REPORT CTSL– 30

AN ACHROMATIC INFLECTOR FOR THE CALTECH SYNCHROTRON

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AUGUST 7, 1961

SYNCHROTRON LABORATORY

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA

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Contents

I. Introduction

The following is an analytic investigation, using first order theory only, of a proposed achromatic inflector for the Caltech Synchrotron. Nothing new in principle is involved in any of the theory presented below, and similar types of analyses have been carried out by others[1)]. The investigation was prompted by the need to find a specific achromatic inflector to couple a 10 MeV electron linac to the Caltech synchrotron. In particular, it was desired to find a system such that the injector was placed in a convenient location, and, if any electrostatic elements were used, that they have reasonably low required field strengths.

II. Approximations and Notation

In the following we will employ x as the radial displacement of a particle from the central orbit of the system, where we will maintain the sense of x throughout the system by reference to the central path. θ will be defined as $\frac{dx}{ds}$, (for small values, the angle of the particle path with respect to the central path), where s is the distance along

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1) See, for example: Lee Teng, "Achromatic Bending Systems", ANLAD-48; F. C. Mills, "Achromatic Beam Bending and Position Shifting Systems", MURA Technical Note, ACC Proposal, File No. P-17; S. Penner, R.S.I. 32, 150 (1961); Raphael Littauer, "Some Design Considerations for Injection Optical Systems", unpublished Cornell report.

the central path, again maintaining a consistent definition of the sign of the quantity by reference to the central path. The quantity k is defined as $\frac{\Delta P}{P}$, the deviation of the particle momentum from the central momentum.

The degree of approximation used here is the usual first order one. That is, the particles are presumed to obey strictly the differential equation

$$\frac{d^2x}{ds^2} + \frac{Q^2}{R^2} x = \frac{1}{R} k$$

in the horizontal plane, where s is the central particle path distance parameter, R is the radius of curvature of the central path, and Q is defined by

$$Q^2 = 1 - n_m \qquad \text{for magnetic elements}$$

where

$$n_m = - \frac{R}{B_o} \frac{\partial B_z}{\partial x}$$

and B_o is the magnetic field at the central path, $(x + R = r)$

$$Q^2 = 2 - n_e \qquad \text{for electrostatic elements,}$$

where

$$n_e = - \frac{R}{E_o} \frac{\partial E_r}{\partial x}$$

and E_o is the radial electric field at the central path. (The latter expression $Q^2 = 2 - n_e$ applies only in the <u>relativistic limit</u>; in general, it is $3 - (\frac{v}{c})^2 - n_e$. Likewise the coefficient of k given above is true only in the relativistic limit for the electric case.) Of course, it is tacitly assumed that $\frac{x}{R}$, Θ, and k are all small quantities.

III. Matrix Representation of a Single Element

To simplify the algebra, we employ here 3 x 3 optical transmission matrices for the system. Furthermore, we henceforth restrict our attention only to constant gradient bending elements with normal incidence. That is, the central path impinges normally upon the face of the bending element, such as in a wedge magnet.

If x_o, Θ_o, and k are the coordinates of a particle entering one side of a bending element, the coordinates of the particle emerging from the other side are given by solution of the differential equation as

$$x = x_o \cos \frac{Qs_o}{R} + \Theta_o \frac{R}{Q} \sin \frac{Qs_o}{R} + k \frac{R}{Q^2} \left[1 - \cos \frac{Qs_o}{R} \right]$$

$$\Theta = -x_o \frac{Q}{R} \sin \frac{Qs_o}{R} + \Theta_o \cos \frac{Qs_o}{R} + k \frac{1}{Q} \sin \frac{Qs_o}{R}$$

$k_{final} = k$ (since, rigorously in a magnetic element, and to first order in an electrostatic one[2]), k is constant.)

[2] Actually, we may say that k = constant rigorously in the electrostatic case as well, if we are interested in the particle parameters only when the particle is outside the bending element, i.e., has returned to the original electrostatic potential. To first order, however, the results are exactly the same for an emerging particle (except for fringing field effects) as though we had used the foregoing differential equation and kept k constant through the system. The actual change in particle energy is taken up in the definition of Q.

In the above, s_o is the path length of the central orbit, and thus $\frac{s_o}{R} = \varphi$, the geometrical angle of the bending element. If we then represent the particle coordinates as a vector $\begin{pmatrix} x \\ \Theta \\ k \end{pmatrix}$, we may then describe the transmission through the element by the optical matrix

$$M_e = \begin{bmatrix} \cos Q\varphi & \frac{R}{Q}\sin Q\varphi & \frac{R}{Q^2}\left[1 - \cos Q\varphi\right] \\ -\frac{Q}{R}\sin Q\varphi & \cos Q\varphi & \frac{1}{Q}\sin Q\varphi \\ 0 & 0 & 1 \end{bmatrix}$$

where, as usual,

$$\begin{pmatrix} x \\ \Theta \\ k \end{pmatrix} = M_e \begin{pmatrix} x_o \\ \Theta_o \\ k \end{pmatrix}$$

Similarly, the transmission of a particle through a field-free section of length ℓ is given by the matrix

$$L = \begin{pmatrix} 1 & \ell & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that through a simple, achromatic lens of focal length f by

$$F = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix of the bending magnet M_e given above may be simplified enormously by taking as the reference planes the principal planes of the system. To illustrate, if we multiply M_e on the right and on the left by the matrix

$$D = \begin{bmatrix} 1 & -d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $d = \frac{R}{Q} \tan\left(\frac{Q\varphi}{2}\right)$, which is equivalent to moving the reference planes a distance d toward the center of the bending element along the lines defined by the direction of the incident and exiting central rays, we find

$$DM_eD = M = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{Q}{R}\sin Q\varphi & 1 & \frac{1}{Q}\sin Q\varphi \\ 0 & 0 & 1 \end{bmatrix}$$

The bending element acts thus as a lens of focal length $\frac{R}{Q \sin Q\varphi}$, except that dispersion is indicated by the non-vanishing term $< M_{23} > = \frac{1}{Q} \sin Q\varphi$. In the case of the uniform wedge magnet or coaxial cylindrical electrostatic element, $Q = 1$, and $d = R \tan(\varphi/2)$. The two planes thus both pass through the intersection of the incident and exiting central rays. For simplicity, we will write the matrix as

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -\rho & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where} \quad \begin{aligned} \rho &= \frac{Q}{R} \sin Q\varphi \\ s &= \frac{1}{Q} \sin Q\varphi \end{aligned}$$

and all reference will be to the principal planes in the subsequent analysis.

IV. Some General Theorems

Below are given some general results, some of which will be found useful in what follows. The first theorem is that the determinant of a transmission matrix must be unity. In the form with which we will be concerned,

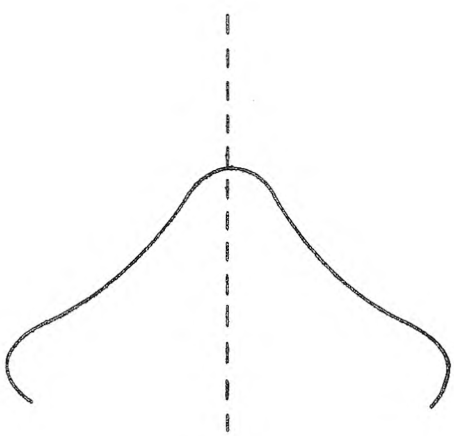
$$T = \begin{bmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{bmatrix}$$

this theorem implies only that AD - CB = 1, and no statements may be made about E and F. However, the result is useful in determining the final form of the matrix (and for checking algebraic and numerical results!).

