

Form Factors in β Decay and μ Capture

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(Received March 7, 1958)

We suppose that β decay and μ capture are described by a universal vector and axial vector Lagrangian and we consider, via dispersion relation techniques, the properties of the corresponding S -matrix elements. Owing to the strong interactions of the nucleons, the structure of the S matrix is expected to be more complicated than that of the Lagrangian. In the former, vector and axial vector terms appear, but with coefficients which in general depend on the invariant nucleon momentum transfer; they can be thought of as Fermi interaction form factors. Moreover, two additional kinds of terms can appear in the S -matrix elements: one which simulates a direct pseudoscalar coupling and one which simulates a direct coupling involving derivatives of the nucleon wave functions. The latter is probably too small to have any experimental significance. The former, though negligible in β decay, may be appreciable in μ capture. We estimate the effective pseudoscalar coupling coefficient there to be about eight times as large as the axial vector coefficient. More generally, we investigate the structure of the various form factors; and we also reconsider, in further refinement, a recent quantitative discussion which we have given of $\pi \rightarrow \mu + \nu$ decay.

I. INTRODUCTION

THE validity of the two-component theory of the neutrino¹ and of the principle of lepton conservation appears to be reasonably well established at the present time.² Beyond this, comparison of β decay and μ decay discloses a remarkably detailed "universality": in the standard way of describing such processes, both seem to be characterized by vector (V) and axial vector (A) couplings³; and in particular, the vector coupling coefficients in β decay and μ decay appear to be almost identical.⁴ As for the other well-known Fermi process, μ -meson capture, one at present knows only that the dominant coupling coefficients must have about the same magnitude as in β decay,⁵ although the types of coupling which occur are not yet established. It does not seem unreasonable to suppose that the universal V , A interaction extends also to the process of μ capture.

The similarity between β decay and μ decay, although very striking, is apparently not quite a precise one. In the latter process, the V and A coupling coefficients appear to be identical in magnitude—as they must be on the two-component neutrino theory. In β decay the axial vector coefficient is slightly larger than the vector coefficient.⁶ This need not be surprising. Even if one assumes a V , A interaction Lagrangian which is truly universal, as between β decay and μ decay, there is a

profound difference between the two processes. The point is that where some of the participating particles are strongly interacting—as is the case for the nucleons in β decay and μ capture—the S -matrix element may have a much more complicated structure than the Lagrangian. Indeed, what is surprising is that any kind of S -matrix universality can persist under such circumstances.

As regards the vector coupling, Feynman and Gell-Mann⁴ have suggested that there may be operative a principle analogous to that of gauge invariance in electrodynamics, where all charged particles interact with static electric fields with the same coupling strength. One could in this way understand that even after "renormalization" the vector coupling coefficients are the same in μ decay and β decay (the nucleon momentum transfer in the latter process is essentially zero; i.e., the situation is analogous to interaction of a charged particle with a static electric field). The slight discrepancy between the axial vector coupling coefficients in μ decay and β decay would then be attributed to "renormalization" effects in the latter process.

Even if these views are correct, the V and A coupling coefficients in the S -matrix element would in general be expected to be functions of the invariant nucleon momentum transfer. In analogy with the problem of nucleon electromagnetic structure, they could be thought of as Fermi interaction form factors. Thus, in the process of μ capture, which involves a not inappreciable momentum transfer, the V and A coefficients might be somewhat different than in β decay, though the variation is very likely small as we shall see in the present paper.

What is more interesting, however, in connection with the role of the strong interactions in Fermi processes, is that coupling types not contained in the Lagrangian may appear in the S -matrix element. As we shall discuss, even if one starts with a Lagrangian which contains only V and A coupling terms, these can generate in the S -matrix element two additional kinds

¹ A. Salam, *Nuovo cimento* **5**, 299 (1957); L. Landau, *Nuclear Phys.* **3**, 127 (1957); T. D. Lee and C. N. Yang, *Phys. Rev.* **105**, 671 (1957).

² The experimental evidence comes from many sources. For a recent compilation see the forthcoming report of the *International Conference on Mesons and Recently Discovered Particles, Padua, 1957* [*Nuovo cimento* (to be published)].

³ Goldhaber, Grodzins, and Sunyar, *Phys. Rev.* **109**, 1015; Hermansfeldt, Maxson, Stähelin, and Allen, *Phys. Rev.* **107**, 641 (1957).

⁴ This quantitative agreement, within the framework of the two-component neutrino theory, was first remarked by R. P. Feynman and M. Gell-Mann, *Phys. Rev.* **109**, 193 (1958).

⁵ See, for example, T. N. K. Godfrey, Princeton University thesis, 1954 (unpublished).

⁶ A. Winther and O. Kofoed-Hansen, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.* (to be published).

of terms: a pseudoscalar coupling, generated by A ; and, generated by V , a term which simulates a direct interaction with derivatives of the nucleon wave functions. This latter coupling is in fact identical in structure with the anomalous magnetic-moment term in the nucleon electromagnetic current. Quantitatively it is of no consequence—it is probably too small to be detected in either β decay or μ capture.* The pseudoscalar term, however, though negligible in β decay, may be quite appreciable in μ capture. In fact, we estimate that the effective pseudoscalar coupling in μ capture is ~ 8 times larger than the axial vector coefficient that generates it. This estimate, and our quantitative discussion in general, is based on dispersion relation techniques of calculation. In the approximation actually adopted, the dispersion relation result concerning the pseudoscalar coefficient is equivalent to what one obtains⁷ in lowest order perturbation theory for the sequence $p \rightarrow n + \pi^+ \rightarrow n + \mu^+ + \nu$, where the amplitude for the last step is obtained from the known rate of pion decay, and where the unrenormalized pion-nucleon coupling constant is replaced by the renormalized one. We invoke the more elaborate dispersion relation methods to show to what extent the approximation is valid, not only for the momentum transfer involved in μ capture—where the approximation is probably quite justified—but also for very large momentum transfers. The reason for our interest in the latter situation has to do with a quantitative discussion of $\pi \rightarrow \mu + \nu$ decay which we have recently given.⁸ In this problem the pseudoscalar term plays a decisive role; but in our earlier discussion, although very large momentum transfers are involved, we made use of results which are presumably only valid at small momentum transfer. We reconsider the problem here, taking into account additional relevant effects; and we find that our earlier result on pion decay is essentially unaltered.

