Frequency Dependent GTD Coders

Ching-Chih Weng, and P. P. Vaidyanathan
Dept. of Electrical Engineering, MC 136-93
California Institute of Technology, Pasadena, CA 91125, USA
E-mail: {cweng@caltech.edu, and ppvnath@systems.caltech.edu}

***Abstract*—This paper proposes the frequency dependent generalized triangular decomposition (FDGTD) coder family for wide-sense-stationary (WSS) vector processes. Under the uniform bit allocation constraint, a set of necessary and sufficient conditions for FDGTD's coding gain optimality is derived. It is shown that one member in the FDGTD family, the frequency dependent geometric mean decomposition (FDGMD) coder, satisfies these conditions and thus is optimal. It is also demonstrated that the FDGMD coders use a simpler uniform quantizer structure and yet achieve a better performance than the conventional optimal orthonormal subband coders with sophisticated bit allocation scheme.**[1]

I. Introduction

The theory of optimal transform coder [4] and its approximations for quantizing vector processes have been well developed and applied in modern data compression systems [3], [5]. The optimal orthogonal solution, Karhunen Loeve transform (KLT), maximizes the coding gain (i.e., minimizes the mean square reconstruction error due to quantization) when high-bit-rate scalar quantizers are used in the transform domain [4]. Another technique, the prediction-based lower triangular transform (PLT), also achieves a similar performance [7]. However, as shown in [14], both KLT and PLT belong to a more general transform coder family known as the *GTD transform coders*. Any member of the GTD family will maximize the coding gain as long as the optimal bit allocation scheme is applied. An interesting member of this family is the GMD transform coder, which uses uniform bit loading thus simplifying the quantizer design process [14].

Generalized from the orthogonal transform coder, the orthonormal subband coders[2] have also been discussed in data compression applications [2], [10], [12], [13]. Here, the signal passes through frequency dependent transforms before being fed into the quantizers. For a given input statistics, the coding gain maximization of the orthonormal subband coder has also been widely discussed. In particular, [12] introduced two conditions: *total-decorrelation*, which means that the filtered subband signals are totally uncorrelated with each other; and *majorization*, which means that the subband signals' power spectra have the same ordering relationship in every frequency. It is proven in [12] that these two conditions are necessary and sufficient for the optimality of the orthonormal subband coders.

Following the concept of generalizing the transform coder to the orthonormal subband coder, in this paper we generalize the GTD transform coder and introduce a novel coder structure for the general WSS vector process, as shown in Fig. 1. We call it the frequency dependent GTD (FDGTD) coder. In the FDGTD coder, the signal first passes through a paraunitary matrix $\mathbf{E}(e^{j\omega})$, and the filtered signal then passes through a frequency dependent PLT stage where $P_{ij}(e^{j\omega})$ denotes the prediction filter from the jth stream to the ith stream. Given any input statistics and under the uniform bit allocation constraint,[3] the set of necessary and sufficient conditions for FDGTD's coding gain optimality can be derived. In the process of deriving FDGTD's coding gain optimality conditions, we find that similar to GMD transform coder's optimal performance in the GTD family, the generalized frequency dependent GMD (FDGMD) satisfies the necessary and sufficient conditions for the optimality of the FDGTD coders. In other words, in the FDGTD family and under the uniform bit allocation constraint, FDGMD coders achieve the optimal coding gain performance.

The FDGMD coders not only achieve the optimal performance in the FDGTD family, they also produce a better coding gain than that of the optimal orthonormal subband coders. This is true even when FDGMD coders use uniform bit loading, which is much simpler than the sophisticated bit allocation scheme used in the optimal orthonormal subband coders.

This paper is structured as follows: Sec. II introduces the signal model and the structure of the FDGTD coder. Sec. III proposes and proves the necessary and sufficient conditions for the optimality of the FDGMD coders under uniform bit allocation. Examples will also be given to illustrate the idea. Sec. IV gives the concluding remarks.

The following notations are used in the paper. Boldface upper-case letters denote matrices, boldface lower-case letters denote column vectors, and italics denote scalars. Superscript $(\cdot)^\dagger$ and $(\cdot)^T$ denote transpose conjugation and transpose respectively. $[\mathbf{A}]_{ij}$ denotes the (i, j)th element of the matrix $\mathbf{A}$. $[\mathbf{A}]_{i\times j}$ denotes the $i \times j$ matrix containing the first i rows and j columns of the matrix $\mathbf{A}$. For vector $\mathbf{x}$, the notation diag$(\mathbf{x})$ denotes the diagonal matrix with diagonal terms equal to the elements in $\mathbf{x}$. For matrix $\mathbf{X}$, the notation diag$(\mathbf{X})$ denotes the column vector whose elements are the diagonal terms of $\mathbf{X}$.