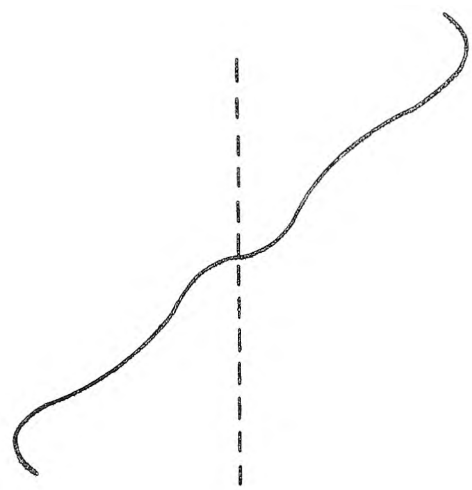
Next, general achromaticity for a transmission matrix requires that both E and F vanish, and thus represents two relations that must be satisfied. This, in general, is tantamount to the removal of two degrees of freedom available in the design of an achromatic system. If one further wishes to specify the entire matrix, one requires only three more relations in the form we employ, since the unity determinant condition takes care of the fourth element. Thus a minimum of five parameters must be available in order for one to specify a system of this form completely.

Although the system finally proposed in this report has no inherent symmetry properties, many achromatic systems constructed have, and there are some general results which are useful for these situations.

Given that a system consists of two similar parts, we treat two cases. First, let us assume that the second part consists of the mirror reflection of the first part about a reference plane normal to the central particle path. We call this case I. We can also consider the same situation except that the curvature of the elements in the second part have been all reversed. This we will call case II. Case I might be called a beam bending system, Case II a beam displacement system, since in the latter case the central ray emerges necessarily parallel to the entering central ray.



CASE I



CASE II

The transmission matrix of a particle traversing the second part of the system can be related to that of the first, in the following way. For Case I the form can be found by requiring that if a particle traverses the first half of the system, has its direction reversed at the central plane, travels backward through the first half, and is reversed again at the beginning, the transmission matrix must be unity. (Of course, we have to imagine also reversing the sign of the magnetic fields in the magnetic elements, but this does not enter the algebra.) Reversal of a particle direction is mathematically equivalent, in this case, to multiplication by

$$N_\Theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(i.e., changing the sign of } \Theta\text{)}$$

Thus if the transmission matrix of a particle going backwards through a system T is T^*, we have, since $N_\Theta = N_\Theta^{-1}$

$$N_\Theta \, T^* \, N_\Theta \, T = 1 \qquad \text{or} \qquad T^* = N_\Theta \, T^{-1} \, N_\Theta$$

For Case II, we perform the same operation except we must reverse the sign of k at the same point we reverse the sign of Θ each time, since the second half of the system is the first half reversed in direction, as in Case I, except that the momentum dispersion is also reversed in sign. Thus, where

$$N_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

we have, since $N_k = N_k^{-1}$, and $N_k N_\theta = N_\theta N_k$

$$N_k N_\theta T_f^* N_k N_\theta T = 1 \qquad \text{or}$$

$$T_f^* = N_k N_\theta T^{-1} N_\theta N_k = N_k T^* N_k$$

where T_f^* the transmission matrix of the second half of the system.

Algebraically, the results are given by the following: if

$$T = \begin{pmatrix} A & B & E \\ C & D & F \\ 0 & 0 & 1 \end{pmatrix} \qquad , \text{ then}$$

$$T^* = \begin{pmatrix} D & B & BF-DE \\ C & A & AF-CE \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_f^* = \begin{pmatrix} D & B & DE-BF \\ C & A & CE-AF \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} D & -B & BF-DE \\ -C & A & CE-AF \\ 0 & 0 & 1 \end{pmatrix}$$

And the final transmission matrices of the systems are given by

Case I

$$U = T^* T = \begin{bmatrix} AD+BC & 2BD & 2BF \\ 2AC & AD+BC & 2AF \\ 0 & 0 & 1 \end{bmatrix}$$

Case II

$$U_f = T_f^* T = \begin{bmatrix} AD+BC & 2BD & 2DE \\ 2AC & AD+BC & 2CE \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, both systems are necessarily of the form

$$\begin{bmatrix} a & b & e \\ c & a & f \\ 0 & 0 & 1 \end{bmatrix}$$

Now, if we require achromaticity, we require $U_{13} = U_{23} = 0$. For Case I, this implies $BF = AF = 0$, and since $AD - BC = 1$, the general requirement is that $F = 0$ (Case I). For Case II, we require $DE = CE = 0$, and, again using $AD - BC = 1$, we find $E = 0$ (Case II). So there are definite requirements on the form of the half system in order to achieve achromaticity.

It is interesting to note, also, that in either Case I or Case II, if we further require that <u>either</u> of the off diagonal elements U_{12} or U_{21} be zero, then the determinental requirement $a^2 = 1$ fixes $a = \pm 1$. This requirement also implies that one of the original four elements A, B, C, or D is also zero.

For our purposes, however, the relevant thing is that, since $U_{11} = U_{22}$ in the symmetric or antisymmetric systems, as well as the requirement on the determinant, then three relations only are necessary on the half system in order to completely specify the final achromatic system. For example, if the final matrix is to be of the form

$$U = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then we require for Case I, upper sign

$B = C = 0, F = 0$

Case I, lower sign

$A = D = 0, F = 0$

Case II, upper sign

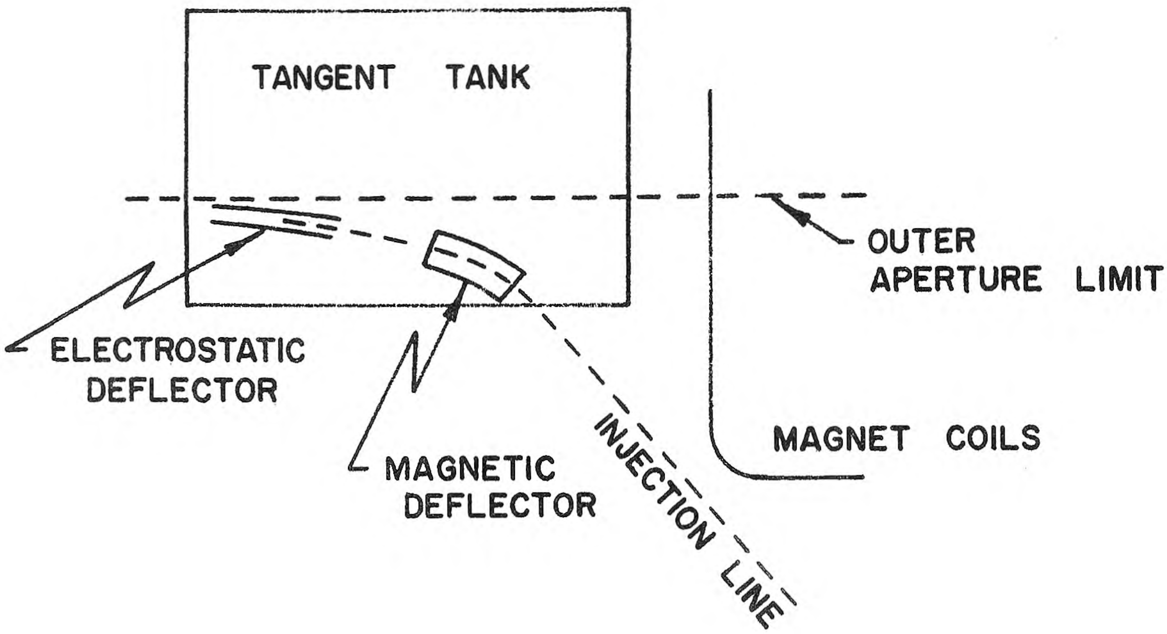
$B = C = 0, E = 0$

Case II, lower sign

$A = D = 0, E = 0$

V. Design Considerations for Caltech Machine

For a satisfactory injection system, the following goals were set. These goals are not necessarily the only possible ones, but were rather chosen on a somewhat personal basis as being convenient for construction and placement of elements. We require, first, that the system be achromatic (to first order). Second, there are difficulties in bringing the beam into the machine, so that only certain paths are suitable. In particular, it was desired to employ an electrostatic deflector for the final bending element in order to avoid problems with stray fields near the septum. However, at 10 MeV, the proposed injection energy, quite high fields are required in order to achieve reasonable radii of curvature. Since the aperture will have to be of the order of a centimeter, large voltages are also called for. In order to avoid sparking problems, we have set an upper limit of 60 kv/cm on the electric field, implying a lower limit of ~ 1.8 meters on the radius of curvature. Such a bending element will not allow the beam to clear the magnet coils, so it is proposed to follow a short electrostatic element with an "H" magnet placed in the tangent tank and close to the beam line to clear the particle path of the magnet coils, as shown in the sketch. It was further decided to restrict the individual elements to uniform field wedge magnets or coaxial cylindrical electrostatic deflectors ($Q = 1$)



for simplicity in construction, and to separate the vertical and horizontal focussing problems. A further goal was the design of the system such that it places the injector in a convenient, somewhat predetermined position.

