BLOCK DIAGONAL GMD FOR ZERO-PADDED MIMO FREQUENCY SELECTIVE CHANNELS WITH ZERO-FORCING DFE

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ABSTRACT

In the class of systems with linear precoder and zero-forcing (ZF) DFE for zero-padded MIMO frequency selective channels, existing optimal transceiver designs present two major drawbacks. First, the optimal system requires a large number of bits to encode the full precoding matrix. Second, the full precoding matrix leads to complex computations. These disadvantages become more severe as bandwidth (BW) efficiency increases. In this article, we propose using the block diagonal geometric mean decomposition (BD-GMD) technique to design an alternative transceiver. The proposed ZF-BD-GMD system uses a block diagonal orthogonal precoder matrix structure to reduce the required number of encoding bits and simplifies the computation. While solving the current optimal system's drawbacks, the ZF-BD-GMD system also produces a similar bit error rate (BER) performance when the block size is large. In other words, the ZF-BD-GMD system is asymptotically optimal in the class of communication systems with linear precoder and ZF-DFE receiver. [1]

Index Terms— Decision Feed Back, Szego's Theorem, Geometric Mean Decomposition, Block Diagonal Matrix, Block Toeplitz Matrix.

I. INTRODUCTION

In high-rate digital communication systems, MIMO frequency selective (FS) channels complicate the transceiver design process because of the inter-block-interference (IBI) effect. However, by applying the zero-padding precoding technique, we can eliminate the IBI and convert the FS channel into an equivalent block channel [7], [1]. From the equivalent block channel matrix, we can derive the optimal system (which we call the ZF-Optimal system) for systems using linear precoder and zero-forcing DFE (ZF-DFE) [8]. However, the ZF-Optimal system suffers from two drawbacks. First, it requires a large number of bits from the receiver to encode the full precoding matrix and feed it back to the transmitter [4]. Second, the full precoding matrix multiplication is computationally complex. These disadvantages become more apparent when the block size is large.

The block diagonal GMD (BD-GMD) is proposed in [3] to design transceivers for MIMO broadcast channels. In this paper, we propose the use of the BD-GMD technique to design a transceiver that solves the above mentioned drawbacks. A ZF-BD-GMD system, which uses block diagonal unitary precoder and ZF-DFE receiver, is proposed. Since the ZF-BD-GMD system's precoder is block diagonal, it requires a much less number of bits to encode the precoding matrix.

In addition, the matrix multiplication at transmitter is much simpler due to the block diagonal structure.

We also analyze the performance and the implementation cost of the proposed system, and find four important properties. First, subchannel gains are non-increasing with channel indices. Second, a tight lower bound for the worst subchannel gains is provided. Third, we prove that as the block size gets larger and approaches infinity, the BER ratio between the ZF-BD-GMD system and the ZF-Optimal system also approaches unity. In this case, the resulting BW efficiency also approaches unity. In other words, the ZF-BD-GMD transceiver performs similarly to the ZF-Optimal system when the block size is large. Fourth, there are many zero elements in the feedforward and the feedback matrices. This leads to simple computations and therefore reduces implementation cost in the receiver.

Summarizing the four properties, the ZF-BD-GMD system is asymptotically optimal in the class of systems with linear precoder and ZF-DFE. With much less complexity in transmitter and receiver implementations, the proposed ZF-BD-GMD system is more desirable than the ZF-Optimal system.

Our finding is presented in the following sections: Section II introduces the signal model and some preliminaries; Section III discusses the proposed ZF-BD-GMD system; Section IV provides numerical simulations. Concluding remarks are given in Section V. [2]

II. SIGNAL MODEL AND PROBLEM FORMULATION

We consider a point-to-point communication system with N_T transmit antennas and N_R receiving antennas. The input-output relation of the frequency selective MIMO channel can be expressed as

$$\mathbf{y}_i = \sum_{k=0}^{L} \mathbf{H}_k \mathbf{x}_{i-k} + \mathbf{n}_i \qquad (1)$$

where $\mathbf{x}_i$ is the $N_T \times 1$ transmitted signal, $\mathbf{H}(z) = \mathbf{H}_0 + \mathbf{H}_1 z^{-1} + \cdots + \mathbf{H}_L z^{-L}$ is the Lth order $N_R \times N_T$ frequency selective FIR MIMO channel, $\mathbf{n}_i$ is the additive channel noise, and $\mathbf{y}_i$ is the $N_R \times 1$ received vector. The noise covariance matrix is assumed to be $\mathbf{R}_n = \sigma_n^2 \mathbf{I}$. The zero-padded system is to transmit N_P zero vectors after every K symbol vectors. That is, in $K + N_P$ symbol durations,