II. Frequency dependent GTD Coders

The proposed coder structure is shown in Fig. 1. The input $\mathbf{x}(n) = [x_0(n), x_1(n), x_{M-1}(n)]^T$ is assumed to be a zero-mean wide sense stationary (WSS) vector process with power spectral density (psd) matrix $\mathbf{S}_{xx}(e^{j\omega})$. The signal $\mathbf{x}(n)$ first passes through a paraunitary filter $\mathbf{E}(e^{j\omega})$ (i.e., $\mathbf{E}(e^{j\omega})$ is unitary for all ω). Let $\mathbf{z}(n) = [z_0(n)\ z_1(n)\ \cdots\ z_{M-1}(n)]^T$ denote the signal after $\mathbf{E}(e^{j\omega})$. Before the quantizers, the

[1]This work is supported in parts by the ONR grant N00014-08-1-0709, and Caltech.

[2]The original notion of subband coder in [12] is used when the input vector $\mathbf{x}(n)$ is a blocked version of the WSS scalar process $x(n)$. Here we use it to denote the entire coder family for the general WSS vector process input, which may or may not be a blocked scalar process. It is not difficult to verify that the two conditions in [12] are still sufficient and necessary for the optimality of the orthonormal coders for the general WSS vector processes. However, the notion of *compaction filters* only exists in the blocked scalar process case. For simplicity, we continue to refer to the substream signals in the vector process as *subband signals*.

[3]If we do not restrict to the uniform bit allocation but use the optimal bit loading scheme for the FDGTD coder, it can still be shown that the coding gain will never exceed the maximized coding gain (12) achieved by the FDGMD coder with uniform bit allocation. That is, *the FDGMD coder with uniform bit allocation is optimal within the family of FDGTD coders with any bit allocation schemes*. However, there could be other members in the FDGTD family also achieving the same maximized coding gain. This will be elaborated in detail in [16].

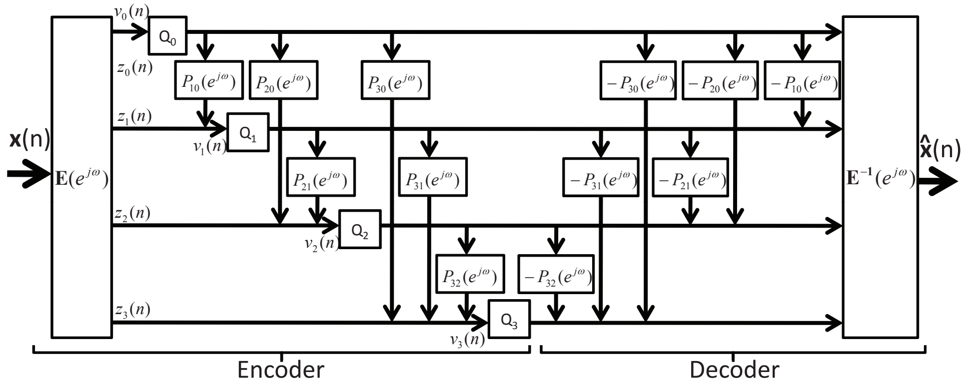

Fig. 1. Schematic of a frequency dependent GMD coder with scalar quantizers. The MINLAB structure is used.

signal $\mathbf{z}(n)$ passes through a frequency-dependent PLT stage, where the kth quantizer input $v_k(n)$ is generated by signal $z_k(n)$ adding the filtered version of the quantized signal $v_0(n)$, $v_1(n)$, up to $v_{k-1}(n)$. The filter $P_{ik}(e^{j\omega})$ is the prediction filter from the kth stream to the ith stream. The decoder performs the inverse operations on the quantized data. Since the overall transform is non-orthonormal due to the predictor structure, in general the coder will have noise gain greater than unity. Thus, we use the MINLAB structure [8] to ensure unity noise gain. The validity of the MINLAB structure must rely on the high-bit-rate assumption where we assume that the prediction based on the quantized data is not too much different from that on the unquantized data. Under this assumption, the signal $\mathbf{v}(n)$ before the quantizer is the filtered version of $\mathbf{x}(n)$ passing through the filter $\mathbf{T}(e^{j\omega})\mathbf{E}(e^{j\omega})$, where $\mathbf{T}(e^{j\omega})$ is the filter used to represent the frequency dependent PLT stage. In particular, $\mathbf{T}(e^{j\omega})$ is a lower triangular matrix with unities on its diagonals for all frequency, and is such that

$$[\mathbf{T}^{-1}(e^{j\omega})]_{i,j} = \begin{cases} 1 & \text{if } i = j \\ P_{ij}(e^{j\omega}) & \text{if } i > j \\ 0 & \text{if } i < j \end{cases}$$

Since $\mathbf{x}(n)$ is zero-mean and WSS, the quantizer inputs $v_i(n)$'s are therefore zero-mean and jointly WSS with psd matrix

$$\mathbf{S}_{vv}(e^{j\omega}) = \mathbf{T}(e^{j\omega})\mathbf{E}(e^{j\omega})\mathbf{S}_{xx}(e^{j\omega})\mathbf{E}^{\dagger}(e^{j\omega})\mathbf{T}^{\dagger}(e^{j\omega}). \quad (1)$$

It is worth noting that $|\det \mathbf{T}(e^{j\omega})| = |\det \mathbf{E}(e^{j\omega})| = 1$. Thus, the determinant of the psd matrix is preserved, i.e., $\det \mathbf{S}_{vv}(e^{j\omega}) = \det \mathbf{S}_{xx}(e^{j\omega})$ for all ω. We call this proposed coder frequency-dependent GTD (FDGTD) coder since it can be seen as a generalization of the GTD transform coder [14].