Having assigned these goals, we may now analyze, in general, the number of parameters required. Each element has two: a radius of curvature R and a geometrical angle φ. Further, there is a drift section between each element. Then a symmetric or antisymmetric system has 3J parameters for the <u>half</u> system, where J is the number of elements in the half system. A non-symmetric system has 3J - 1 parameters, where J is the number of elements. In the system we desire, 5 parameters have already been determined; i.e., we have specified the radii of curvature and geometrical angles of the first two elements, and the distance between them[3]. Now, the achromatic requirement provides two relationships for the non-symmetric system, or one relationship for the half system in the symmetric or antisymmetric cases. In order to specify completely the transmission matrix, we require three more relationships in the non-symmetric case, or two on the half system of the symmetric ones.

For our purposes, however, it was decided that it is unnecessary to specify completely the transmission matrix. Instead, it was decided that the matrix should be of the form

$$T_A = \begin{bmatrix} m & -\Lambda m & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and not to specify the value of Λ, since the element $-\Lambda m$ can be made zero by simply translating the input reference plane a distance Λ upstream from the point originally chosen for the analysis.

With the somewhat less stringent requirement given above, the total number of determined parameters or relationships is given below:

Symmetric or Antisymmetric System:

1. Number of requirements on half system = 7
2. Minimum number of elements in half system such that

   $3J > 7$; $J = 3$

---

[3] There is, to be sure, some latitude in assigning these parameters, but to be safe, we assume they are fixed.

3. Conclusion: a six element system is required. (Or 5 with symmetric plane in center of No. 3.)

Non-Symmetric System:

1. Number of requirements on system = 9

2. Minimum number of elements such that $3J - 1 > 9$; $J = 4$

3. Conclusion: a four element system is required.

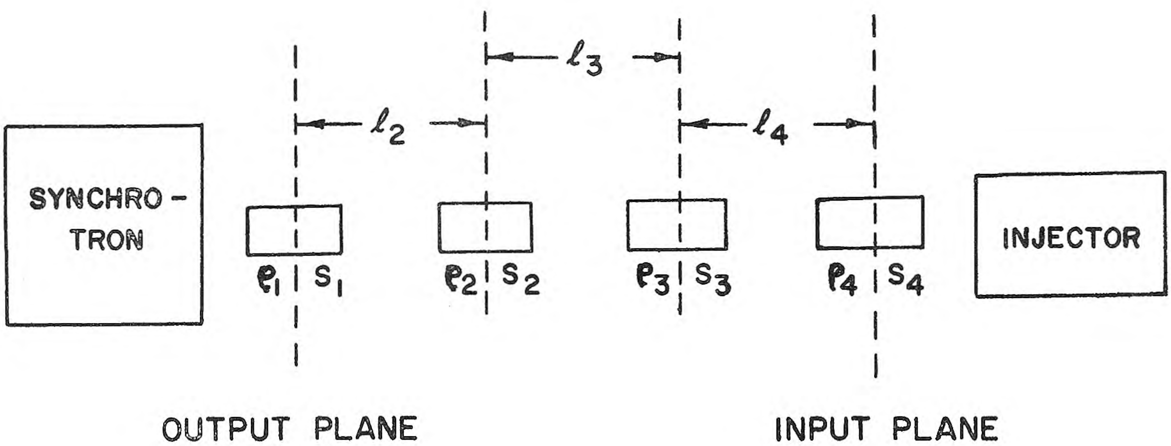
Thus it is seen that in either a 6 element symmetric system or a four element non-symmetric system, there remain two free parameters with which we are allowed to determine the final placement of the injector.

VI. Specific Choice of System

Although the symmetric system suggested above has many pleasing features, such as requiring only three distinct types of elements, and there may be some reduction in geometrical aberrations, it was decided that the reduction in total number of elements to four, possible with the non-symmetric system, was sufficiently attractive to warrant its adoption. The algebra involved is somewhat more severe, since the 6 element system requires the consideration of only the half system, or the multiplication of 6 matrices (3 bending elements, 3 drift sections) while the four element system requires the multiplication of 7 (4 bending elements, 3 drift sections).

VII. Derivation of the Transmission Matrix

The calculation is given below in steps. We retain the terminology of ρ and s as defined earlier, so that the final matrix allows a free choice of $Q = \sqrt{1-n}$. However, we will actually use only Q = 1. Furthermore, it is to be noted that either right or left bending elements are allowed, as this is simply determined by the sign of s; if $s > 0$ is taken as the "normal" bending direction, an $s < 0$ corresponds to a reverse magnet curvature (i.e., to changing the sign of both φ and R, so that ρ is always positive definite). We assume here that always $-\pi < Q\varphi < \pi$. The input and output planes of the bending elements are always taken as the principal planes, and the drift section length between the elements is measured from those planes. The sketch below clarifies the notation



That is, ρ_1, s_1 are the parameters of the electrostatic element, and ρ_2, s_2, etc., are the parameters of the magnets taken in sequential order from the machine to the injector. The distance ℓ_2 is the drift space between the electrostatic element and the first magnet, and ℓ_2, ℓ_3, are the other drift spaces in the same order. (Taken between principal planes of the elements!) Then if M_i is the transmission matrix of the $i^{\underline{th}}$ bending element, and L_i that of the $L^{\underline{th}}$ drift region, the matrix we seek is given by

$$T_A = M_1\, L_2\, M_2\, L_3\, M_3\, L_4\, M_4$$

The results follow.

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\rho_1 & 1 & s_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_2\, M_2 = \begin{bmatrix} 1-\ell_2\rho_2 & \ell_2 & \ell_2 s_2 \\ -\rho_2 & 1 & s_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_1\, L_2\, M_2 = \begin{bmatrix} 1-\ell_2\rho_2 & \ell_2 & \ell_2 s_2 \\ -\rho_1(1-\ell_2\rho_2)-\rho_2 & -\ell_2\rho_1+1 & -\ell_2 s_2\rho_1+s_1+s_2 \\ 0 & 0 & 1 \end{bmatrix}$$