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[2]The following notations are used in the paper. Boldface upper-case letters denote matrices, boldface lower-case letters denote column vectors, and italics denote scalars. The superscript $(\cdot)^H$ denotes transpose conjugation. $[\mathbf{A}]_{ij}$ denotes the (i, j)th element of the matrix $\mathbf{A}$. $[\mathbf{A}]_{i \times j}$ denotes the $i \times j$ matrix containing the first i rows and j columns of the matrix $\mathbf{A}$. By $\mathbf{A} \succeq \mathbf{B}$, we mean $\mathbf{A} - \mathbf{B}$ is positive semi-definite.

the following is transmitted: $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_K, \mathbf{0}, \cdots, \mathbf{0}\}$. In order to prevent contamination from previous blocks, one must choose $N_P \geq L$. The bandwidth efficiency is defined as

$$\epsilon = \frac{K}{K + N_P} \qquad (2)$$

Note that as long as $N_P \geq L$, the I/O relation is not affected even if we choose larger N_P. Therefore it is desirable to choose $N_P = L$, so that the BW efficiency is maximized to be $K/(K + L)$. In this case, the I/O relation of the ZP system can be expressed as an equivalent block channel:

$$\underbrace{\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_{K+L} \end{bmatrix}}_{\mathbf{y}_{ZP,K}} = \mathbf{H}_{ZP,K} \underbrace{\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_K \end{bmatrix}}_{\mathbf{x}_{ZP,K}} + \underbrace{\begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \\ \vdots \\ \mathbf{n}_{K+L} \end{bmatrix}}_{\mathbf{n}_{ZP,K}} \qquad (3)$$

where

$$\mathbf{H}_{ZP,K} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{H}_1 & \mathbf{H}_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{H}_L & \vdots & \ddots & \mathbf{H}_0 \\ \mathbf{0} & \mathbf{H}_L & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_L \end{bmatrix} \qquad (4)$$

where K in the subscript denotes that $\mathbf{H}_{ZP,K}$ has KN_T columns. Here we assume $(K + L)N_R \geq KN_T$, so that it is possible to achieve the zero-forcing condition.

We consider the system where the transmitted vector is linearly precoded by a $N_T K \times N_T K$ matrix $\mathbf{P}$: $\mathbf{x}_{ZP,K} = \mathbf{Ps}$, where $\mathbf{s} = \{\mathbf{s}_1^T, \mathbf{s}_2^T, \cdots, \mathbf{s}_K^T\}^T$, and $\mathbf{s}_i$ is the $N_T \times 1$ transmitted symbol vector. Here we assume the transmitted signal is zero-mean and uncorrelated, with covariance $E[\mathbf{s}_i \mathbf{s}_j^H] = \delta(i - j)\sigma_s^2 \mathbf{I}$. Each symbol is selected from the same constellation. We define a constant α, which stands for the noise to signal symbol power ratio:

$$\alpha \doteq \sigma_n^2 / \sigma_s^2. \qquad (5)$$

The power on the transmitted vector $\mathbf{x}_{ZP,K}$ is restricted to be $\leq KN_T\sigma_s^2$. Since the transmitted symbol is white, the constraint on the precoder becomes $\text{Tr}[\mathbf{PP}^H] \leq KN_T$. Note that the power constraint is increasing linearly with K, which is crucial to make fair comparison for systems with different value of K. We consider the QAM constellation. In this case the BER will be the function of SINR of the input to the decision device [6], i.e.,

$$\text{BER}(\text{SINR}_i) = \gamma Q(\beta \sqrt{\text{SINR}_i}) \qquad (6)$$

where γ and β are constants which depend on the constellation, and $Q(\cdot)$ is the Q-function defined as $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-\lambda^2/2} d\lambda$.

We can treat the I/O relation (3) as an effective block channel communication system. For the zero-forcing case, the optimal solution for minimizing the average BER under the total power constraint is suggested by the Theorem 1 in [8]. The optimal precoder is with no loss of generality a unitary matrix. The optimal receiver is the corresponding optimal ZF-DFE solution suggested in section III in [8]. Based on the no-error-propagation assumption, the resulting system

acts similar to parallel independent Gaussian channels with channel gains $[\mathbf{L}]_{ii}$, where $\mathbf{L}$ is the matrix such that the QR decomposition of $\mathbf{HP}$ is $\mathbf{HP} = \mathbf{QL}^H$. Here $\mathbf{P}$ is the optimal precoder, $\mathbf{Q}$ is a unitary matrix, and $\mathbf{L}^H$ is a upper triangular matrix.