To derive the coding gain expression, we model the quantizers with additive noise sources $q_i(n)$ as in [11]. We assume these noise sources are jointly WSS with zero mean and variances of the form

$$\sigma^2_{q_i} = c2^{-2b_i}\sigma^2_{v_i} \quad (2)$$

where b_i is the number of bits assigned to the ith quantizer, and $\sigma^2_{v_i}$ is the variance of the ith quantizer input. This assumption is called *high-bit-rate* assumption, and is widely used in the literature [4], [12], [7]. The constant c, which depends on the nature of the pdf of the quantizer input, is assumed to be the same for all streams. This model does not require that each $q_i(n)$ be white or that any two noise sources be uncorrelated. The quantity $b = \frac{1}{M}\sum_{i=0}^{M-1} b_i$, which is the average bit rate, is assumed to be fixed. The coding gain of a coder is defined by comparing the average mean square value ε_{coder} of the reconstruction error $\mathbf{x}(n) - \hat{\mathbf{x}}(n)$ with the mean square value ε_{direct} of the direct quantization error (roundoff quantizer) with the same bit rate b. Using the high-bit-rate noise model, an expression for the coding gain G_C can be written as

$$G_C = \frac{\varepsilon_{direct}}{\varepsilon_{coder}} \quad (3)$$

To maximize the coding gain in (3), we need to minimize the mean square error ε_{coder}. In this paper, we shall assume uniform bit allocation, i.e., $b_i = b$ for all i, which simplifies the quantizer design. Similar to [14], it will be shown in [16] that this is not a loss of generality.

Under the uniform bit allocation scheme, we have

$$\varepsilon_{coder} = \sum_{i=0}^{M-1} \sigma^2_{q_i} = \sum_{i=0}^{M-1} c2^{-2b_i}\sigma^2_{v_i} = c2^{-2b}\sum_{i=0}^{M-1}\sigma^2_{v_i}$$

Note that by AM-GM inequality, we have

$$\frac{1}{M}\sum_{i=0}^{M-1}\sigma^2_{v_i} \geq \left(\prod_{i=0}^{M-1}\sigma^2_{v_i}\right)^{\frac{1}{M}} \quad (4)$$

The equality can be achieved when $\sigma^2_{v_i} = \sigma^2_v$ for all i for some constant σ^2_v. We will show in Sec. III that

$$\left(\prod_{i=0}^{M-1}\sigma^2_{v_i}\right)^{\frac{1}{M}} \geq \underbrace{\int_0^{2\pi} \sqrt[M]{\det S_{xx}(e^{j\omega})}\frac{d\omega}{2\pi}}_{\text{call this } \sigma^2_x} \quad (5)$$

where σ^2_x is a fixed quantity that depends only on the input statistics. We will also show later that both equalities in (4) and (5) are achievable by using the coder call the *FDGMD coder*, which is designed by performing frequency-wise GMD of the spectral factor of the input psd matrix. Thus, the FDGMD coder is optimal in maximizing the coding gain within the class of FDGTD coders under the uniform bit allocation constraint!

III. Sufficient and Necessary Conditions for FDGTD Coder Optimality Under Uniform Bit Allocation

In the following we derive a set of sufficient and necessary conditions for the optimality of the FDGTD coder under uniform bit allocation. We first assume the equality in (4) holds true and $\sigma^2_{v_i}$ is constant for all i. Thus, the problem of maximizing the coding gain is *equivalent to minimizing the product of the subband variances* (5).

A. The First Necessary Condition: Total Decorrelation of Subbands

For orthonormal subband coders, Vaidyanathan proved that total decorrelation is necessary for the optimal coders with some bit allocation scheme [12]. The same condition is also necessary for the FDGTD coder structure with uniform bit allocation.

Theorem 3.1: Total Decorrelation Is Necessary: For fixed input psd $\mathbf{S}_{xx}(e^{j\omega})$, suppose a coder is optimal (in the coding gain sense) among the class of all FDGTD coders with uniform bit allocation. Then, the random processes before each quantizer are uncorrelated with each other, that is

$$E[v_i(n)v_k^*(m)] = 0$$

for $i \neq k$, and for all $n,\ m$. This condition will also be referred to as *total decorrelation* of the subband. $\diamondsuit$

Thus, for optimality, the random processes $v_i(\cdot)$ and $v_k(\cdot)$ must be decorrelated, not just random variables $v_i(n)$ and $v_k(n)$ for each time n. Equivalently, the psd matrix of the vector process $\mathbf{v}(n) = [v_0(n)\ v_1(n)\ \cdots\ v_{M-1}(n)]^T$ must be diagonal, i.e.,

$$\mathbf{S}_{vv}(e^{j\omega}) = \mathrm{diag}([S_{v_0}(e^{j\omega}), \cdots, S_{v_{M-1}}(e^{j\omega})]^T) \tag{6}$$