L_3M_3 is L_2M_2 given above with subscripts changed from 2 to 3. Then

$$M_1\, L_2\, M_2\, L_3\, M_3 = \begin{bmatrix} (1-\ell_2\rho_2)(1-\ell_3\rho_3) - \ell_2\rho_3 & \ell_3(1-\ell_2\rho_2) + \ell_2 & \ell_3 s_3(1-\ell_2\rho_2) + \ell_2(s_2+s_3) \\ -(1-\ell_3\rho_3)\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right] - \rho_3(1-\ell_2\rho_1) & -\ell_3\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right] + (1-\ell_2\rho_1) & -\ell_3 s_3\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right] + s_1 + (1-\ell_2\rho_1)(s_2+s_3) \\ 0 & 0 & 1 \end{bmatrix}$$

then L_4M_4 is given by the expression L_2M_2 with the obvious subscripts changed, so finally

$$M_1 L_2 M_2 L_3 M_3 L_4 M_4 = T_A =$$

$$
\Bigg[\begin{aligned}
&(1-\ell_2\rho_2)(1-\ell_3\rho_3)(1-\ell_4\rho_4)-\ell_2\rho_3(1-\ell_4\rho_4)-\ell_3\rho_4(1-\ell_2\rho_2)-\ell_2\rho_4\\
&-(1-\ell_4\rho_4)(1-\ell_3\rho_3)\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right]-\rho_3(1-\ell_4\rho_4)(1-\ell_2\rho_1)\\
&\qquad +\ell_3\rho_4\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right]-\rho_4(1-\ell_2\rho_1)\\
&0
\end{aligned}
$$

$$
\begin{aligned}
&\ell_4(1-\ell_2\rho_2)(1-\ell_3\rho_3)-\ell_2\ell_4\rho_3+\ell_3(1-\ell_2\rho_2)+\ell_2\\
&-\ell_4(1-\ell_3\rho_3)\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right]-\ell_4\rho_3(1-\ell_2\rho_1)-\ell_3\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right]\\
&\qquad +(1-\ell_2\rho_1)\\
&0
\end{aligned}
$$

$$
\begin{aligned}
&\ell_4 s_4(1-\ell_2\rho_2)(1-\ell_3\rho_3)-\ell_4 s_4\ell_2\rho_3+\ell_3 s_4(1-\ell_2\rho_2)+\ell_3 s_3(1-\ell_2\rho_2)\\
&\qquad +\ell_2(s_2+s_3+s_4)\\
&-\ell_4 s_4(1-\ell_3\rho_3)\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right]-\ell_4 s_4\rho_3(1-\ell_2\rho_1)-\ell_3 s_4\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right]\\
&\qquad +(1-\ell_2\rho_1)(s_2+s_3+s_4)+s_1-\ell_3 s_3\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right]\\
&1
\end{aligned}\Bigg]
$$

This is now the matrix upon which we must impose the transmission conditions.

In order to simplify the subsequent algebra, we will consider s_1, ρ_1, ℓ_2, s_2, ρ_2 as predetermined constants. Then we define

$$\alpha = (1 - \ell_2 \rho_2) \tag{1a}$$

$$\beta = \rho_1(1 - \ell_2 \rho_2) + \rho_2 \tag{1b}$$

$$\gamma = (1 - \ell_2 \rho_1) \tag{1c}$$

Then the matrix we require is

$$T_A = \begin{bmatrix} m & -\Lambda m & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where we will not specify Λ. This leads to the following four relations:

Achromatic Condition $(T_A)_{13} = (T_A)_{23} = 0$:

$$\ell_4 s_4(1-\ell_3 s_3)\alpha - \ell_4 s_4 \ell_3 \rho_3 + \ell_3 s_4 \alpha + \ell_3 s_3 \alpha + \ell_2(s_2+s_3+s_4) = 0 \tag{2a}$$

$$-\ell_4 s_4(1-\ell_3 s_3)\beta - \ell_4 s_4 \rho_3 \gamma - \ell_3 s_4 \beta - \ell_3 s_3 \beta + s_1$$
$$+(s_2+s_3+s_4)\gamma = 0 \tag{2b}$$

Requirement $(T_A)_{21} = 0$:

$$-(1-\ell_4\rho_4)(1-\ell_3\rho_3)\beta - \rho_3(1-\ell_4\rho_4)\gamma + \rho_4\ell_3\beta - \rho_4\gamma = 0 \tag{2c}$$

Requirement $(T_A)_{22} = \frac{1}{m}$:

$$-\ell_4(1-\ell_3\rho_3)\beta - \ell_4\rho_3\gamma - \ell_3\beta + \gamma = \frac{1}{m} \tag{2d}$$

Four somewhat simpler relations can be derived from the foregoing by taking suitable linear combinations. If (2a) is multiplied by β and (2b) by α and the results added, we find

$$s_2 + s_3 + s_4 + \alpha s_1 = \ell_4 \, s_4 \, \rho_3 \tag{3a}$$

To obtain this result, we must observe that

$$\alpha\gamma + \beta\ell_2 = (1-\ell_2\rho_2)(1-\ell_2\rho_1) + \ell_2\left[\rho_1(1-\ell_2\rho_2)+\rho_2\right] = 1$$

Then if we multiply (2a) by γ and (2b) by $-\ell_2$, add, and using $\alpha\gamma + \beta\ell_2 = 1$, we find

$$\ell_4 s_4(1-\ell_3\rho_3) + \ell_3(s_3+s_4) = \ell_2 s_2 \tag{3b}$$

Further, if we multiply (2c) by $-\ell_4$, (2d) by $(1 - \ell_4\,\rho_4)$, add, we obtain

$$\frac{1}{m}(1 - \ell_4\rho_4) + \ell_3\beta = \gamma \tag{3c}$$

Finally, multiplication of (2d) by ρ_4 and addition to (2c) yields

$$\frac{\rho_4}{m} + \rho_3 (\gamma - \beta \ell_3) + \beta = 0 \tag{3d}$$

The final element $-\Lambda m$, is given by

$$-\Lambda m = \alpha \ell_4 (1-\ell_3 \rho_3) + \ell_2 (1-\ell_4 \rho_3) + \alpha \ell_3 \tag{4}$$

The relations 3a,b,c,d, are the ones we shall employ in a specific design, and we will discover the value of Λ only after the design is made.

VIII. Example of Actual Design

For purely numerical reasons, it is convenient to rewrite the relations given above in a somewhat different way. Since there are four equations and six unknowns (ℓ_3, ρ_3, s_3, ℓ_4, ρ_4, s_4), two of them must be specified before the system is fully determined. One of the arbitrary conditions which we set is that the orientation of the injector is such that its long dimension (the beam line) be made parallel to the wall of the laboratory. This is really a condition on the sum $\varphi_3 + \varphi_4$, since φ_1 and φ_2 are predetermined, but it is more convenient to actually specify s_3 and s_4 separately so as to satisfy the angular condition, and solve for the other parameters. Then by choosing several such sets of s_3 and s_4, we may find the system most satisfactory in the other parameters.

Therefore, considering s_3 and s_4 as known quantities, we write the following four expressions derived from 3a,b,c,d.

$$\ell_3 = \frac{\frac{s_4}{m} + s_1 + \gamma\,(s_2 + s_3)}{\beta\, s_3} \tag{5a}$$

$$\ell_4 = \frac{\ell_2 s_1 + (\alpha s_1 + s_2)\,\ell_3}{s_4} \tag{5b}$$

$$\rho_3 = \frac{\alpha s_1 + s_2 + s_3 + s_4}{\ell_4 s_4} \tag{5c}$$

$$\rho_4 = m\left[\rho_3(\beta\ell_3 - \gamma) - \beta\right] \tag{5d}$$

Determination of s_3 and s_4 then allows these relations to be employed in numerical order to yield the other parameters.

A specific example is given below. A set of parameters which allow the beam to be carried to the septum are

$r_1 = 1.875$ meters

$s_1 = 0.15 \quad (\varphi_1 \approx 8.6^\circ)$

$\ell_2 = 0.38$ meters

$r_2 = 0.35$ meters

$s_2 = 0.6428 \quad (\varphi_2 = 40^\circ)$

thus: $\rho_2 = 1.836$

$\rho_1 = 0.08$

$\alpha = 0.302$

$\beta = 1.860$

$\gamma = 0.970$

We now take m = +1, (for unity transfer matrix except for the "drift" term $-m\Lambda$). We set $\varphi_3 = \varphi_4 = 44.7^{\circ}$, $s_3 = s_4 = 0.7034$.