II. STRUCTURE OF THE S-MATRIX ELEMENT

A

We suppose that β decay and μ capture are described by a direct interaction Lagrangian of the Fermi type, with axial vector and vector couplings,

$$\mathcal{L}_I = Z_2 f_A \bar{\psi} i \gamma_\lambda \gamma_5 (1 + \gamma_5) \psi_\nu \bar{\psi}_p i \gamma_\lambda \gamma_5 \psi_n + Z_2 f_V \bar{\psi} i \gamma_\lambda (1 + \gamma_5) \psi_\nu \bar{\psi}_p \gamma_\lambda \psi_n + \text{c.c.}, \quad (1)$$

where f_A and f_V are the unrenormalized coupling constants, Z_2 is the nucleon wave-function renormalization constant (a corresponding constant for the lepton is set

* *Note added in proof.*—It should be emphasized that this statement is based on the assumption that one is dealing with a conventional Fermi interaction theory, described, for example, by the Lagrangian of Eq. (1). In the theory of Feynman and Gell-Mann,⁴ the presence of additional terms such as a direct pion-lepton interaction would cause the anomalous moment to be very much larger; experimental tests to detect it have been proposed by Gell-Mann, Phys. Rev. (to be published).

⁷ S. Weinberg, Phys. Rev. **106**, 1301 (1957).

⁸ M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).

equal to unity, since we will treat the weak interaction to lowest order and also neglect electromagnetic effects); ψ_i is the electron or μ -meson field. We shall consider the processes

$$\begin{pmatrix} e^- \\ \mu^- \end{pmatrix} + p \rightarrow n + \nu.$$

To lowest order in the weak interaction, the S -matrix element is given by

$$S = i(2\pi)^4 \delta(n + p_\nu - p - p_l) M, \quad (2)$$

where n , p_ν , p , and p_l are, respectively, the neutron, neutrino, proton, and electron (or μ meson) four-momenta; and

$$M = \bar{u}_\nu (1 - \gamma_5) i \gamma_\lambda \gamma_5 u_l \langle n | P_\lambda(0) | p \rangle + \bar{u}_\nu (1 - \gamma_5) \gamma_\lambda u_l \langle n | V_\lambda(0) | p \rangle; \quad (3)$$

$|n\rangle$ and $|p\rangle$ are the physical neutron and proton states, and

$$P_\lambda = Z_2 f_A \bar{\psi}_n i \gamma_\lambda \gamma_5 \psi_p, \quad (4)$$

$$V_\lambda = Z_2 f_V \bar{\psi}_n \gamma_\lambda \psi_p.$$

For simplicity, the lepton spinors are normalized according to

$$\bar{u}_l \gamma_4 u_l = \bar{u}_\nu \gamma_4 u_\nu = 1. \quad (5)$$

If we were now to neglect the strong interactions of the nucleons, the matrix elements $\langle n | P_\lambda | p \rangle$ and $\langle n | V_\lambda | p \rangle$ would have exactly the same structure as the lepton covariants with which they are, respectively, contracted. The actual expressions, however, may be more complicated. Nevertheless, from general invariance principles which concern the strong interactions we know that the structure of the matrix elements must be given by

$$\langle n | P_\lambda | p \rangle = \left(\frac{m^2}{p_0 n_0} \right)^{\frac{1}{2}} \bar{u}_n \{ a i \gamma_\lambda \gamma_5 - b (\not{p} - \not{n}) \gamma_5 \} u_p; \quad (6)$$

$$\langle n | V_\lambda | p \rangle = \left(\frac{m^2}{p_0 n_0} \right)^{\frac{1}{2}} \bar{u}_n \{ c \gamma_\lambda - d \sigma_{\lambda\mu} (\not{p} - \not{n})_\mu \} u_p; \quad (7)$$

where m is the nucleon mass and the nucleon spinors have the invariant normalization $\bar{u}u = 1$. That the momentum factors n and p appear above only in the combination $(\not{p} - \not{n})$ follows from charge independence and time-reversal invariance for the strong interactions. For the rest, the indicated structure of the matrix elements is determined by the requirement of Lorentz invariance. Finally, we note that the coefficients a , b , c , and d may in general be functions of the invariant nucleon momentum transfer $(n - p)^2$.

Collecting terms, and using the Dirac equation and momentum conservation to carry out reductions, we find for the matrix element M

$$M = \left(\frac{m^2}{p_0 n_0} \right)^{\frac{1}{2}} \{ a \bar{u}_\nu (1 - \gamma_5) i \gamma_\lambda \gamma_5 u_l \bar{u}_n i \gamma_\lambda \gamma_5 u_p + m_l b \bar{u}_\nu (1 - \gamma_5) \gamma_5 u_l \bar{u}_n \gamma_5 u_p + c \bar{u}_\nu (1 - \gamma_5) \gamma_\lambda u_l \bar{u}_n \gamma_\lambda u_p + i d \bar{u}_\nu (1 - \gamma_5) \gamma_\lambda (\not{p}_l - \not{p}_\nu)_\mu \bar{u}_n \sigma_{\lambda\mu} u_p. \quad (8)$$

The functions a and c are evidently the usual axial vector and vector coupling coefficients and for zero value of the argument $(n-p)^2$ are to be identified with the coupling constants g_A and g_V of β decay; for μ -capture the argument $(n-p)^2$ has the value m_μ^2 . The second term in Eq. (8) has the form of a conventional pseudoscalar coupling, with effective pseudoscalar coupling coefficient m_b . The last term has the form which one would obtain with a direct interaction involving derivatives of the spinor fields. It is well known that terms of this type are not present to any appreciable extent in β decay. We shall indeed see in the following section that, although there is no theoretical reason to doubt the existence of such a term, the coefficient d is probably extremely small, of order $(1/2m)(m_\pi/2m)^2(G^2/4\pi)g_V$, where G is the pion-nucleon coupling constant. On the other hand, the effective pseudoscalar coefficient m_b may well be appreciable in μ capture ($m_l=m_\mu$), though it is presumably quite negligible in β processes ($m_l=m_e$).

