Lemma 15].

**Proposition 31** *Let  $\rho$  and  $\sigma$  be faithful states acting on a separable Hilbert space  $\mathcal{H}$ . The following Chebyshev-like bound holds for all  $\varepsilon \in (0, 1)$  and all  $n \geq 1$ :*

$$\frac{1}{n} D_H^\varepsilon(\rho^{\otimes n} \| \sigma^{\otimes n}) \leq D(\rho\|\sigma) + \sqrt{\frac{2V(\rho\|\sigma)}{n(1-\varepsilon)}} + \frac{C(\varepsilon)}{n}, \quad (\text{E.2})$$

where  $C(\varepsilon) \equiv \ln 6 + 2 \ln([1+\varepsilon]/[1-\varepsilon])$ .

**Proof.** Let  $T^n$  be any test satisfying  $\text{Tr}\{(I^{\otimes n} - T^n)\rho^{\otimes n}\} \leq \varepsilon$ . By applying the above proposition (making the replacements  $\rho \rightarrow \rho^{\otimes n}$  and  $\sigma \rightarrow \sigma^{\otimes n}$ , so that  $X_n$  is a random variable with mean  $nD(\rho\|\sigma)$  and variance  $nV(\rho\|\sigma)$ ), we find that

$$\text{Tr}\{T^n \sigma^{\otimes n}\} \geq e^{-\theta_n} \left( \frac{\Pr\{X_n \leq v_n\}}{1 + e^{v_n - \theta_n}} - \text{Tr}\{(I - T^n)\rho^{\otimes n}\} \right) \quad (\text{E.3})$$

$$\geq e^{-\theta_n} \left( \frac{\Pr\{X_n \leq v_n\}}{1 + e^{v_n - \theta_n}} - \varepsilon \right). \quad (\text{E.4})$$

Setting  $v_n = nD(\rho\|\sigma) + \sqrt{\frac{2nV(\rho\|\sigma)}{1-\varepsilon}}$ , we find as a consequence of the Chebyshev inequality that

$$\Pr\{X_n > v_n\} = \Pr\left\{X_n - nD(\rho\|\sigma) > \sqrt{\frac{2nV(\rho\|\sigma)}{1-\varepsilon}}\right\} \quad (\text{E.5})$$

$$= \Pr\left\{[X_n - nD(\rho\|\sigma)]^2 > \frac{2nV(\rho\|\sigma)}{1-\varepsilon}\right\} \quad (\text{E.6})$$

$$< \frac{1-\varepsilon}{2}, \quad (\text{E.7})$$

implying that  $\Pr\{X_n \leq v_n\} \geq (1+\varepsilon)/2$ . Substituting above and taking  $\theta_n = v_n + C_0(\varepsilon)$  for  $C_0(\varepsilon)$  a constant such that  $\frac{1+\varepsilon}{2(1+e^{-C_0(\varepsilon)})} - \varepsilon > 0$ , we find that

$$\text{Tr}\{T^n \sigma^{\otimes n}\} \geq e^{-\theta_n} \left( \frac{1+\varepsilon}{2(1+e^{-C_0(\varepsilon)})} - \varepsilon \right) \quad (\text{E.8})$$

$$= \exp\left\{-\left[nD(\rho\|\sigma) + \sqrt{\frac{2nV(\rho\|\sigma)}{1-\varepsilon}} + C_0(\varepsilon)\right]\right\} \left( \frac{1+\varepsilon}{2(1+e^{-C_0(\varepsilon)})} - \varepsilon \right). \quad (\text{E.9})$$

Since this holds for every test  $T^n$  satisfying  $\text{Tr}\{(I^{\otimes n} - T^n) \rho^{\otimes n}\} \leq \varepsilon$ , we can apply a negative logarithm and divide by  $n$  to conclude that

$$\frac{1}{n} D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq D(\rho\|\sigma) + \sqrt{\frac{2V(\rho\|\sigma)}{n(1-\varepsilon)}} + \frac{C_0(\varepsilon)}{n} - \frac{1}{n} \ln\left(\frac{1+\varepsilon}{2(1+e^{-C_0(\varepsilon)})} - \varepsilon\right). \quad (\text{E.10})$$

The above bound implies (E.2).

To get the constant  $C(\varepsilon)$  in the upper bound, consider that the condition  $\frac{1+\varepsilon}{2(1+e^{-C_0(\varepsilon)})} - \varepsilon > 0$  is equivalent to the condition  $C_0(\varepsilon) > \ln(2\varepsilon/[1-\varepsilon])$ . So we can pick  $C_0(\varepsilon) = \ln(3\varepsilon/[1-\varepsilon])$  and this choice implies that

$$-\ln\left(\frac{1+\varepsilon}{2(1+e^{-C(\varepsilon)})} - \varepsilon\right) = \ln\left(\frac{2(1+2\varepsilon)}{\varepsilon(1-\varepsilon)}\right). \quad (\text{E.11})$$

We then find that

$$C_0(\varepsilon) - \ln\left(\frac{1+\varepsilon}{2(1+e^{-C(\varepsilon)})} - \varepsilon\right) = \ln 6 + \ln\left(\frac{1+2\varepsilon}{(1-\varepsilon)^2}\right) \quad (\text{E.12})$$

$$\leq \ln 6 + 2 \ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right) = C(\varepsilon). \quad (\text{E.13})$$

Putting everything together, we get the bound stated in (E.2). ■

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