Then we find

$\ell_3 = 1.649$ meters

$\ell_4 = 1.695$ meters

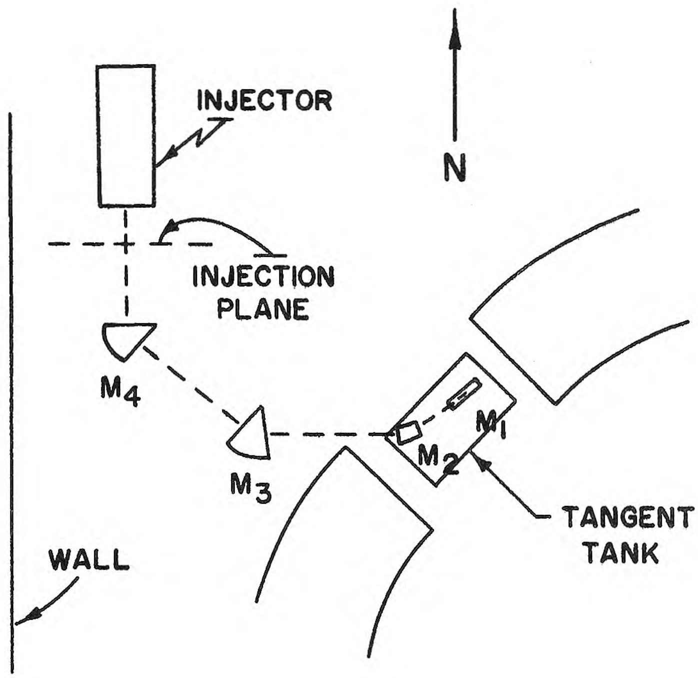
$\rho_3 = 1.758$ or $r_3 = 0.400$ meters

$\rho_4 = 1.828$ or $r_4 = 0.385$ meters

and also $-\Lambda m = -\Lambda = -1.226$ m, that is, Λ is a positive number. This implies that we must add 1.226 meters to the flight path to achieve true unity transfer. This is satisfactory, since we already have to move the exit (output) plane about 14 cm toward the machine (to the end of the electrostatic section) for proper injection, and the entrance (input) plane about 15 cm toward the injector to get it to the front face of the entrance magnet (M_4). Then we merely have to consider the input plane to exist some 93 cm before the entrance magnet (M_4) pole face.

The system seems an acceptable one, as the injector ends up parallel with the laboratory wall, and well removed from the machine. A rough sketch is shown below.

It is also to be noted that a radius of curvature of 1/3 meter for 10 Mev electrons requires a magnetic field of about 1000 gauss, a convenient figure for magnet design.



Other values can, of course, be found using different values of s_3 and s_4, (and by changing s_1, s_2, ρ_1, ρ_2, ℓ_2 slightly), as the above is only one specific example. One can also adjust m.

IX. Some General Observations

It might also be convenient to find a set of parameters such that the injector ends up pointed in the opposite direction, i.e., it would be west of the machine, pointed north, and injecting into the N.W. tangent tank. This turns out to be hard to do with the stipulations we have made, as can be seen by examination of (5b). All quantities in the numerator on the right are necessarily positive except α, and s_1 is required to be a small number. Thus it is difficult to make s_4 a negative quantity without also making ℓ_4 negative. Such a system without a negative s_4 would be very clumsy, as all the reversal in beam direction would have to be done by M_3.

Aberrations

Only rough estimates have been made of the aberrations of the system. These estimates indicate that, owing to the small beam and beam divergence, the detrimental effect of the geometrical and chromatic aberrations should be negligible. This does not, however, relieve the requirements on the goodness of the elements themselves, or the control of the magnetic fields.

X. Synchrotron Acceptance and Emittance Matching

It would appear that the output of the proposed injector (~ 0.6 cm diameter spot, 1 mr spread) is roughly in the proper range for the synchrotron acceptance. If, however, we decide to choose, for example, a smaller spot and larger angular spread in the median plane for the injection phase space, this can be done in two ways. First, we could adjust the value of m in the formulas (5a,b,c,d), to the proper value, or we could simply focus the proper emittance on the input plane of an m = 1 inflector with auxiliary lenses. The latter course would involve some increase in the spacing between the injector and the inflector input plane to accomodate the lenses, but probably not much. We will adopt the latter course, owing to its much greater flexibility.

Preferably, the emittance matching lens system should be adjustable over a moderate range, as the optimum matching must eventually be determined experimentally. However, as an initial guide we will assume the synchrotron acceptance as calculated by Peck*, and we will assume the worst case for the output of the linac, which is that of a diffuse source with the angular limits ± 1 mr and spot size 6 mm diameter. In designing the system, however, we must keep in mind that the linac output may, in fact, be a more correlated source, so that improvement may be achieved in capture by taking advantage of the correlation of divergence angle with radial position. This is the primary reason for requiring that the matching system be flexible.

For the purpose of the following discussion, we will assume that all emittance matching elements must be placed between the injector and the input of the inflector. This separates the vertical from the horizontal requirements, which are quite different, and also separates, in first order, the emittance matching problem from the requirement of first order achromaticity in the beam bending system. Thus we consider that we must carry the output of the linac in the horizontal plane to the horizontal input plane of the inflector with the proper optical

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* Charles Peck, "Capture Efficiency of a Constant Gradient Synchrotron", California Institute of Technology Synchrotron Laboratory Report CTSL-13.

adjustments. For this purpose, we will assume that the inflector has unit transfer to the synchrotron septum. The vertical emittance, on the other hand, must be transferred, with the proper adjustments, from the injector to the synchrotron septum. We will consider the beam bending system to have no vertical focussing properties, (which is true to first order). The distance through the inflector is about 5 meters in the proposed system.

XI. Sample Transfer Requirements

From the calculations of Peck, it can be seen that, for 10 Mev injection, a spot size of 6 mm is somewhat too large in the horizontal plane, while divergence angles somewhat greater than ± 1 mr could be tolerated. Also, the Cornell linac appeared to perform somewhat better with respect to divergence angle than the ± 1 mr quoted. Thus for purposes of initial design, we will assume that a magnification of 0.5 is required in the horizontal plane. That is, we nominally require a transfer matrix of

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$$

(or its negative) in the horizontal plane between the injector output and the inflector input. In the vertical plane, the acceptance requirements are much less severe. A transfer matrix of

$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

in the vertical plane between output of the injector and the inflector septum would probably be best, but we probably do not require such a stringent condition. The primary thing is actually to keep the vertical spot size small enough to clear the 1.5 cm gap planned for the tangent tank magnet. On the other hand, it is probably desirable to allow the beam to become as large as possible vertically in order to reduce space charge effects.

XII. Quadrupole Lenses

Because of their flexibility and relatively small aberrations, we propose to employ quadrupole magnetic lenses for the emittance matching. For reasons of completeness, we will develop here the required equations to first order.

We consider a quadrupole lens to have the following properties: (1) the magnetic field vanishes along the axis, (2) the magnetic field gradient in the x direction and in the y direction is constant, and (3) there is no axial component of the magnetic field. Assumption (3) is, of course, not valid in the fringing region, but is approximately true in the interior of a lens. Then, if the magnetic field gradient is defined by

$$\frac{\partial B_y}{\partial x} = \Gamma = \frac{\partial B_x}{\partial y}$$

where we are considering x in the horizontal plane and y in the vertical plane, (and in a right-handed coordinate system, z is the particle direction) the paraxial equations of motion of a particle in the lens are given by

$$x = x_o \cos \kappa z + \frac{x_o'}{\kappa} \sin \kappa z$$

$$y = y_o \cosh \kappa z + \frac{y_o'}{\kappa} \sinh \kappa z$$

where

$$\kappa^2 = \pm \Gamma \frac{\beta c}{\mathcal{E}} \quad ,$$

and the sign refers to the sign of charge of the particle, $\mathcal{E}$ is the total energy of the particle, (e.g., in MKS units $\mathcal{E}$ is in electron volts)

$$x_o,\ y_o,\ x_o' = \left.\frac{dx}{dz}\right|_o \ ,\ y_o' = \left.\frac{dy}{dz}\right|_o$$

refer to the values at $z = 0$. Then if the quadrupole has a length L, the optical transmission matrices are given by

Horizontal: $$\begin{pmatrix} \cos \kappa L & \frac{1}{\kappa} \sin \kappa L \\ -\kappa \sin \kappa L & \cos \kappa L \end{pmatrix}$$

Vertical: $$\begin{pmatrix} \cosh \kappa L & \frac{1}{\kappa} \sinh \kappa L \\ \kappa \sinh \kappa L & \cosh \kappa L \end{pmatrix}$$

Of course, the designation "horizontal" and "vertical" are merely convenient, and henceforth we shall call the matrix having the form given for the horizontal the "focussing plane", and that given for the vertical the "defocussing plane", and assume that the magnetic fields are so oriented that κ is real.