B

It may be of some interest to consider how the matters discussed above would apply in the case of Fermi interactions involving a hyperon. For ease of comparison with the foregoing, consider for example the process

$$\begin{pmatrix} e^- \\ \mu^- \end{pmatrix} + \Sigma^+ \rightarrow n + \nu.$$

Suppose that such a process is described by a direct V, A Fermi interaction Lagrangian. In this case the matrix elements analogous to $\langle n|P_\lambda|\Sigma\rangle$ and $\langle n|V_\lambda|\Sigma\rangle$ would again contain terms similar to those in (6) and (7), with the proton momentum p , of course, replaced by the hyperon momentum Σ . But there can now appear additional terms, which before were ruled out by the principle of charge independence and the observation that p and n are members of a charge multiplet. In $\langle n|P_\lambda|\Sigma\rangle$ there can appear a term $\bar{u}_n i b' \sigma_{\lambda\mu} (p+\Sigma)_\mu \gamma_5 u_\Sigma$; and in $\langle n|V_\lambda|\Sigma\rangle$ a term $\bar{u}_n i d' (\Sigma-n)_\lambda u_\Sigma$. This latter term can be reduced, by use of the Dirac equation and momentum conservation, to yield in the over-all S -matrix element a contribution which simulates a direct scalar coupling. We do not pursue the discussion of hyperon β processes any further, however. There is as yet no experimental evidence for hyperon β decay.⁹

III. DISPERSION RELATION APPROACH

A dispersion relation discussion of the Fermi interaction form factors a , b , c , and d of Eqs. (6) and (7) could be carried out in a manner similar to that for the electromagnetic form factors of the nucleon. For the latter problem there now exists considerable experi-

⁹ J. Steinberger (private communication). The upper limit on beta decay of the Λ^0 is experimentally an order of magnitude below the rate expected if the Fermi couplings are the same as in nucleon beta decay.

mental information on the form factors over a wide range of momentum transfers, so that there is a strong motivation for detailed discussion. In the Fermi interaction case one has experimental information only at zero momentum transfer (β processes) and, to some extent, at the momentum transfer $(n-p)^2 \simeq m_\mu^2$ of μ capture. It is only for processes where Fermi interactions play an intermediate role, as presumably in $\pi \rightarrow \mu + \nu$ decay, that large momentum transfers are of significance. We shall then only indicate in outline how a dispersion relation treatment of the full problem could be carried out; but in detail we attempt only approximate quantitative estimates.

A

We start with a discussion of the matrix element $\langle n|P_\lambda|p\rangle$. Following standard procedures,¹⁰ we write

$$\begin{aligned} \left(\frac{p_0 n_0}{m^2}\right)^{\frac{1}{2}} \langle n|P_\lambda|p\rangle &= i \left(\frac{p_0}{m}\right)^{\frac{1}{2}} \bar{u}_n \\ &\times \int d^4x e^{-in \cdot x} \langle 0|T(P_\lambda F(x))|p\rangle, \end{aligned} \quad (9)$$

where $T(\)$ denotes the Wick product and $F(x)$ is the source of the neutron field, defined by

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m\right) \psi_n(x) = F(x). \quad (10)$$

We have dropped an equal-time commutator term which would have ultimately yielded a constant addition to our expression for the form factor a . This is permissible since we will eventually write down a dispersion relation for a in which a constant is subtracted. The equal-time commutator contributes nothing to the coefficient b , however, and for this coefficient we shall assume a dispersion relation with no subtraction. We now observe that

$$T(P_\lambda F(x)) = [P_\lambda, F(x)]\theta(-x) + F(x)P_\lambda. \quad (11)$$

The second term makes no contribution to the matrix element. In the first term θ is the step function. Thus, we have that

$$\begin{aligned} \left(\frac{p_0 n_0}{m^2}\right)^{\frac{1}{2}} \langle n|P_\lambda|p\rangle \\ = i \left(\frac{p_0}{m}\right)^{\frac{1}{2}} \bar{u}_n \int d^4x e^{-in \cdot x} \langle 0|[P_\lambda, F(x)]\theta(-x)|p\rangle. \end{aligned} \quad (12)$$

We shall not discuss in detail the difficulties of giving a rigorous derivation of the desired dispersion relations for the form factors a and b . But we notice that the integral of Eq. (12) has the form of a Fourier transform

¹⁰ Lehmann, Symanzik, and Zimmerman, *Nuovo cimento* 2, 425 (1955).

of an advanced commutator—a structure which at least makes plausible that a and b are analytic functions of $\xi = (n-p)^2$ in the upper half ξ plane. At this point we simply state the dispersion relations which we shall use. For b we assume that no subtraction is necessary and we have

$$b(\xi) = -\frac{1}{\pi} \int_0^\infty d\xi' \frac{\text{Im}b(-\xi')}{\xi' + \xi - i\epsilon}, \quad (13)$$

where the instruction “ $-i\epsilon$ ” shows how the function is defined for negative ξ . For a we make one subtraction and write

$$a(\xi) = g_A - \frac{\xi}{\pi} \int_0^\infty d\xi' \frac{\text{Im}a(-\xi')}{\xi'(\xi' + \xi - i\epsilon)}, \quad (14)$$

where $g_A = a(0)$.

Our next task is to evaluate the imaginary parts of a and b . To do this we return to Eq. (12) and express the absorptive part, call it A_λ , as a sum over states; A_λ is the coefficient of $\frac{1}{2}i$ in $i\theta(-x_0) = \frac{1}{2}i - \frac{1}{2}i\epsilon(x_0)$. We find

$$A_\lambda = \pi(\rho_0/m)^{\frac{1}{2}} \sum_s \bar{u}_n(0|P_\lambda|s) \times \langle s|F(0)|p\rangle \delta(p_s + n - p). \quad (15)$$

The term with F and P_λ interchanged makes no contribution. The spatial part of the δ function must, with our normalization, be regarded as a Kronecker δ symbol. In order to maintain the proper reality conditions at all stages of approximation we shall understand, though we do not write it explicitly, that the sum over intermediate states is one-half the sum over “in” and “out” states.