However, the optimal precoder for ZF DFE suffers from two disadvantages mentioned in Sec. I. To resolve these two drawbacks, we propose using BD-GMD theory [3] to design the transceiver.

III. ZERO FORCING BD-GMD SYSTEM

The block-diagonal geometric mean decomposition (BD-GMD) technique was introduced in [3] to design transceivers that use dirty-paper coding for MIMO broadcast channels. The schemes in [3] decompose each user's MIMO channel into parallel subchannels with identical SNRs/SINRs, thus equal-rate coding can be applied across the subchannels of each user.

In this section we introduce the Zero-Forcing BD-GMD (ZF-BD-GMD) system, which uses block diagonal unitary linear precoder and zero-forcing DFE for zero-padded MIMO FS channels. Let us consider the BD-GMD of the matrix $\mathbf{H}_{ZP,K}^H$, i.e., (the algorithm for computing the decomposition is in Sec. III-A in [3])

$$\mathbf{H}_{ZP,K}^H = \underbrace{\begin{bmatrix} \mathbf{P}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{P}_K \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} \mathbf{L}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \times & \mathbf{L}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \times & \cdots & \times & \mathbf{L}_K \end{bmatrix}}_{\mathbf{L}} \mathbf{Q}^H$$

where $\mathbf{P}_i$'s are $N_T \times N_T$ unitary matrices, $\mathbf{Q}$ is a $(K + L)N_R \times KN_T$ matrix with orthonormal columns, each $N_T \times N_T$ matrix $\mathbf{L}_i$ is lower triangular with equal diagonal terms r_i within itself, and '$\times$' refers to some nonzero entries. The proposed BD-GMD transceiver is based on this decomposition. The block diagonal precoder is chosen as the block diagonal matrix $\mathbf{P}$, and the receiving forward filter is chosen as $\mathbf{Q}^H$. Since $\mathbf{Q}$ has orthonormal columns, the channel noise after $\mathbf{Q}^H$ is still white with variance σ_n^2. The effective channel after the linear precoder $\mathbf{P}$ and feedforward filter $\mathbf{Q}^H$ acts like a triangular channel matrix $\mathbf{L}^H$ with additive white Gaussian channel noise. This triangular structure facilitates simple decision feedback equalization. Fig. 1 shows the transceiver structure of the ZF-BD-GMD system.
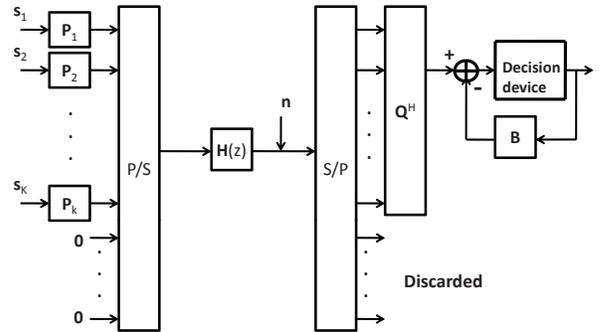

Fig. 1. The ZF-BD-GMD transceiver.

As in many analyses of DFE systems [9], we assume that there is no error propagation. Based on this assumption,

the overall system behaves similar to a system with KN_T independent parallel SISO AWGN channels. Each channel has identical noise variance σ_n^2 but with a different channel gain $[\mathbf{L}]_{ii}$. Since the transmitted symbol has energy σ_s^2, the SINR in ith stream before the detection device is $\text{SINR}_i = |[\mathbf{L}]_{ii}|^2 \sigma_s^2 / \sigma_n^2 = |[\mathbf{L}]_{ii}|^2 / \alpha$. The BER of ith stream will be the function of SINR [6], i.e., $\text{BER}(\text{SINR}_i) = \gamma Q(\beta \sqrt{\text{SINR}_i}) = \gamma Q(\beta \sqrt{|[\mathbf{L}]_{ii}|^2 / \alpha})$. Therefore, to analyze the performance of the ZF-BD-GMD system, we have to study the diagonal terms of $\mathbf{L}$.