Proof: Suppose a pair of the subband processes, say $v_0(\cdot)$ and $v_1(\cdot)$, are not uncorrelated. Then, $E[v_0(n)v_1^*(n-k)] \neq 0$ for some k. We now show how to decrease the product of the variances by re-designing the predictors. Suppose we use a delay z^{-k} and an additional predictor $-r$ from the 0th stream to the 1st stream to produce the uncorrelated pair $w_0(n)$ and $w_1(n)$ (see Fig. 2, where $\mathbf{T}_0(e^{j\omega})$ denotes the remaining frequency dependent PLT part). Note that this fixed predictor $-r$ works for all n by the WSS property. The delay element can be absorbed into the paraunitary filter $\mathbf{E}(e^{j\omega})$, which follows the similar argument made in the proof of Theorem 1 of [12]. The additional predictor can be absorbed into $\mathbf{T}(e^{j\omega})$ without destroying the structure of $\mathbf{T}(e^{j\omega})$ (i.e., lower triangular and with 1's on the diagonal). Also, since $w_1(n)$ is different from $v_1(n)$, the predictors $P_{i1}(e^{j\omega})$ for $i \geq 2$ needs to be changed correspondingly. However, it can be seen that it is possible to make $w_i(n) = v_i(n)$ for $i \geq 2$ by just changing the predictors $P_{i1}(e^{j\omega})$. Thus the structure in Fig. 2 is the same as using a modified pair of filters $\{\mathbf{E}_{new}(e^{j\omega}), \mathbf{T}_{new}(e^{j\omega})\}$ where $\mathbf{E}_{new}(e^{j\omega})$ is still paraunitary and $\mathbf{T}_{new}(e^{j\omega})$ is still lower triangular with diagonal entries all equal to unity. We now check if the product of the first two subband variances has been reduced, i.e., if $\sigma_{w_0}^2\sigma_{w_1}^2 < \sigma_{v_0}^2\sigma_{v_1}^2$.

Let $\mathbf{R}_w$ and $\mathbf{R}_v$ be the correlation matrices of the vectors $[w_0(n)\ w_1(n)]^T$ and $[v_0(n)\ v_1(n-k)]^T$. Note that by using $\{\mathbf{E}_{new}(e^{j\omega}), \mathbf{T}_{new}(e^{j\omega})\}$, the determinant is preserved, and thus $\det \mathbf{R}_w = \det \mathbf{R}_v$. Note that the diagonal elements of $\mathbf{R}_w$ and $\mathbf{R}_v$ are the quantities $\sigma_{w_i}^2$ and $\sigma_{v_i}^2$. Since $w_0(n)$ and $w_1(n)$ are uncorrelated, we have

$$\sigma_{w_0}^2\sigma_{w_1}^2 = \det \mathbf{R}_w = \det \mathbf{R}_v = \sigma_{v_0}^2\sigma_{v_1}^2 - |r_0|^2 < \sigma_{v_0}^2\sigma_{v_1}^2$$

where r_0 denotes the non-zero cross-correlation between $v_0(n)$ and $v_1(n-k)$. Thus, we have shown that for optimality $v_0(\cdot)$ and $v_1(\cdot)$ need to be totally decorrelated. For the case where $(i,k) \neq (0,1)$, if $v_i(\cdot)$ and $v_k(\cdot)$ are not totally decorrelated, similar arguments can be used. However, one has to be careful about the predictor filters, since by adding additional predictor filter $-r$ from ith stream to kth stream, all the predictors $P_{gh}(e^{j\omega})$ for all $g,\ h \geq i$ need to be changed correspondingly. It can be verified that after the changes, the overall structure is still representable by Fig. 1, thus is still inside the FDGTD coder class we are discussing. Therefore, we have proved optimality implies the total decorrelation between $v_i(\cdot)$ and $v_k(\cdot)$ for $i \neq k$. This completes the proof. ∎

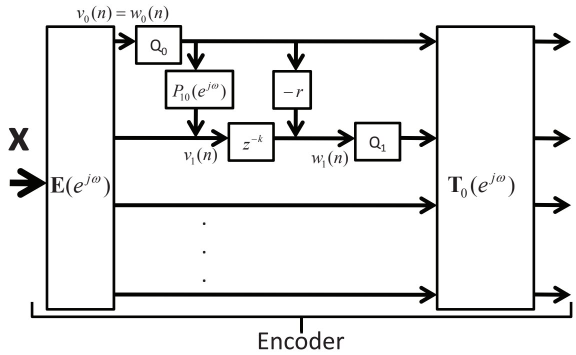

Fig. 2. Increasing the coding gain by exploiting residual correlation.

B. The Second Necessary Condition: Spectrum Equalizing

We say that the set of the subband signals, or the set of subband signal power spectra $\{S_{v_k}(e^{j\omega})\}$, has the *spectrum equalizing* (SE) property if

$$S_{v_0}(e^{j\omega}) = S_{v_1}(e^{j\omega}) = \cdots = S_{v_{M-1}}(e^{j\omega}),\ \text{for all } \omega. \tag{7}$$

From the consequence of the total decorrelation property, the optimal coders must have $\prod_{k=0}^{M-1} S_{v_k}(e^{j\omega}) = \det \mathbf{S}_{xx}(e^{j\omega})$. If all $S_{v_k}(e^{j\omega})$ are equal, then all of them are equal to their geometric mean $\left(\det \mathbf{S}_{xx}(e^{j\omega})\right)^{1/M}$. Now we will prove that this property is necessary for the optimal FDGTD coders under uniform bit allocation.