It can be easily shown that for any optical matrix of the form,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

if $C \neq 0$, then we may reduce it to a general thick lens form

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

if we move the reference planes to the principal planes of the system. The first principal plane is located a distance $P_1 = \frac{1-D}{C}$ ahead of the original first reference plane, and the second principal plane a distance $P_2 = \frac{1-A}{C}$ beyond the original second reference plane. Then in this case the focal length $f = -\frac{1}{C}$.

Thus for the quadrupole lens, the focal length in the focussing plane is given by $f_+ = \frac{1}{\kappa \sin \kappa L}$, and in the defocussing plane $f_- = -\frac{1}{\kappa \sinh \kappa L}$, and the principal planes are located a distance

$$\frac{\tan \frac{\kappa L}{2}}{\kappa}$$

in the focussing plane and

$$\frac{\tanh \frac{\kappa L}{2}}{\kappa}$$

in the defocussing plane toward the center of the lens (from the edges).

In essentially all practical cases, however, a simplifying approximation may be made. If it happens that κL is small, then $\sin \kappa L \approx \sinh \kappa L$, and $\frac{1}{f_+} \approx \kappa^2 L \approx -\frac{1}{f_-}$. More explicitly, if we look more carefully at the expansion for sin and sinh, we find the condition that, if

$$(\kappa L)^2 \approx \frac{L}{f} \ll 3$$

then

$$f_+ \approx -f_- \quad ,$$

and

$$\frac{\tan \frac{\kappa L}{2}}{\kappa} \approx \frac{\tanh \frac{\kappa L}{2}}{\kappa} \approx \frac{L}{2}$$

or the two principal planes coincide in the center of the quadrupole. Thus we may treat the quadrupole lens as a <u>thin</u> lens, whose focal lengths in the horizontal and vertical planes are equal in magnitude and opposite in sign. This is the approximation we shall employ in all further considerations.

XIII. Suggested Systems

We will now derive the applicable relations for some systems which satisfy our requirements. It will be seen that although one quadrupole triplet system exists, it is very clumsy, and the most satisfactory one is a quadruplet.

For the purpose of the derivation, we assume that we require a matrix

$$\begin{pmatrix} M & L_H \\ 0 & \frac{1}{M} \end{pmatrix}$$

in the horizontal plane, and

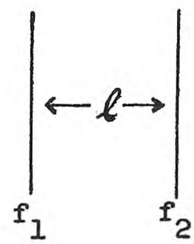
$$\begin{pmatrix} N & L_V \\ 0 & \frac{1}{N} \end{pmatrix}$$

in the vertical plane, where $M = \pm 1/2$ and $N = \pm 2$. Systems of this nature, with the vanishing M_{21} element we shall term "telescopic", whereas those in which the element M_{12} vanishes, we shall term "imaging". What we would actually prefer is an imaging, telescopic system in the horizontal plane between injector output and inflector input, and an imaging, telescopic system in the vertical plane between injector output and inflector septum. However, it is easily seen that a telescopic system can be converted to an imaging, telescopic one by a simple translation of reference planes, whereas a non-telescopic system cannot be converted to telescopic by any such translation. It will also become apparent that, if we choose $M = + 1/2$ and $N = -2$, it is in general comparatively easy to arrange that the required translations are commensurate with the actual placement of injector, inflector, and septum in our case. Thus for now, we require only double telescopy with the correct vertical and horizontal magnifications.

XIV. Doublet

It is trivial to show that a doublet cannot satisfy the requirements. The transmission matrix for a doublet is given by

$$\begin{pmatrix} 1-\ell g_1 & \ell \\ \ell g_1 g_2 - g_1 - g_2 & 1-\ell g_2 \end{pmatrix}$$



where $g_1 = 1/f_1$, $g_2 = 1/f_2$, the focal length reciprocals of lens one and lens two, respectively. The transmission in the orthogonal plane is obtained by changing the signs of g_1 and g_2. Thus if we require double telescopy, we require

$$\ell g_1 g_2 - g_1 - g_2 = \ell g_1 g_2 + g_1 + g_2 = 0$$

$$\text{or} \quad \ell g_1 g_2 = 0$$

This is nonphysical.

XV. Triplet

The transmission matrix for a triplet is given below. It can be obtained from the transmission matrix given above for the four magnet system by letting

$$s_1,\ s_2,\ s_3,\ s_4 \longrightarrow 0$$

$$\rho_4 \longrightarrow g_1$$

$$\rho_3 \longrightarrow g_2$$

$$\rho_2 \longrightarrow g_3$$

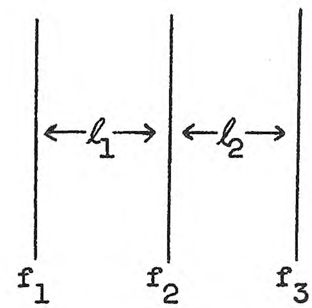
$$\rho_1 \longrightarrow 0$$

$$\ell_4 \longrightarrow \ell_1$$

$$\ell_3 \longrightarrow \ell_2$$

$$\ell_2,\ \ell_1 \longrightarrow 0$$

$\Longrightarrow$ transmission



The input and output planes of the system are taken at the principal planes of lens one and lens three respectively. $g_1 = 1/f_1$, etc.

$$\begin{bmatrix} 1-\ell_2 g_2-\ell_1 g_1-\ell_2 g_1+\ell_1\ell_2 g_1 g_2 & \ell_1+\ell_2-\ell_1\ell_2 g_2 \\ -g_1-g_2-g_3+\ell_1 g_1 g_2+\ell_1 g_1 g_3+\ell_2 g_1 g_3+\ell_2 g_2 g_3-\ell_1\ell_2 g_1 g_2 g_3 & 1-\ell_1 g_2-\ell_2 g_3-\ell_1 g_3+\ell_1\ell_2 g_2 g_3 \end{bmatrix}$$

We now force the matrix conditions we wish to impose, i.e.,

$$\begin{bmatrix} M & L_H \\ 0 & \frac{1}{M} \end{bmatrix}$$

for the matrix given above, and

$$\begin{bmatrix} N & L_V \\ 0 & \frac{1}{N} \end{bmatrix}$$

for this matrix with the signs of g_1, g_2, and g_3 reversed. Since we are not specifying L_H and L_V the unity determinant condition makes it necessary to specify only N, M, and the two zero elements. Thus we have four equations. These four equations can be combined linearly to yield four simpler ones, which are given below:

$$1 + \ell_1\ell_2 g_1 g_2 = \frac{N+M}{2} \equiv \alpha_1 \tag{6a}$$

$$\ell_1 g_1 + \ell_2 g_1 + \ell_2 g_2 = \frac{N-M}{2} \equiv \alpha_2 \tag{6b}$$

$$\ell_1 g_1 g_2 + \ell_1 g_1 g_3 + \ell_2 g_1 g_3 + \ell_2 g_2 g_3 = 0 \tag{6c}$$

$$g_1 + g_2 + g_3 + \ell_1\ell_2 g_1 g_2 g_3 = 0 \tag{6d}$$

Since there are four equations and five unknowns, we employ one (g_2) as a scale factor, and write the dimensionless quantities

$$\lambda_1 = \ell_1 g_2$$

$$\lambda_2 = \ell_2 g_2$$

$$\gamma_1 = g_1/g_2$$

$$\gamma_3 = g_3/g_2$$

Some extensive algebraic manipulations yield the following set of relations

$$\gamma_3 = -\frac{\gamma_1 + 1}{\alpha_1} \tag{7a}$$

$$\gamma_1 = \frac{\alpha_2 - \lambda_2}{\lambda_1 + \lambda_2} \tag{7b}$$

$$\lambda_2 = \frac{\alpha_2}{\alpha_1} \frac{(\alpha_1 - 1)\lambda_1 - \alpha_2}{\lambda_1} \tag{7c}$$

$$(\alpha_1-1)(\alpha_1^2-\alpha_2^2)\lambda_1^2 + \alpha_2\left[(\alpha_1-2)(\alpha_1^2-\alpha_2^2) + \alpha_1\right]\lambda_1 + \left[\alpha_1 - (\alpha_1^2-\alpha_2^2)\right]\alpha_2^2 = 0 \tag{7d}$$