The states $|s\rangle$ which contribute must have zero nucleon number. For low-energy phenomena like β decay and μ capture the most significant states are expected to be those of smallest mass. One can see this from Eqs. (13) and (14), where ξ' represents the square of the mass of the intermediate states $|s\rangle$ in Eq. (12). The state of lowest mass is just the one-pion state. Next comes the three-pion state (the two-pion state is ruled out by charge independence and charge conjugation invariance). Ultimately one encounters nucleon-antinucleon states, hyperon pair states, etc.

The one-pion state, as it turns out, contributes only to the coefficient b . Its contribution can be easily expressed in terms of the renormalized pion-nucleon coupling constant and the experimental lifetime for $\pi \rightarrow \mu + \nu$ decay. In the latter process we can write the S -matrix element, to lowest order in the weak-interaction Lagrangian of Eq. (1), in the form

$$S = i(2\pi)^4 \delta(p_\mu + p_\nu - p_\pi) \times (m_\mu/p_{\mu 0})^{\frac{1}{2}} \bar{u}_\mu i\gamma_\lambda \gamma_5 (1 + \gamma_5) u_\nu \langle 0|P_\lambda|\pi\rangle. \quad (16)$$

It is clear that $\langle 0|P_\lambda|\pi\rangle$ must be proportional to $(p_\pi)_\lambda$ and, in terms of our earlier notation,⁸ we write

$$\langle 0|P_\lambda|\pi\rangle = -i(p_\pi)_\lambda F(p_\pi^2)/(2p_{\pi 0})^{\frac{1}{2}}, \quad (17)$$

where, of course, $p_\pi^2 = -m_\pi^2$ for actual pion decay. The numerical value of $F(-m_\pi^2)$ is known from the experimental pion lifetime.

The other relevant matrix element is $\langle \pi|F|p\rangle$ and here, where $(n-p)^2 = p_\pi^2 = -m_\pi^2$, we may express this as

$$\bar{u}_n \langle \pi|F|p\rangle = \left(\frac{m}{p_0}\right)^{\frac{1}{2}} \frac{1}{(2p_{\pi 0})^{\frac{1}{2}}} \sqrt{2} G \bar{u}_n i\gamma_5 u_p, \quad (18)$$

where G is the renormalized pion-nucleon coupling constant. We now substitute (17) and (18) into (15) and compare the result with the definition (6). We find, as regards the contribution from the one-pion state,

$$A_\lambda = \pi \sqrt{2} G F(-m_\pi^2) \times (p-n)_\lambda \bar{u}_n \gamma_5 u_p \delta((n-p)^2 + m_\pi^2); \quad (19)$$

and thus, with $\xi = (n-p)^2$,

$$\text{Im}a(\xi) = 0, \quad (20)$$

$$\text{Im}b(\xi) = -\pi \sqrt{2} G F(-m_\pi^2) \delta(\xi + m_\pi^2).$$

Since the next least massive state which can contribute is the three-pion state, our dispersion relations may now be written

$$a(\xi) = g_A - \frac{\xi}{\pi} \int_{(3m_\pi)^2}^\infty d\xi' \frac{\text{Im}a(-\xi')}{\xi'(\xi' + \xi - i\epsilon)}, \quad (21)$$

$$b(\xi) = \frac{-\sqrt{2} G F(-m_\pi^2)}{\xi + m_\pi^2} + \frac{1}{\pi} \int_{(3m_\pi)^2}^\infty d\xi' \frac{\text{Im}b(-\xi')}{\xi' + \xi - i\epsilon}. \quad (22)$$

We now want to obtain some idea of the contributions from states of higher mass than the one-pion state already considered. The next state which properly should be considered is the three-pion state, but this appears to be too difficult to treat in any meaningful way. Since in a perturbation expansion sense the Fermi interactions always proceed through direct couplings of leptons with nucleons, we instead turn directly to the nucleon-antinucleon intermediate state as representative of the important higher mass states.

It becomes convenient now to consider, in place of $\langle n|P_\lambda|p\rangle$, the matrix element $\langle 0|P_\lambda|\bar{n}p\rangle$, where \bar{n} denotes an antineutron. In terms of the coefficients a and b already defined in (6), we have

$$(p_0 \bar{n}_0/m^2)^{\frac{1}{2}} \langle 0|P_\lambda|\bar{n}p\rangle = \bar{v}_{\bar{n}} [a i\gamma_\lambda \gamma_5 - b(p+\bar{n})_\lambda \gamma_5] u_p, \quad (23)$$

where v is the antiparticle spinor and the coefficients a and b are now regarded as functions of $\xi = (\bar{n}+p)^2$. For an “out” state one simply replaces a and b by their complex conjugates. We again form the absorptive part A_λ , introduce a sum over intermediate states, and this time select the contribution from intermediate nucleon pair states $|\bar{N}P\rangle$. We then have

$$A_\lambda = \pi \left(\frac{p_0}{m}\right)^{\frac{1}{2}} \sum_{\bar{N}P} \bar{v}_{\bar{n}} \langle 0|P_\lambda|\bar{N}P\rangle \times \langle \bar{N}P|F|p\rangle \delta(\bar{N}+P-\bar{n}-p), \quad (24)$$

where as usual one-half the sum over "in" and "out" states is understood.