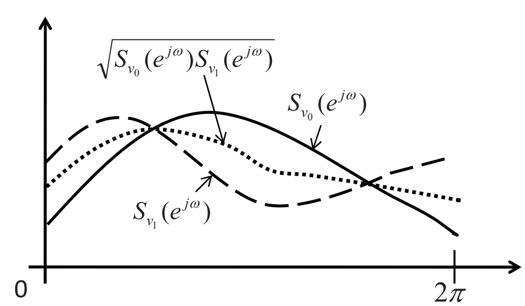

Fig. 3. An example of the power spectra of two subbands which do not have the SE property. $S_{v_0}(e^{j\omega})$ are not equal to $S_{v_1}(e^{j\omega})$ for some ω.

Theorem 3.2: SE Property Is Necessary: For fixed input psd $\mathbf{S}_{xx}(e^{j\omega})$ and under the uniform bit allocation constraint, suppose a FDGTD coder is optimal (in maximizing the coding gain) among the class of all FDGTD coders. Then, the power spectras of the signals before the each quantizers have the SE property (and thus are all equal).

Proof: Here we first consider the case of $M = 2$. Assume $S_{v_0}(e^{j\omega}) \neq S_{v_1}(e^{j\omega})$ for some ω (see Fig. 3 for an example.) Note that $\sigma_{v_i}^2 = \frac{1}{2\pi}\int_0^{2\pi} S_{v_i}(e^{j\omega})d\omega$. Suppose we are able to produce $w_0(n)$ and $w_1(n)$ from $v_0(n)$ and $v_1(n)$ such that $S_{w_0}(e^{j\omega}) = S_{w_1}(e^{j\omega}) = \sqrt{S_{v_0}(e^{j\omega})S_{v_1}(e^{j\omega})}$. Then, we have

$$\begin{aligned}\sigma_{v_0}^2\sigma_{v_1}^2 &= \frac{1}{4\pi^2}\int_0^{2\pi} S_{v_0}(e^{j\omega})d\omega \int_0^{2\pi} S_{v_1}(e^{j\omega})d\omega \\ &\geq \frac{1}{4\pi^2}\left(\int_0^{2\pi}\sqrt{S_{v_0}(e^{j\omega})S_{v_1}(e^{j\omega})}d\omega\right)^2 \quad (8) \\ &= \sigma_{w_0}^2\sigma_{w_1}^2\end{aligned}$$

where the inequality (8) is from the Cauchy-Schwarz inequality on square-integrable real-value functions. The equality holds when $S_{v_0}(e^{j\omega}) = \alpha S_{v_1}(e^{j\omega})$ for all ω where α is some constant. If $\alpha = 1$, this contradicts the assumption $S_{v_0}(e^{j\omega}) \neq S_{v_1}(e^{j\omega})$ for some ω, so $\alpha \neq 1$. In this case, $\sigma_{v_0}^2 = \alpha\sigma_{v_1}^2 \neq \sigma_{v_1}^2$, which leads to violation of the equality in (4), which is necessary for the optimality. This shows the inequality in (8) has to be strict, and thus $\sigma_{v_0}^2\sigma_{v_1}^2 > \sigma_{w_0}^2\sigma_{w_1}^2$.

It only remains to prove that such $[w_0(n)\ w_1(n)]^T$ can be obtained from $[v_0(n)\ v_1(n)]^T$ with permissible transformations in the proposed coder structure. By Theorem 3.1 we know that the psd matrix of $[v_0(n)\ v_1(n)]^T$ is diagonal for all frequency since they are totally decorrelated. Taking the determinant at both sides of (1), we have

$$S_{v_0}(e^{j\omega})S_{v_1}(e^{j\omega}) = \det \mathbf{S}_{vv}(e^{j\omega}) = \det \mathbf{S}_{xx}(e^{j\omega})$$

Consider the following decomposition:[4]

$$\mathbf{S}_{xx}^{\dagger/2}(e^{j\omega}) = \mathbf{Q}^\dagger(e^{j\omega})\mathbf{R}(e^{j\omega})\mathbf{P}(e^{j\omega}) \quad (9)$$

where

$$\mathbf{R}(e^{j\omega}) = \sqrt[4]{S_{v_0}(e^{j\omega})S_{v_1}(e^{j\omega})}\begin{bmatrix}1 & r(e^{j\omega})\\ 0 & 1\end{bmatrix} \quad (10)$$

is an upper triangular matrix with diagonals equal to the geometric mean of $\sqrt{S_{v_0}(e^{j\omega})}$ and $\sqrt{S_{v_1}(e^{j\omega})}$. Here $\mathbf{Q}(e^{j\omega})$ and $\mathbf{P}(e^{j\omega})$ are both 2×2 unitary matrix for all frequency ω. The existence of this decomposition is ensured by the GMD theory [6] for every frequency ω. Let $[w_0(n)\ w_1(n)]^T$ be the signal constructed by passing $[x_0(n)\ x_1(n)]^T$ through filter $\mathbf{P}(e^{j\omega})$ and the predictor matrix $\mathbf{R}_1(e^{j\omega})$, where

$$\mathbf{R}_1(e^{j\omega}) = \begin{bmatrix}1 & 0\\ -r^*(e^{j\omega}) & 1\end{bmatrix}.$$