The final equation is a quadratic in λ_1, which can be solved directly. The solution, where we also note that $\alpha_1^2 - \alpha_2^2 = MN$, is given by

$$\lambda_1 = - \frac{\alpha_2\left[(\alpha_1-2)MN + \alpha_1\right]}{2(\alpha_1-1)MN}$$

$$\pm \alpha_2 \sqrt{\frac{\left[(\alpha_1-2)MN + \alpha_1\right]^2}{4(\alpha_1-1)^2 M^2 N^2} - \frac{\alpha_1 - MN}{(\alpha_1-1)MN}} \qquad (8)$$

We may now investigate the solutions. For N = 2, M = 1/2, we find

$$\lambda_1 = \pm \sqrt{-\frac{9}{16}}$$

which is, of course, nonphysical.

For N = -2, M = - 1/2, we find

$$\lambda_1 = \frac{2}{3} \pm \frac{\sqrt{-17}}{12}$$

which is again nonphysical. The last case to investigate is N = -2, M = 1/2. (The case of N = 2, M = - 1/2 is physically the same, as it is just the N = -2, M = + 1/2 system reversed in direction, and with the horizontal and vertical planes interchanged.) This case yields the two real solutions

$$\lambda_1 = \frac{5}{28} \qquad \text{or} \qquad \lambda_1 = \frac{35}{28}$$

However, only the first is physical, since ℓ_1 and ℓ_2 must both be positive. This infers that λ_1 and λ_2 must have the <u>same sign</u>. It can be seen from the previous equations that the values M = 1/2, N = -2, λ_1 = 35/28 yield a <u>negative</u> value for λ_2. The first solution, however, gives the physical values, expressed in terms of the focal length ($f_2 = 1/g_2$) of lens two in the horizontal plane

$$\ell_1 = \frac{5}{28} f_2$$

$$\ell_2 = \frac{35}{4} f_2$$

$$f_1 = \frac{1}{\gamma_1 g_2} = -\frac{25}{38} f_2$$

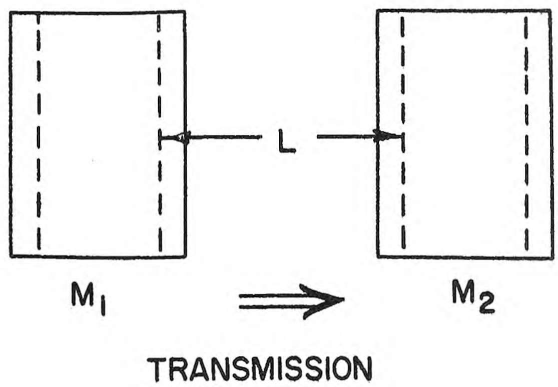
$$f_3 = \frac{1}{\gamma_3 g_2} = -\frac{25}{4} f_2$$

This system is the only one of the combinations of magnification which is physically realizable, and it is very clumsy. The very large difference between ℓ_1 and ℓ_2 ($\ell_2/\ell_1 = 49$) implies that the system will take up a large space with real lenses of significant thickness. We thus go on to the quadruplet.

XVI. Quadruplet

To proceed with the quadruplet case in the straightforward way as was done with the triplet leads to impressive algebraic difficulties. Instead, we have found satisfactory solutions by pre-grouping the quadruplet into two doublets with special properties of their own. This system is amenable to fairly simple algebraic treatment.

First, let us consider two systems of any kind with a drift region L between them. We consider the two systems M_1 and M_2 between their reference planes.



Then the transmission between the input plane of M_1 and the output plane of M_2 is given by

$$M_2 \, L \, M_1 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

$$= \begin{bmatrix} A_2A_1 + B_2C_1 + LA_2C_1 & A_2B_1 + B_2D_1 + LA_2D_1 \\ A_1C_2 + C_1D_2 + LC_1C_2 & B_1C_2 + D_1D_2 + LC_2D_1 \end{bmatrix}$$

Now the system we are seeking is defined by setting the proper elements to the required values. We define A_1^*, B_1^*, etc., as the above elements with the signs of all lens focal lengths reversed, then M_1^*, M_2^*, are the transmissions in the opposite (vertical) plane if M_1, M_2 are systems of quadrupoles. Then the defining relations are

$$A_1A_2 + B_2C_1 + LA_2C_1 = M \qquad (9a)$$

$$A_1C_2 + C_1D_2 + LC_1C_2 = 0 \tag{9b}$$

$$A_1^*A_2^* + B_2^*C_1^* + LA_2^*C_1^* = N \tag{9c}$$

$$A_1^*C_2^* + C_1^*D_2^* + LC_1^*C_2^* = 0 \tag{9d}$$

We now assume that M_1 and M_2 are quadrupole doublets, with the reference planes taken at the centers of the lenses. We further now assume that the two doublets are themselves telescopic in the horizontal plane, i.e., $C_1 = C_2 = 0$. For a doublet, this condition, along with the above defining relations, yields

$$A_1A_2 = M = mn$$

$$D_1D_2 = \frac{1}{M} = \frac{1}{mn}$$

$$m = A_1$$

$$n = A_2$$

$$\frac{1}{m} = D_1$$

$$\frac{1}{n} = D_2$$

when m and n are the magnifications of the telescopic doublets M_1 and M_2 respectively. Then, again referring to the doublet matrix previously given, we may find

$$A_1^* = 1 + ag_1 = 2 - m \tag{10a}$$

$$D_1^* = 1 + ag_2 = 2 - \frac{1}{m} \tag{10b}$$

$$A_2^* = 1 + bh_1 = 2 - n \tag{10c}$$

$$D_2^* = 1 + bh_2 = 2 - \frac{1}{n} \tag{10d}$$

$$A_1 = m = 1 - ag_1 \tag{10e}$$

$$A_2 = n = 1 - bh_1 \tag{10f}$$

where a = spacing between lenses in first doublet; b = spacing between lenses in second doublet; g,h are reciprocals of focal length in the horizontal plane; and g_1, g_2 refer to first and second lenses in first doublet; h_1, h_2 refer to first and second lenses in second doublet.

$$B_1 = B_1^* = a \tag{11a}$$

$$B_2 = B_2^* = b \tag{11b}$$

Also, since

$$C_1 = ag_1g_2 - g_1 - g_2 = 0$$

$$C_1^* = ag_1g_2 + g_1 + g_2 \quad ,$$

we see that

$$C_1^* = 2ag_1g_2 = 2(g_1 + g_2)$$

Similarly,

$$C_2^* = 2bh_1h_2 = 2(h_1 + h_2)$$

Now, using $m = 1 - ag_1$, $n = 1 - bh_1$, we may find that

$$C_1^* = - \frac{2}{a} \frac{(m-1)^2}{m} \tag{12a}$$

$$C_2^* = - \frac{2}{b} \frac{(n-1)^2}{n} \tag{12b}$$

We now develop the equation for the vertical magnification N. By eliminating L between (9c) and (9d), we find

$$C_2^* N = (B_2^* C_2^* - A_2^* D_2^*) C_1^*$$

The matrix determinental condition requires

$$A_2^* D_2^* - B_2^* C_2^* = 1 \quad , \qquad \text{so}$$

$$N = - \frac{C_1^*}{C_2^*} = - \frac{b}{a} \frac{n}{m} \frac{(m-1)^2}{(n-1)^2} \tag{13}$$