Since the small lower triangular $\mathbf{L}_i$ has identical diagonal terms, we use r_i to denote the diagonal terms in $\mathbf{L}_i$. From the property of BD-GMD (Eq. (21) in [3]), we have

$$r_i = \left(\frac{\det(\mathbf{H}_{ZP,i}^H \mathbf{H}_{ZP,i})}{\det(\mathbf{H}_{ZP,i-1}^H \mathbf{H}_{ZP,i-1})} \right)^{\frac{1}{2N_T}} \qquad (7)$$

We notice that r_i is independent of the block length K. That is, even if we increase K, it does not change the previous symbol stream performance. The following is our first theorem.

Theorem 3.1: r_m is non-increasing. That is, for $m \geq 2$, $r_m \leq r_{m-1}$.
   Proof: We define

$$\tilde{\mathbf{H}}_m \doteq \begin{bmatrix} \mathbf{H}_m^H & \mathbf{H}_{m+1}^H & \cdots & \mathbf{H}_L^H \end{bmatrix} \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \\ \vdots \\ \mathbf{H}_{L-m} \end{bmatrix} \qquad (8)$$

for $m = 0, 1, \cdots, L$, and $\tilde{\mathbf{H}}_m = \mathbf{0}$ for $m > L$. We also define

$$\tilde{\mathbf{H}}_{-k} = \tilde{\mathbf{H}}_k^H$$

Based on the Block Toeplitz structure of $\mathbf{H}_{ZP,m}$, we can write

$$\mathbf{H}_{ZP,m+1}^H \mathbf{H}_{ZP,m+1} = \begin{bmatrix} \mathbf{H}_{ZP,m}^H \mathbf{H}_{ZP,m} & \mathbf{B}_m^H \\ \mathbf{B}_m & \tilde{\mathbf{H}}_0 \end{bmatrix} \qquad (9)$$

where $\mathbf{B}_m = \begin{bmatrix} \tilde{\mathbf{H}}_{-m} & \tilde{\mathbf{H}}_{-(m-1)} & \cdots & \tilde{\mathbf{H}}_{-1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{H}}_{-m} & \mathbf{B}_{m-1} \end{bmatrix}$. By taking the Schur form [2] of (9) and taking the determinant, we have $\det(\mathbf{H}_{ZP,m+1}^H \mathbf{H}_{ZP,m+1}) = \det(\mathbf{H}_{ZP,m}^H \mathbf{H}_{ZP,m}) \det(\tilde{\mathbf{H}}_0 - \mathbf{B}_m(\mathbf{H}_{ZP,m}^H \mathbf{H}_{ZP,m})^{-1} \mathbf{B}_m^H)$.
   Using (7), r_m can be written as

$$r_m = \det(\tilde{\mathbf{H}}_0 - \mathbf{B}_{m-1}(\mathbf{H}_{ZP,m-1}^H \mathbf{H}_{ZP,m-1})^{-1} \mathbf{B}_{m-1}^H)^{\frac{1}{2N_T}}$$

If we define $\mathbf{C}_m = \begin{bmatrix} \tilde{\mathbf{H}}_1 & \tilde{\mathbf{H}}_2 & \cdots & \tilde{\mathbf{H}}_m \end{bmatrix}$, we have

$$\mathbf{B}_m(\mathbf{H}_{ZP,m}^H \mathbf{H}_{ZP,m})^{-1} \mathbf{B}_m^H$$
$$= \begin{bmatrix} \tilde{\mathbf{H}}_{-m} & \mathbf{B}_{m-1} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{H}}_0 & \mathbf{C}_{m-1} \\ \mathbf{C}_{m-1}^H & \mathbf{H}_{ZP,m-1}^H \mathbf{H}_{ZP,m-1} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{H}}_{-m}^H \\ \mathbf{B}_{m-1}^H \end{bmatrix}$$
$$\succeq \mathbf{B}_{m-1}(\mathbf{H}_{ZP,m-1}^H \mathbf{H}_{ZP,m-1})^{-1} \mathbf{B}_{m-1}^H$$

where the last inequality follows from Lemma 1 in [10].
   Therefore, we are able to establish the inequality

$$\tilde{\mathbf{H}}_0 - \mathbf{B}_m(\mathbf{H}_{ZP,m}^H \mathbf{H}_{ZP,m})^{-1} \mathbf{B}_m^H$$
$$\preceq \tilde{\mathbf{H}}_0 - \mathbf{B}_{m-1}(\mathbf{H}_{ZP,m-1}^H \mathbf{H}_{ZP,m-1})^{-1} \mathbf{B}_{m-1}^H$$

Taking the determinant, we arrive at $r_{m+1} \leq r_m$. ∎

Theorem 3.1 states that the subchannel gains of the ZF-BD-GMD system are in a non-increasing order. Thus, when we increase K to increase the BW efficiency, the additional sub-streams will never have better performance than the existing ones.