Now the first factor, $\langle 0|P_\lambda|\bar{N}P\rangle$, can be expressed as in (23). The factor $\langle \bar{N}P|F|p\rangle$, on the other hand, is—for $|\bar{N}P\rangle$ an "out" state—just the matrix element for proton-antineutron scattering; and the delta function in (24) is just such that we require this matrix element only for physical values of the momenta. In fact, by evaluating (24) in the center-of-mass system for the nucleon pair, one sees that only the 3P_1 and 1S_0 scattering amplitudes are relevant. The matrix element $\langle 0|P_\lambda|\bar{N}P\rangle$ is proportional to the S -matrix element for production of a lepton pair by a nucleon pair. It turns out that the coefficient a is proportional to the amplitude for production by a nucleon pair in the 3P_1 state; and the combination $\{a - [(\bar{n}+p)^2/2m]b\}$ is proportional to the 1S_0 amplitude.

Let us denote by δ_1 the complex 3P_1 phase shift and by δ_0 the complex 1S_0 phase shift for \bar{n}, p scattering. We express $\langle \bar{N}P|F|p\rangle$ in terms of these phase shifts and now carry out the operations implied in (24).

For the nucleon pair contribution to $\text{Im}a$, we find

$$(\text{Im}a)_{\text{pair}} = [\text{Im}f_1 \text{Im}a + \text{Re}f_1 \text{Re}a]\theta(-\xi - 4m^2), \quad (25)$$

where $f_1 = e^{i\delta_1} \sin\delta_1$. The left-hand side denotes the nucleon pair contribution to $\text{Im}a$; on the right-hand side the true coefficient a is involved. The step function is inserted to remind us that the pair state contributes only for values of ξ corresponding to physical scattering: $\xi \leq -4m^2$. In the present case, since the one-pion state considered earlier makes no contribution to $\text{Im}a$, we can write $(\text{Im}a)_{\text{pair}} = \text{Im}a$, and therefore

$$\text{Im}a(\xi) = \tan\varphi_1(-\xi) \text{Re}a(\xi)\theta(-\xi - 4m^2), \quad (26)$$

where

$$\tan\varphi_1(-\xi) = \frac{\text{Re}e^{i\delta_1} \sin\delta_1}{1 - \text{Im}e^{i\delta_1} \sin\delta_1}. \quad (27)$$

The argument of φ_1 has been set equal to $-\xi$ for later convenience. The phase shift δ_1 is to be regarded as a function of the center-of-mass wave number for \bar{n}, p scattering: $k = (-\frac{1}{4}\xi - m^2)^{\frac{1}{2}}$.

Proceeding in the same way for the combination $[a - (\xi/2m)b]$ which is involved in 1S_0 scattering, we find

$$\begin{aligned} & \left[\text{Im} \left(a - \frac{\xi}{2m} b \right) \right]_{\text{pair}} \\ &= \left[\text{Im}f_0 \text{Im} \left(a - \frac{\xi}{2m} b \right) + \text{Re}f_0 \text{Re} \left(a - \frac{\xi}{2m} b \right) \right] \\ & \quad \times \theta(-\xi - 4m^2), \quad (28) \end{aligned}$$

where $f_0 = e^{i\delta_0} \sin\delta_0$. For later reference we define

$$\tan\varphi_0(-\xi) = \frac{\text{Re}e^{i\delta_0} \sin\delta_0}{1 - \text{Im}e^{i\delta_0} \sin\delta_0}. \quad (29)$$

We can now substitute the nucleon-pair contributions into the dispersion relations (21) and (22) and, treating these as integral equations, solve for a and b . The solutions are readily obtained and one finds^{11,12}

$$a(\xi) = g_A \exp \left\{ -\frac{\xi}{\pi} \int_{4m^2}^{\infty} dy \frac{\varphi_1(y)}{y(y+\xi-i\epsilon)} \right\}; \quad (30)$$

$$\begin{aligned} b(\xi) &= -\sqrt{2}GF(-m_\pi^2) \frac{1}{\xi + m_\pi^2} \\ & \times \exp \left\{ -\left(\frac{m_\pi^2 + \xi}{\pi} \right) \int_{4m^2}^{\infty} dy \frac{\varphi_0(y)}{(y - m_\pi^2)(y + \xi - i\epsilon)} \right\} \\ & + \frac{2m}{\xi} g_A \left[\exp \left\{ -\frac{\xi}{\pi} \int_{4m^2}^{\infty} dy \frac{\varphi_1(y)}{y(y+\xi-i\epsilon)} \right\} \right. \\ & \quad \left. - \exp \left\{ -\frac{\xi}{\pi} \int_{4m^2}^{\infty} dy \frac{\varphi_0(y)}{y(y+\xi-i\epsilon)} \right\} \right]. \quad (31) \end{aligned}$$

To evaluate these at the momentum transfer $\xi = m_\mu^2$ relevant for μ capture, it is legitimate to expand in powers of ξ and retain only the first nonvanishing terms. One finds

$$a(m_\mu^2) \approx g_A \left\{ 1 - \frac{m_\mu^2}{\pi} \int_{4m^2}^{\infty} dy \frac{\varphi_1(y)}{y^2} \right\}; \quad (32)$$

$$\begin{aligned} b(m_\mu^2) &\approx -\frac{\sqrt{2}GF(-m_\pi^2)}{m_\mu^2 + m_\pi^2} \\ & \times \left\{ 1 - \left(\frac{m_\pi^2 + m_\mu^2}{\pi} \right) \int_{4m^2}^{\infty} dy \frac{\varphi_0(y)}{y^2} \right\} \\ & + \frac{2mg_A}{\pi} \int_{4m^2}^{\infty} dy \frac{\varphi_0(y) - \varphi_1(y)}{y^2}. \quad (33) \end{aligned}$$

We notice that the value of $F(-m_\pi^2)$, as determined from the experimental pion-decay lifetime, can be represented by⁸

$$F(-m_\pi^2) = -0.13\sqrt{2}Gmg_A/(2\pi^2), \quad (34)$$

if we identify g_A with the Gamow-Teller coupling constant of β decay. (The algebraic sign follows from our earlier discussion of pion decay.) Now from the definitions (27) and (29) one sees that, provided there is

¹¹ These solutions are more easily obtained than might at first appear to be the case. Consider for example the form factor $a(\xi)$. We know this is a function analytic in the complex ξ plane, cut from $-4m^2$ to $-\infty$. We also know from (26) that just above the cut $\text{Im}a = \tan\varphi_1 \text{Re}a$; and we know that $a(0) = g_A$. Finally, we demand that the dispersion integral (14) shall exist. These demands fully specify the solution given. One proceeds likewise to solve for the combination $[a - (\xi/2m)b]$, except that here there is an additional requirement: namely, b has an isolated singularity at $\xi = -m_\pi^2$; that is, $\text{Im}b$ must have the δ function singularity indicated in (20).