Thus, we can calculate the psd of $[w_0(n)\ w_1(n)]^T$ as follows:

$$\begin{aligned}&\mathbf{S}_{ww}(e^{j\omega}) \\ =\ &\mathbf{R}_1(e^{j\omega})\mathbf{P}(e^{j\omega})\mathbf{S}_{xx}(e^{j\omega})\mathbf{P}^\dagger(e^{j\omega})\mathbf{R}_1^\dagger(e^{j\omega}) \\ =\ &\begin{bmatrix}1 & 0\\ -r^*(e^{j\omega}) & 1\end{bmatrix}\mathbf{R}^\dagger(e^{j\omega})\mathbf{R}(e^{j\omega})\begin{bmatrix}1 & -r(e^{j\omega})\\ 0 & 1\end{bmatrix} \\ =\ &\begin{bmatrix}\sqrt{S_{v_0}(e^{j\omega})S_{v_1}(e^{j\omega})} & 0\\ 0 & \sqrt{S_{v_0}(e^{j\omega})S_{v_1}(e^{j\omega})}\end{bmatrix}\end{aligned}$$

where in the derivation we have substituted in (9) and (10). Therefore, if we use $\{\mathbf{E}(e^{j\omega}), \mathbf{T}(e^{j\omega})\} = \{\mathbf{P}(e^{j\omega}), \mathbf{R}_1(e^{j\omega})\}$, we are able to decrease the product of the stream signal variances. This completes the proof for the case $M = 2$. For greater M, a similar proof technique can be used and thus is left to the reader. ■

[4]We denote the spectral factor of $\mathbf{S}_{xx}(e^{j\omega})$ to be $\mathbf{S}_{xx}^{1/2}(e^{j\omega})$, i.e., $\mathbf{S}_{xx}(e^{j\omega}) = \mathbf{S}_{xx}^{1/2}(e^{j\omega})\mathbf{S}_{xx}^{\dagger/2}(e^{j\omega})$

C. A Set of Necessary and Sufficient Conditions

We will show in [16] by providing examples that although the total decorrelation and the SE property are necessary for the optimality of the FDGTD coders when uniform bit loading is applied, neither of them is individually sufficient. However, in the following we will prove that *if we put them together, that turns out to be optimal!*

Theorem 3.3: Optimal FDGTD Coders: When uniform bit loading is applied, the coding gain of a FDGTD coder is maximum for a given input psd $\mathbf{S}_{xx}(e^{j\omega})$ if and only if the signals before the each quantizers $v_i(n)$ satisfy the following two properties:

1) They are totally uncorrelated. That is,

$$E[v_i(n)v_k^*(m)] = 0 \text{ for } i \neq k, \text{ and for all } n,\ m.$$

2) Their psd are all equal, that is,

$$S_{v_0}(e^{j\omega}) = S_{v_1}(e^{j\omega}) = \cdots = S_{v_{M-1}}(e^{j\omega}) \text{ for all } \omega.$$

Proof: In view of earlier theorems, it only remains to prove that the total decorrelation and the SE property together imply optimality. If the filter pair $\{\mathbf{E}(e^{j\omega}), \mathbf{T}(e^{j\omega})\}$ performs total decorrelation, $\mathbf{S}_{vv}(e^{j\omega})$ must be diagonal. If this filter pair also results in the SE property, then we must have $\mathbf{S}_{vv}(e^{j\omega}) = c(e^{j\omega}) \cdot \mathbf{I}$ for some fixed $c(e^{j\omega})$. Since $\det \mathbf{S}_{vv}(e^{j\omega}) = \det \mathbf{S}_{xx}(e^{j\omega})$, we can conclude that $c(e^{j\omega}) = \sqrt[M]{\det \mathbf{S}_{xx}(e^{j\omega})}$. Therefore for each frequency, the value of $c(e^{j\omega})$ is *uniquely* determined by $\det \mathbf{S}_{xx}(e^{j\omega})$. Since the total decorrelation and the SE property are necessary for the optimality and since there is only one set of subband power spectra satisfying these two conditions simultaneously, it follows that these two conditions leads to optimality. This completes the proof. ■