Although the subchannel gains are non-increasing, in Theorem 3.2 below we provide the tight lower bound for the worst subchannel gain.

Theorem 3.2: Suppose $\tilde{\mathbf{H}}(e^{j\omega}) \doteq \sum_{k=-\infty}^{\infty} \tilde{\mathbf{H}}_k e^{-jk\omega}$. If $-\infty < \int_{-\pi}^{\pi} \log \det \tilde{\mathbf{H}}(e^{j\omega}) d\omega$, then the worst substream channel gain is lower bounded as follows:

$$\lim_{M \to \infty} r_M \doteq r = \exp\left(\frac{1}{2N_T} \int_{-\pi}^{\pi} \log \det \tilde{\mathbf{H}}(e^{j\omega}) \frac{d\omega}{2\pi} \right)$$

The proof can be found in [10]. The above two theorems facilitate the derivation of the third theorem, which states the asymptotic optimality of the ZF-BD-GMD transceiver. The proof can be found in [10] as well.

Theorem 3.3: The average BER of the ZF-BD-GMD transceiver approaches the average BER of the ZF-Optimal system. That is,

$$\lim_{K \to \infty} \frac{\text{BER}_{ZFBDGMD}(K)}{\text{BER}_{ZFoptimal}(K)} = 1$$

where $\text{BER}_{ZFBDGMD}(K)$ and $\text{BER}_{ZFoptimal}(K)$ denote the average BER of the ZF-BD-GMD system and the ZF-Optimal system, respectively, when the block size is K. Thus, the ZF-BD-GMD transceiver is asymptotically optimal when $K \to \infty$.

III-A. Implementation Cost of the ZF-BD-GMD system

For the transmitter side, the total cost of forming the transmitted vector $\mathbf{x}_{ZP,K}$ is K matrix (with size $N_T \times N_T$) multiplications, which is in the order of $O(KN_T^2)$. Compared to $O(K^2 N_T^2)$ in the ZF-Optimal system, there will be K times saving. Now let us look at the receiver side. The lower triangular feedback matrix $\mathbf{L}$ consists of $K \times K$ blocks and each block is an $N_T \times N_T$ matrix. The matrix $\mathbf{Q}$ consists of $(K+L) \times K$ blocks and each block is an $N_R \times N_T$ matrix. Both $\mathbf{L}$ and $\mathbf{Q}$ contains many zero elements:

Theorem 3.4: In the ZF-BD-GMD system, $\mathbf{L}$ and $\mathbf{Q}$ both have lower block bandwidth[3] L, where L is the order of the frequency selective channel. That is, whenever $i > j + L$, the (i, j)th block in $\mathbf{L}$ is a $\mathbf{0}_{N_T \times N_T}$ zero matrix, and the (i, j)th block in $\mathbf{Q}$ is a $\mathbf{0}_{N_R \times N_T}$ zero matrix.

The proof can be found in [10]. Since $\mathbf{L}$ is a lower triangular matrix, this theorem implies $\mathbf{L}$ is a block banded matrix with $(L + 1)$ bands (including the main block diagonal). We can calculate the approximate number of non-zero entries in $\mathbf{L}$: $\left(\frac{(2K-L-1)L}{2} \right) N_T^2 + K \left(\frac{N_T^2 + N_T}{2} \right) \approx K \left((L + 1/2)N_T^2 + N_T/2 \right)$, which grows linearly with K when K is large. In the ZF-Optimal system, the number of non-zero entries in $\mathbf{L}$ is $(K^2 N_T^2 + KN_T)/2$. The number of non-zero entries in $\mathbf{L}$ corresponds to the number of feedback paths in the DFE. Therefore, the ZF-BD-GMD systems saves tremendously in the number of feedback paths. The number of non-zero entries in $\mathbf{Q}$ corresponds to the number of

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[3]The block bandwidth for a block matrix is defined similarly to the bandwidth defined in p. 152 of [2] originally for matrix with scalar entries.

operations when the signal is passed through the feedforward filter. We can also calculate the number of non-zero entries in $\mathbf{Q}$: $N_T N_R (K + L)K - N_T N_R (K^2 - K)/2 \approx N_T N_R K^2/2$ when K is large. In contrast to the ZF-Optimal system, in which the number of non-zero entries in $\mathbf{Q}$ is about $N_T N_R K^2$, the ZF-BD-GMD feedforward part saves half of the operations.