¹² See also Federbush, Goldberger, and Treiman (to be published).

always some absorption in the nucleon-antinucleon interaction, the phase angles $\phi_{1,0}$ are confined to the range $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. Furthermore, on physical grounds we expect $\phi(4m^2)=0$.

One sees then that at the momentum transfer of μ capture the contributions from the intermediate nucleon-pair state are of order $(m_\mu^2+m_\pi^2)/m^2$ relative to the contribution from the one-pion state; i.e., the coefficients a and b are well approximated by the leading terms in (32) and (33). This result is not surprising. If the complex phase shifts δ_0 and δ_1 were better known than is the case we could, of course, compute the small corrections, but too little is known to justify any detailed calculations.

The "effective" pseudoscalar coupling constant in μ capture is just given by $m_\mu b(m_\mu^2) = "g_P"$. From (34), we find

$$"g_P" \approx \frac{0.5}{\pi} \left(\frac{G^2}{4\pi} \right) \left(\frac{mm_\mu}{m_\mu^2 + m_\pi^2} \right) g_A \approx 8g_A. \quad (35)$$

This is large enough so that the pseudoscalar contributions to μ -capture effects should be comparable to those coming from the axial vector and vector couplings.

B

One of the purposes of the present paper is to investigate the validity of certain assumptions which we adopted in a recent quantitative discussion of $\pi \rightarrow \mu + \nu$ decay.⁸ We pictured this process as occurring through pion dissociation into a nucleon pair, the latter annihilating to produce the leptons. Only the axial vector and pseudoscalar couplings in the Fermi interaction Lagrangian are relevant here and, as in the present discussion, we assumed that the Lagrangian contains no pseudoscalar coupling. However, as we have seen, the S -matrix element does contain a pseudoscalar term, as well as an axial vector term. In the discussion of pion decay, a knowledge of the form factors $a(\xi)$ and $b(\xi)$ is required for values of $\xi \leq -4m^2$. Nevertheless, in our earlier discussion we adopted for these the expressions obtained in the one-pion intermediate state approximation. What we want to show now is that our results on pion decay are essentially unaltered if we adopt the more complicated expressions (30) and (31), which include also the nucleon-pair contributions. These latter expressions are, of course, still not guaranteed to be accurate for large values of momentum transfer, since we are still neglecting a great many other intermediate states in computing a and b . But we feel that the pion decay discussion can now be put on a more firm footing.

In our work on pion decay we encountered an expression

$$R \equiv \text{Re} \left\{ K^*(\xi) \left[a(\xi) - \frac{\xi}{2m} b(\xi) \right] \right\}, \quad (36)$$

where a and b are the same form factors considered in the present paper, and $K(\xi)$ is related to the pion-

nucleon vertex function according to

$$\begin{aligned} \langle n\bar{p} \text{ out} | (m_\pi^2 - \square) \phi_\pi | 0 \rangle \\ = \left(\frac{m^2}{n_0 p_0} \right)^{\frac{1}{2}} \bar{u}_n i \gamma_5 v_{\bar{p}} K[(n+\bar{p})^2]. \end{aligned} \quad (37)$$

In reference 8 both a and b were real in the approximation considered there. In a certain approximation whose validity is not here under discussion, we found that

$$\begin{aligned} K(\xi) = \sqrt{2}G \exp \left\{ - \left(\frac{\xi + m_\pi^2}{\pi} \right) \right. \\ \left. \times \int_{4m^2}^{\infty} dy \frac{\varphi_0(y)}{(y - m_\pi^2)(y + \xi - i\epsilon)} \right\}, \end{aligned} \quad (38)$$

where $\varphi_0(y)$ is the same phase angle as in (29).

Collecting all our present results we find for R the expression

$$\begin{aligned} R = \left[g_A \exp \left(\frac{m_\pi^2}{\pi} \int_{4m^2}^{\infty} dy \frac{\varphi_0(y)}{y(y - m_\pi^2)} \right) \right. \\ \left. + \frac{\xi}{2m} \sqrt{2}GF(-m_\pi^2) \frac{1}{\xi + m_\pi^2} \right] H(\xi), \end{aligned} \quad (39)$$

where

$$\begin{aligned} H(\xi) = \sqrt{2}G \exp \left\{ - \frac{2}{\pi} (\xi + m_\pi^2) P \right. \\ \left. \times \int_{4m^2}^{\infty} dy \frac{\varphi_0(y)}{(y - m_\pi^2)(y + \xi)} \right\}; \end{aligned} \quad (40)$$

the symbol P denotes a principle value integration. The expression (39) differs in two respects from the one adopted in our earlier discussion, where we neglected the nucleon pair contribution to the form factors a and b : (1) The exponential which multiplies g_A in (39) was previously in effect set equal to unity; but the exponential is in fact not very different from unity, so this correction is quite insignificant. (2) The factor $H(\xi)$ in (39) was earlier replaced simply by $\text{Re}K(\xi)$. For the relevant values of ξ , namely $\xi \leq -4m^2$, we have

$$\begin{aligned} \text{Re}K(\xi) = \cos \varphi_0(-\xi) \sqrt{2}G \exp \left\{ - \left(\frac{\xi + m_\pi^2}{\pi} \right) \right. \\ \left. \times \int_{4m^2}^{\infty} dy \frac{\varphi_0(y)}{(y - m_\pi^2)(y + \xi)} \right\}. \end{aligned} \quad (41)$$

The two functions, $H(\xi)$ and $\text{Re}K(\xi)$, certainly differ significantly. But our final expression⁸ for the pion decay rate involves a certain integral over $H(\xi)$; in the earlier treatment, over $\text{Re}K(\xi)$. Provided only that the value of this integral is large compared to $(G^2/2\pi^2)^{-1} \approx 0.1$, the decay rate expression is in fact essentially independent of the value of the integral in question. In

our earlier discussion, in which $H(\xi)$ was replaced by $\text{Re}K(\xi)$, we showed that, for reasonable assumptions concerning the complex phase shift δ_0 , this circumstance obtains. A similar discussion can be made to show that the same holds true if we use the more accurate expression $H(\xi)$. To summarize: in our earlier formula for the pion decay rate, $\text{Re}K$ must be replaced by H ; but the final answer—for reasonable behavior of δ_0 —is independent of either. The numerical result quoted earlier stands unchanged.