The expression for L is obtained by solving (9d) for L and substituting in the expressions (10) and (12). The result is

$$L = \frac{a}{2} \frac{m(2-m)}{(m-1)^2} + \frac{b}{2} \frac{(2n-1)}{(n-1)^2} \tag{14}$$

Or, if we eliminate b with the aid of (13), we have

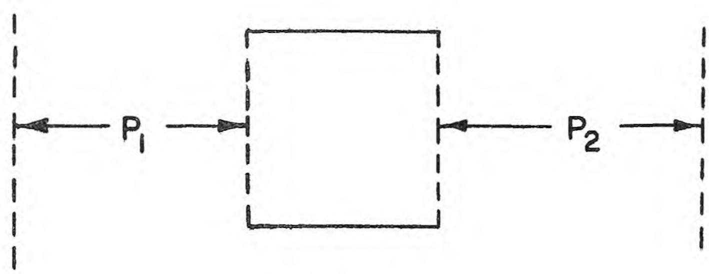
$$L = \frac{am}{2(m-1)^2} \left[2 + \frac{N}{n} - m - 2N \right] \tag{15a}$$

The above equation, along with

$$M = mn$$

plus (13), and the set (10), completely defines the system.

We still have at our disposal, however, the individual doublet magnifications, subject to the condition that $mn = M$, the scaling factor "a", and the actual placement of the system with respect to the bending system and the injector. In order to examine these effects, we require an expression for the position of the input and output planes for which the system is both imaging and telescopic. Of course, the vertical and horizontal planes are not necessarily the same. To obtain this expression, we multiply the transmission matrix on the right and left by drift distances P_1 and P_2 in the horizontal plane, and P_1^* and P_2^* in the vertical plane, and set equal to an imaging matrix. Thus



SYSTEM

$$\begin{pmatrix} 1 & P_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M & E \\ 0 & \frac{1}{M} \end{pmatrix} \begin{pmatrix} 1 & P_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \frac{1}{M} \end{pmatrix}$$

Thus we require

$$P_1 M + \frac{P_2}{M} = -E \qquad \text{(horizontal)} \tag{16a}$$

$$P_1^* N + \frac{P_2^*}{N} = -E^* \qquad \text{(vertical)} \tag{16b}$$

and

$$E = A_2 B_1 + B_2 D_1 + LA_2 D_1 = an + \frac{b}{m} + L\frac{n}{m} \tag{17a}$$

$$E^* = A_2^* B_1^* + B_2^* D_1^* + LA_2^* D_1^*$$

$$= a(2-n) + b(2 - \frac{1}{m}) + L(2-n)(2 - \frac{1}{m}) \tag{17b}$$

XVII. Sample Quadruplet System

We choose $N = -2$, $M = +1/2$. We also set, somewhat arbitrarily, $m = n = 1/\sqrt{2}$. Then

$$L = \frac{(6\sqrt{2} - 5)a}{2(3 - 2\sqrt{2})} \approx 10.16a$$

and

$$g_1 = \frac{2 - \sqrt{2}}{2a} \approx \frac{0.2929}{a} \quad ; \qquad f_1 \approx 3.415\ a$$

$$g_2 = \frac{1 - \sqrt{2}}{a} \approx \frac{0.4142}{a} \quad ; \qquad f_2 \approx -2.414\ a$$

$$b = 2a$$

$$h_1 = \frac{2 - \sqrt{2}}{2b} \quad ; \qquad e_1 \approx 6.83\ a$$

$$h_2 = \frac{1 - \sqrt{2}}{b} \quad ; \qquad e_2 \approx -4.828\ a$$

where

$$e_1 = \frac{1}{h_1} \quad ; \quad e_2 = \frac{1}{h_2} \quad ,$$

the focal lengths of the lenses in the second doublet.

Such a system seems acceptable. For example, if $a = 0.1$ meter, then $L \approx 1$ meter and the strongest lens required has a focal length of ≈ -24 cm, which seems readily obtainable for 10 Mev electrons.

To find the reference planes for which this system becomes focussing, we refer to equations (16) and (17). We find

$$E \approx 13.7\ a$$

$$E^* \approx 10.16\ a$$

For purposes of comparison, we set $P_1 = P_1^* = 0$; i.e., the injector output is focussed on the input lens. Then

$$P_1 = -\ 6.52\ a$$

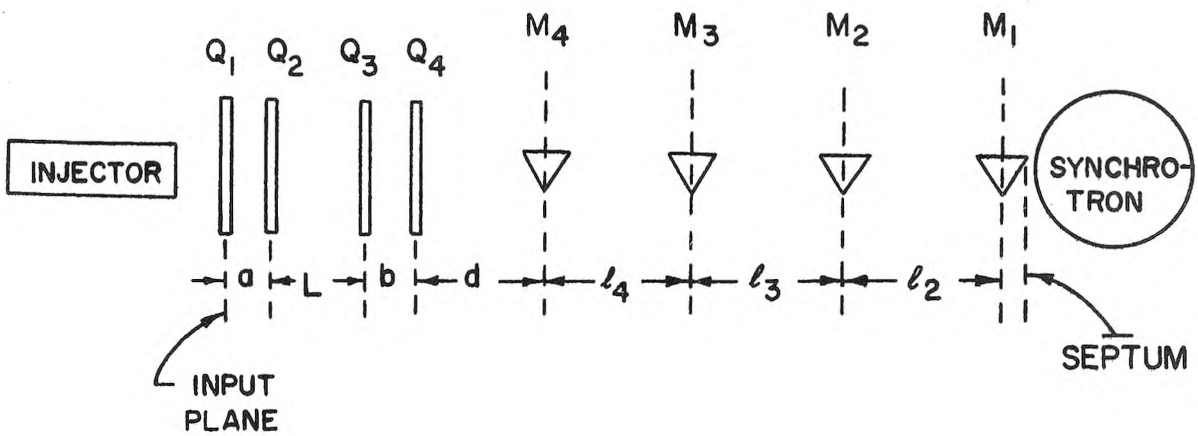
$$P_2^* = +\ 20.32\ a$$

Since the horizontal input plane for the proposed inflector appears approximately 90 cm in front of the input magnet, it is quite easy to make P_1 negative. If we make $a = 0.1$, this allows ~ 25 cm clearance between the end of the magnet and the first quadrupole. In this case $P_2^* \approx 2.03$ m. This does not quite meet the required condition, as the total distance between the final quadrupole and the septum would be about 4.3 meters.

However, the 2.3 meter discrepancy is not at all serious. For the <u>worst case</u>, beam divergence after the magnification of 2 in the vertical plane is ± 0.5 mr, and it may be considerably better. This means a maximum (worst case) spot spread of an additional ~ 2.3 mm out of a spot size in the vertical of 12 mm. Even this can be compensated somewhat by changing the adjustment of the quadrupoles slightly to produce a slight focussing action in the vertical plane. This solution seems quite satisfactory.

XVIII. Summary

Below are given the relevant dimensions and parameters for the complete suggested injection system. The system is shown "straightened out", as it were, for clarity.



M's are magnets (except that M_1 is an electrostatic bending element); Q's are quadrupoles. Lengths (meters):

$\ell_1 = 0.141$

$\ell_2 = 0.38$

$\ell_3 = 1.6493$

$\ell_4 = 1.6945$

$d = 0.432$

$b = 0.2$

$L = 1.016$

$a = 0.1$

Magnets: uniform field wedge: deflection angles and radii of curvature (degrees, meters):

M_1 ;	8.627°	1.875
M_2 ;	40.0°	0.350
M_3 ;	44.7°	0.4002
M_4 ;	44.7°	0.3846

Quadrupoles: focal lengths in horizontal planes: (meters):

Q_1 :	+ 0.3415
Q_2 :	- 0.2414
Q_3 :	+ 0.683
Q_4 :	- 0.4828

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