To summarize, the proposed ZF-BD-GMD system is asymptotically optimal when K is large. In addition, it has much less complexity in the transmitter and receiver implementations.

III-B. Lazy Precoder for SISO FS Channels

For the SISO case ($N_T = N_R = 1$), the precoder in the ZF-BD-GMD system will be diagonal and unitary. It can be proved (see [10]) that the ZF-BD-GMD system has the same BER performance as the lazy precoder system, (i.e., with identity precoder). Thus the lazy precoder system inherits all the benefits from the ZF-BD-GMD system. Therefore, *the lazy precoder system is asymptotically optimal when $K \to \infty$*. This makes the lazy precoder system a more favorable design than the ZF-Optimal system, since it requires no channel information and no precoding in the transmitter.

IV. NUMERICAL SIMULATIONS

In the numerical simulations, symbols are generated using gray encoded QPSK constellations with each symbol power σ_s^2. For each case, 10^3 Rayleigh fading channels are used for the Monte Carlo simulations. Those channels have the entries coming from i.i.d complex zero-mean Gaussian distributions with unit variance. The additive channel noise has covariance matrix $\mathbf{R}_n = \mathbf{I}$.

In Fig. 2 we show the simulation results for the case of two transmitting antennas and two receiving antennas. The MIMO channels have order $L = 2$. The zero-forcing system performances for $K = 3$, $K = 10$, and $K = 20$ are shown. The ZF-Optimal system appears to have the best performance for all K. For a large K, the ZF-BD-GMD system performs similarly to the ZF-Optimal system. This is consistent with Theorem 3.3. The performance of systems with lazy precoder and ZF DFE is also plotted for comparison.

V. CONCLUDING REMARKS

The ZF-BD-GMD system has been proposed to address the two well-known drawbacks in the optimal system for zero-padded MIMO frequency selective channels. The ZF-BD-GMD system is shown to be asymptotically optimal when the bandwidth efficiency approaches unity. In addition, it has much lower implementation cost than the optimal system. Thus, it appears to be a favorable candidate for practical implementation. We also discussed the tradeoff between the BW efficiency and the BER performance for the ZF-BD-GMD system and the ZF-Optimal systems. Numerical simulations were provided to confirm the theoretical findings in this paper. This paper focus on the ZF-DFE. The case with MMSE-DFE is under investigation. It appears that by performing the BD-GMD on $\left[\mathbf{H}_{ZP,K}^H \quad \sqrt{\alpha}\mathbf{I} \right]$, similar results for the MMSE-DFE case can be obtained.

VI. REFERENCES

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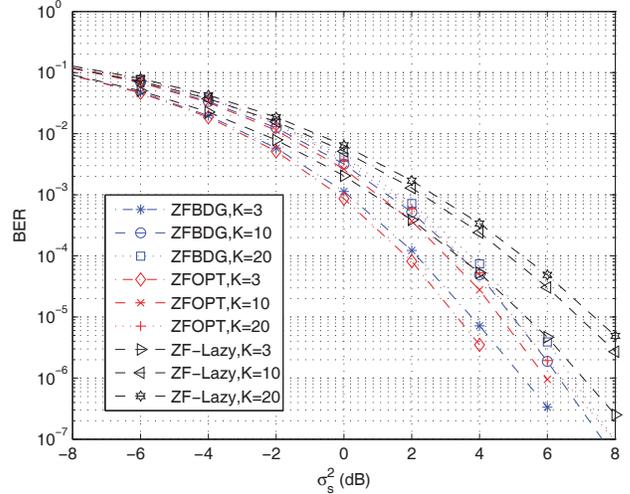
Fig. 2. The BER performance of the zero-forcing systems for MIMO ($N_T = N_R = 2$) Rayleigh channels of order 3, with $K = 3$, $K = 10$, and $K = 20$. "ZFBDG" represents the ZF-BD-GMD system; "ZFOPT" represents the ZF-Optimal system; and "ZF-Lazy" represents the lazy precoder with zero-forcing DFE.

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