C

We now turn to a discussion of the form factors c and d defined in Eq. (7). There is no strong motivation here to make a very careful analysis, since we expect that the form factor d is in any case too small to be detected in β decay or μ capture and that the coefficient c does not differ appreciably in μ capture from the value g_V relevant in β decay. After briefly setting up the problem, we shall therefore resort to rather crude approximations to estimate the magnitude of the effects involved.

As in (12), we can write the matrix element $\langle n | V_\lambda | p \rangle$ in the form

$$\left(\frac{p_0 m_0}{m^2}\right)^{\frac{1}{2}} \langle n | V_\lambda | p \rangle = i \left(\frac{p_0}{m}\right)^{\frac{1}{2}} \bar{u}_n \int d^4x e^{-in \cdot x} \langle 0 | [V_\lambda, F(x)] \theta(-x_0) | p \rangle. \quad (42)$$

The assumed dispersion relations are

$$c(\xi) = g_V - \frac{\xi}{\pi} \int d\xi' \frac{\text{Im}c(-\xi')}{\xi'(\xi' + \xi - i\epsilon)}, \quad (43)$$

and

$$d(\xi) = \frac{1}{\pi} \int d\xi' \frac{\text{Im}d(-\xi')}{\xi' + \xi - i\epsilon}, \quad (44)$$

where again we assume that no subtraction is required for the form factor d . In these expressions $\xi = (n - p)^2$, as before.

The absorptive part of (42), call it B_λ , has the same structure as in (15), and with one half the sum of "in" and "out" states understood, we have that

$$B_\lambda = \pi (p_0/m)^{\frac{1}{2}} \sum_s \bar{u}_n \langle 0 | V_\lambda | s \rangle \langle s | F | p \rangle \delta(p_s + n - p). \quad (45)$$

In the present case, since V_λ transforms like a four-vector, the state s of lowest mass which can contribute is the two-pion state; and for the small momentum transfers relevant for β decay and μ capture we expect this lowest mass state to be the most significant one. To evaluate this two-pion contribution, we need to know the matrix elements $\langle 0 | V_\lambda | k_j q_i \rangle$ and $\langle k_j q_i | F | p \rangle$, where k and q denote the momenta of the two intermediate pions and the indices i and j are charge labels.

The present situation is similar in structure to the one

encountered in a study of the electromagnetic form factors for nucleons. The only difference is that V_λ is there replaced by the isotopic vector part of the current density operator. The matrix element $\langle k_j q_i | F | p \rangle$ is an analytic continuation of the pion-nucleon scattering matrix element; in connection with the problem of nucleon electromagnetic structure it has been shown that it may, with sufficient accuracy, be treated in lowest order perturbation theory.^{12,13} We shall so treat it here. The vertex $\langle 0 | V_\lambda | k_j q_i \rangle$ may itself now be studied via dispersion relation techniques, again in analogy with our treatment in the electromagnetic structure problem. The only difference is that here we assume that there is no point interaction analogous to the direct production of a meson pair by a photon.¹⁴ The general structure of the matrix element is given by

$$(4k_0 q_0)^{\frac{1}{2}} \langle 0 | V_\lambda | k_j q_i \text{ in} \rangle = \left(\frac{1}{2}\right)^{\frac{1}{2}} \epsilon_{-,ij} (q-k)_\lambda W[(q+k)^2], \quad (46)$$

where $\epsilon_{-,ij} = \epsilon_{1ij} - i\epsilon_{2ij}$. We take for $W(\xi)$ the dispersion relation

$$W(\xi) = -\frac{\xi}{\pi} \int_{(2m_\pi)^2}^{\infty} d\xi' \frac{\text{Im}W(-\xi')}{\xi'(\xi' + \xi - i\epsilon)}, \quad (47)$$

which is in line with our assumption that there is no direct interaction coupling two pions to a lepton pair.

The absorptive part of (46), call it C_λ , can be obtained in the standard way and we find

$$C_\lambda = \frac{\pi(2q_0)^{\frac{1}{2}}}{\sqrt{2}} \sum_s \langle 0 | V_\lambda | s \rangle \langle s | J_j | q_i \rangle \delta(p_q - q - k), \quad (48)$$

where J_j is the source of the pion field. We shall now content ourselves with evaluating this in lowest order perturbation theory, in which case the only relevant intermediate state is that consisting of a nucleon-antinucleon pair.

To effect the evaluation of (48), we substitute for the matrix elements the following perturbation approximations:

$$(N_0 \bar{N}_0 / m^2)^{\frac{1}{2}} \langle 0 | V_\lambda | N \bar{N} \rangle = g_V \bar{v}_{\bar{N}} \gamma_\lambda u_N, \quad (49)$$

$$(N_0 \bar{N}_0 / m^2)^{\frac{1}{2}} \langle N \bar{N} | J_j | q_i \rangle$$

$$= \frac{-G^2}{(2q_0)^{\frac{1}{2}}} \bar{u}_N \frac{1}{2} [\tau_i, \tau_j] \frac{i\gamma \cdot (q-k)}{2} v_{\bar{N}}$$

$$\times \left[\frac{1}{(N-k)^2 + m^2} + \frac{1}{(N-q)^2 + m^2} \right], \quad (50)$$

where N and \bar{N} denote nucleon and antinucleon (the charges are not specified here since we are now including

¹³ Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. **110**, 265 (1958).