Notice in particular that the two properties in Thm 3.3 imply equality in (4) and (5). Consider next the following frequency dependent GMD:

$$\mathbf{S}_{xx}(e^{j\omega})^{\frac{\dagger}{2}} = \mathbf{Q}^\dagger(e^{j\omega})\ \sqrt[M/2]{\det \mathbf{S}_{xx}(e^{j\omega})}\mathbf{R}_1(e^{j\omega})\mathbf{P}(e^{j\omega}) \quad (11)$$

where for all frequency ω, $\mathbf{Q}(e^{j\omega})$ and $\mathbf{P}(e^{j\omega})$ are both unitary matrices, and $\mathbf{R}_1(e^{j\omega})$ is an upper triangular matrix with all the diagonal entries equal to unity. If we use the filter pair $\{\mathbf{E}(e^{j\omega}), \mathbf{T}(e^{j\omega})\} = \{\mathbf{P}(e^{j\omega}), \mathbf{R}_1^{-\dagger}(e^{j\omega})\}$, the transformed signal $\mathbf{v}(n)$ will have psd $\mathbf{S}_{vv}(e^{j\omega}) = \sqrt[M]{\det \mathbf{S}_{xx}(e^{j\omega})}\cdot\mathbf{I}$. That is, the signal $\mathbf{x}(n)$, which is transformed to totally decorrelated streams with identical psd, satisfies the two properties in Thm 3.3. Therefore, this filter pair is one of the coding-gain-maximizing FDGTD coders[5] under uniform bit allocation. We call this coder the *FDGMD coder* since it is designed by performing frequency-dependent GMD on the spectral factor of the input psd. It can also be seen that

$$\sigma_{v_i}^2 = \int_0^{2\pi}\sqrt[M]{\det \mathbf{S}_{xx}(e^{j\omega})}\frac{d\omega}{2\pi} = \sigma_x^2, \text{ for } i = 0, \cdots, M-1,$$

where σ_x^2 is defined in (5). The resulting minimized MSE equals

$$\varepsilon_{fdgmd} = \sum_{i=0}^{M-1}\frac{c}{M}2^{-2b}\sigma_{v_i}^2 = \frac{c2^{-b}}{2\pi}\int_0^{2\pi}\sqrt[M]{\det \mathbf{S}_{xx}(e^{j\omega})}d\omega$$

[5]Note that the coding-gain-maximizing FDGTD may not be unique.

Since the MSE of the direct quantization is

$$\begin{aligned}\varepsilon_{direct} &= c2^{-b}\frac{1}{M}\sum_{k=0}^{M-1}\int_0^{2\pi}[\mathbf{S}_{xx}(e^{j\omega})]_{kk}\frac{d\omega}{2\pi}\\ &= c2^{-b}\int_0^{2\pi}\frac{\mathrm{Tr}(\mathbf{S}_{xx}(e^{j\omega}))}{M}\frac{d\omega}{2\pi},\end{aligned}$$

the maximized coding gain (3) can thus be calculated as

$$G_C = \frac{\int_0^{2\pi}\frac{1}{M}\mathrm{Tr}(\mathbf{S}_{xx}(e^{j\omega}))d\omega}{\int_0^{2\pi}\sqrt[M]{\det \mathbf{S}_{xx}(e^{j\omega})}d\omega} \tag{12}$$

D. Comparison With Optimal Orthonormal Subband Coders

In [12], the optimal orthonormal subband coders are developed for a WSS scalar process. In this case, the input vector of the coder is obtained from blocking signals of a scalar WSS process into vectors. In the following example we consider the FDGMD coder for such application. We will show the FDGMD coder with uniform bit allocation achieves even better coding gain compared to the optimal orthonormal coder with optimal (typically nonuniform) bit allocation scheme.

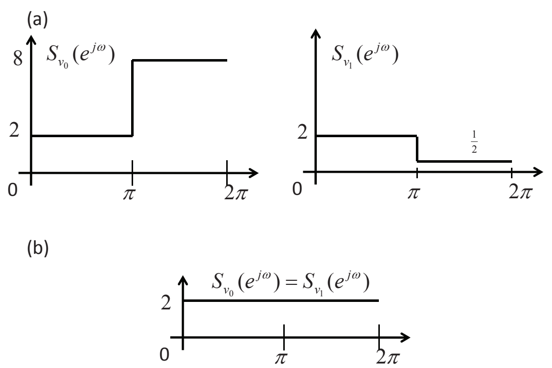

Fig. 4. Example: (a) the psd of the subbands obtained by the optimal subband coders [12]. (b) the psd of the subbands obtained by the FDGMD coder.

Example: Consider the subband coder for a WSS scalar process with 2-fold blocked version

$$\mathbf{x}(n) = [x(2n)\ x(2n-1)]^T$$

which can be seen in Fig. 1(b) in [12] for $M = 2$. It is shown in [9] that the vector process $\mathbf{x}(n)$ is WSS, and the psd matrix $\mathbf{S}_{xx}(e^{j\omega})$ of $\mathbf{x}(n)$ is pseudocirculant. Suppose the psd matrix $\mathbf{S}_{xx}(e^{j\omega})$ is given as the following:

$$\mathbf{S}_{xx}(e^{j\omega}) = \begin{cases}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} & \text{if } 0 < \omega \le \pi;\\ \frac{1}{4}\begin{bmatrix}17 & 15e^{-j\omega/2}\\ 15e^{j\omega/2} & 17\end{bmatrix} & \text{if } \pi < \omega \le 2\pi.\end{cases}$$