¹⁴ In the model of Fermi interactions recently discussed by Feynman and Gell-Mann (reference 4), a direct point interaction is assumed to exist. This would imply in Eq. (47) an additive constant, $W(0)$, of known strength.

isotopic-spin variables in our sum over states). Carrying out the operations indicated in (48) and comparing the result with (46) we find, with neglect of terms of order $(m_\pi/m)^2$,

$$\text{Im}W(-\xi) = -g_V \left(\frac{G^2}{4\pi}\right) \left(\frac{\xi-4m^2}{\xi}\right)^{\frac{1}{2}} \theta(\xi-4m^2). \quad (51)$$

From (47) we now find

$$\text{Re}W(-\xi) = \frac{-2\xi}{mm_\pi^2} \left(\frac{f^2}{4\pi}\right) g_V \frac{4m^2}{\xi} \left\{ 1 - \frac{(1-\xi/4m^2)^{\frac{1}{2}}}{(\xi/4m^2)^{\frac{1}{2}}} \times \tan^{-1} \left(\frac{(\xi/4m^2)^{\frac{1}{2}}}{(1-\xi/4m^2)^{\frac{1}{2}}} \right) \right\}. \quad (52)$$

We have introduced $(f^2/4\pi) = (G^2/4\pi)(m_\pi^2/4m^2) \approx 0.08$. The above result holds for $\xi < 4m^2$. For $\xi > 4m^2$ the arc tangent is to be replaced by a logarithm. We now have from (46) and (52) the matrix element $\langle 0 | V_\lambda | k_j q_i \rangle$ required for evaluation of the two-pion contribution to B_λ in (45). For the other matrix element, $\langle k_j q_i | F | p \rangle$, we adopt the perturbation result, which, except for trivial changes, is given by (50). We now evaluate B_λ and hence $\text{Im}c$ and $\text{Im}d$. Finally the form factors c and d are obtained from the dispersion relations (43) and (44). Although in these dispersion integrals the variable ξ runs from $(2m_\pi)^2$ all the way to infinity, the important contributions come from values of ξ near the lower limit $(2m_\pi)^2$. We therefore approximate the complicated expression (52) by evaluating W at $\xi = (2m_\pi)^2$ and treating it as a constant in the dispersion integral. Furthermore, the behavior of $\text{Re}W(-\xi)$ for large ξ is undoubtedly given incorrectly by our perturbation approach: it goes as $\ln \xi$ instead of to zero as one would expect.

If one makes the above replacement of $\text{Re}W(-\xi)$ by its value at $\xi = (2m_\pi)^2$, the problem is reduced to exactly the one that has already been treated for the electromagnetic structure.¹³ One needs only replace in the isotopic vector charge and magnetization density form factors, at the appropriate places, the charge e by $2 \text{Re}W(-4m_\pi^2)$. The factor of two arises from the differences in isotopic-spin structure. We record here for convenience the electromagnetic form factors as computed from perturbation theory, calling F_1 the charge density and F_2 the magnetization density:

$$F_1(\xi) \simeq \frac{e}{2} - \frac{e}{12} \frac{(0.24)}{m_\pi^2} \xi + \dots, \quad (53a)$$

$$F_2(\xi) \simeq 1.7 \frac{e}{2m} \left[1 - \frac{1}{6} \frac{(0.12)}{m_\pi^2} \xi + \dots \right]. \quad (53b)$$

Making the replacement $e \rightarrow -2(8/3\pi)(f^2/4\pi)g_V$, we find

$$c(\xi) = g_V \left[1 + \frac{4}{9\pi} \left(\frac{f^2}{4\pi}\right) (0.24) \frac{\xi}{m_\pi^2} + \dots \right], \quad (54a)$$

$$d(\xi) = 1.7 \frac{g_V}{2m} \times \frac{16}{3\pi} \left(\frac{f^2}{4\pi}\right) \left[1 - \frac{1}{6} (0.12) \frac{\xi}{m_\pi^2} + \dots \right]. \quad (54b)$$

We see that for the values of ξ of interest in β decay and μ capture, $c(\xi) \approx g_V$ and $d(\xi) \approx d(0)$. Further, the value of $d(0)$ is exceedingly small. Reference to Eq. (8) shows that $d(0)$ is multiplied by lepton momenta which even in μ -capture are no larger than m_μ . Hence $m_\mu d(0) \approx 1/100$, which is far too small to have any measurable effects.

It is our feeling that in spite of the crudity of these estimates they are not in error by the several orders of magnitude which would be required to make the effects significant.

D

We return here briefly to the basic dispersion relation (14) for the form factor a . If one supposes that a subtraction is not required, then the previously neglected contribution from the equal time commutator must be taken into account and one would have that

$$a(\xi) = f_A + \frac{1}{\pi} \int d\xi' \frac{\text{Im}a(-\xi')}{\xi' + \xi - i\epsilon}. \quad (14')$$

However, there still remains the possibility of an additional additive constant in the above equation, even if $\text{Im}a(-\xi')$ approaches zero as $\xi' \rightarrow +\infty$. The presence or absence of such a constant cannot be argued. If, for example, one says that both the real and imaginary parts of a approach zero at infinity, the constant must, of course, be precisely $-f_A$. If $\text{Im}a$ alone is supposed to approach zero, but $\text{Re}a$ is allowed to approach a constant, then, unless this limiting value is specified, the problem of an additive constant above is still unsettled.

If one were to conjecture that $\text{Re}a \rightarrow f_A$ as $\xi \rightarrow \infty$, the solution of (14') would be, in our approximation of taking into account only the one-pion and the nucleon-pair intermediate states,

$$a(\xi) = f_A \exp \int_{4m^2}^{\infty} d\xi' \frac{\varphi_1(\xi')}{\xi' + \xi - i\epsilon}. \quad (14'')$$

For our practical purposes we have avoided these speculations by using the subtracted dispersion relation (14). We have therefore made no attempt to evaluate, in any sense, the renormalized coupling constant in terms of the unrenormalized f_A .