Note that $\mathbf{S}_{xx}(e^{j\omega})$ is indeed pseudocirculant and positive definite, hence it is a valid psd matrix for a 2-fold blocked scalar WSS process. It can be readily verified that the paraunitary matrix $\mathbf{E}(e^{j\omega})$:

$$\mathbf{E}(e^{j\omega}) = \begin{cases}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \text{if } 0 < \omega \le \pi;\\ \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{j\omega/2}\end{bmatrix} & \text{if } \pi < \omega \le 2\pi.\end{cases}$$

performs total decorrelation and is such that the power spectras of the subband signals become as in Fig. 4(a). It can be seen that the majorization property is satisfied. Therefore, this $\mathbf{E}(e^{j\omega})$ is the optimal orthonormal subband coder. The subband variance is $[\sigma_{v_0}^2\ \sigma_{v_1}^2]^T = [5\ \ 5/4]^T$. Thus, the optimal bit allocation scheme is $[b+1\ \ b-1]^T$, and the corresponding MSE is

$$\varepsilon_{orthonorml} = \frac{1}{2}c2^{-b-1}\sigma_{v_0}^2 + \frac{1}{2}c2^{-b+1}\sigma_{v_1}^2 = \frac{5}{2}c2^{-b}$$

On the other hand, suppose we use the FDGMD coder with uniform bit allocation scheme, it can be verified the subband power spectra are as in Fig. 4. Thus the MSE can be calculated as

$$\varepsilon_{fdgmd} = \frac{1}{2}c2^{-b}(\sigma_{v_0}^2 + \sigma_{v_1}^2) = 2c2^{-b} < \varepsilon_{orthonorml}$$

Thus, *FDGMD with uniform bit allocation achieves even better coding gain than the optimal orthonormal subband coders with optimal bit allocation!*

IV. Concluding Remarks

In this paper, we proposed the FDGTD coder which can be seen as a generalization of the GTD transform coder. We showed that the two properties – total decorrelation and the spectrum equalizing property, are sufficient and necessary for the optimality of the proposed coder under uniform bit loading constraint. We also showed that the FDGMD coder in fact maximizes the coding gain with uniform bit allocation. It has been observed that this coder has strong connection with the notion of PCFB [1], and the optimal communication systems with linear precoder and ZF-DFE [15]. These findings will be elaborated in greater detail in [16].

References

[1] S. Akkarakaran and P. P. Vaidyanathan, “Filter bank optimization with convex objectives, and the optimality of principal component forms,” *IEEE Trans. Sig. Proc.,* vol. 49, pp. 100-114, Jan. 2001.
[2] I. Djokovic, and P. P. Vaidyanathan, “On optimal analysis/synthesis filters for coding gain optimization,” *IEEE Trans. Sig. Proc.,* vol. 44, pp. 1276-1279, May. 1996.
[3] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression.* Norwell, MA: Kluwer Academic Publshers, 1992.
[4] J. J. Y. Huang and P. M. Schultheiss, “Block quantization of correlated Gaussian random variables,” *IEEE Trans. Communication Systems,* vol. 11, pp. 289-296, Sept. 1963.
[5] N. S. Jayant, and P. Noll, *Digital coding of waveforms,* Englewood Cliffs, NJ: Prentice-Hall, 1984.
[6] Y. Jiang, W. W. Hager, and J. Li “Generalized triangular decomposition, ” *Mathematics of computation.,* Oct. 2007.
[7] S. M. Phoong, and Y. P. Lin, “Prediction-based lower triangular transform,” *IEEE Trans. Sig. Proc.,* vol. 48, No. 7, pp.1947-1955, Jul. 2000.
[8] S. M. Phoong, and Y. P. Lin, “MINLAB: minimum noise structure for ladder-based biorthogonal filter banks,” *IEEE Trans. Sig. Proc.,* vol. 48, No. 2, pp.465-476, Feb. 2000.
[9] V. Sathe and P. P. Vaidyanathan, “Effects of multirate systems on the statistical properties of random signals,” *IEEE Trans. Sig. Proc.,* vol. 41, pp. 131-146, Jan. 1993.
[10] M. K. Tsatsanis, and G. B. Giannakis, “Principal component filter banks for optimal multiresolution analysis,” *IEEE Trans. Sig. Proc.,* vol. 43, pp. 1766-1777, Aug. 1995.
[11] P. P. Vaidyanathan, *Multirate systems and filter banks,* Englewood Cliffs, NJ: Prentice-Hall, 1993.
[12] P. P. Vaidyanathan, “Theory of optimal orthonormal subband coders,” *IEEE Trans. Sig. Proc.,* vol. 46, pp. 1528-1543, Jun. 1998.
[13] M. Vetterli, and J. Kovacevic, *Wavelets and subband coding,* Englewood Cliffs, NJ: Prentice-Hall, 1995.
[14] C. C. Weng, C. Y. Chen, and P. P. Vaidyanathan, “Generalized triangular decomposition in transform coding,” *IEEE Trans. on Sig. Proc.,* vol. 58, No. 2, pp. 566-574, Feb. 2010.
[15] C. C. Weng, C. Y. Chen, and P. P. Vaidyanathan, “MIMO transceivers with decision feedback and bit loading: theory and optimization,” *IEEE Trans. Sig. Proc.,* vol. 58, pp. 1334-1346, Mar. 2010.
[16] C. C. Weng, and P. P. Vaidyanathan, “Theory and applications of frequency dependent generalized triangular decomposition,” in preparation.