

Bulk Connectedness and Boundary Entanglement

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Abstract

We prove, for any state in a conformal field theory defined on a set of boundary manifolds with corresponding classical holographic bulk geometry, that for any bipartition of the boundary into two non-clopen sets, the density matrix cannot be a direct product of the reduced density matrices on each region of the bipartition. In particular, there must be entanglement across the bipartition surface. We extend this no-go theorem to general, arbitrary partitions of the boundary manifolds into non-clopen parts, proving that the density matrix cannot be a direct product. This result gives a necessary condition for states to potentially correspond to holographic duals.

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1 Introduction

The question of which conformal field theory states can correspond to smooth, classical gravity duals must be answered in order to determine the limitations of AdS/CFT [1–3]. Approaches thus far have included constraints on the theories [4] and on the entropies [5, 6], though less work has been done to directly constrain the states themselves.

Folklore in the community has suggested that perhaps multipartite entanglement (entanglement that cannot be distilled into Bell pairs alone) may not be well suited for smooth gravity duals without some amount of bipartite entanglement, due to tension between classes of known multipartite entangled states and the Ryu-Takayanagi (RT) formula for entanglement entropy [5–7]. Nevertheless, multiboundary wormhole geometries where the multipartite entanglement is provably existent are known, at least in three dimensions [8]. Again, however, less work has been done on constraining what classes of bipartite entangled states can have classical gravity duals.

In this work, we will focus on excluding certain patterns of entanglement from corresponding to smooth, classical gravity duals. In particular, we will demonstrate that, for conformal field theory (CFT) states defined on a set of boundaries, only those with nonzero entanglement entropy across any nontrivial subdivision are permitted. In Sec. 2, we will gain intuition by considering a CFT defined on a set of three boundaries. We prove our main results in Sec. 3, first considering some requisite properties of bulk connectedness between sets of CFT boundaries and then interpreting and applying these results to study entanglement between those boundaries. In particular, we establish a set of necessary conditions on the boundary state for the existence of a holographic bulk geometry, generalized to any number of boundaries. We conclude and discuss implications of our result in Sec. 4.

2 Three Boundary Case

Let us begin by describing a bulk geometry that could hypothetically exist but that may be problematic from the perspective of information theory and holography. Consider three manifolds A , B , and C on which we define a CFT and some pure state $|\psi\rangle$. Permit A and B to be connected through the bulk and similarly for A and C , but keep B and C disconnected in the bulk; see Figs. 1 and 2. Now partition A into regions A_1 and A_2 such that A_1 is the subset of A containing geodesics that enter the region of spacetime connected with B and A_2 is the subset that contains geodesics entering the region of spacetime connected with C . Holographically, the leading-order term in the entanglement entropy of a region is the area of the RT surface that subtends that region. Now construct the RT surface subtending A_1B . This RT surface should have zero area, giving vanishing entanglement entropy. This would therefore be a nontrivial partition of the full Hilbert space that would have one side unentangled with the other. Writing the full pure state density matrix on the

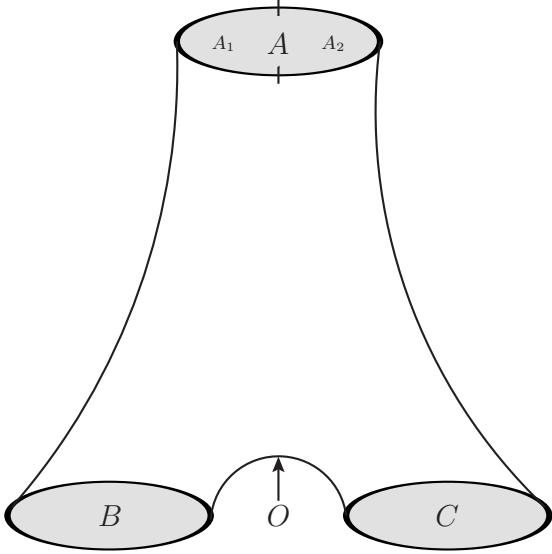


Figure 1: Situation in which A , B , and C define the boundaries of a holographic geometry, in which there exist bulk paths connecting each pairwise combination of A , B , and C .

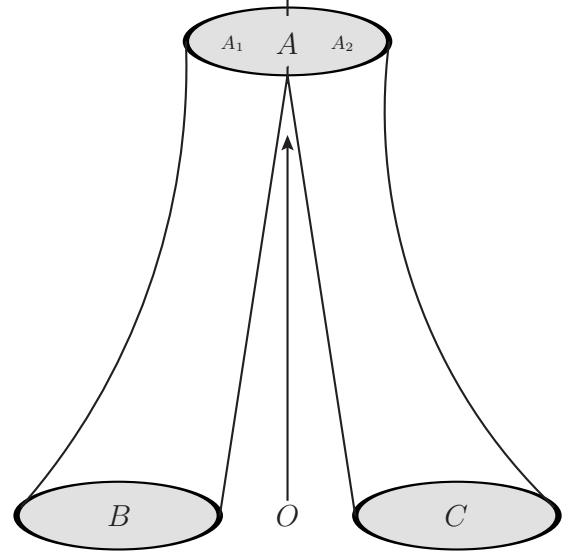


Figure 2: Situation in which A is partitioned between A_1 and A_2 , with $A_1 \cup B$ and $A_2 \cup C$ disconnected in the bulk.

three CFT boundaries as $\rho = |\psi\rangle\langle\psi|$, we are disallowing the situation in which $\rho = \rho_{A_1B} \otimes \rho_{A_2C}$, where ρ_{A_1B} and ρ_{A_2C} are the reduced density matrices on A_1B and A_2C , respectively. In other words, ρ cannot be a separable pure state.

The reason why this scenario would be problematic from a holographic perspective is that there exist subregions a_1 and a_2 of A_1 and A_2 , respectively, that can be chosen to be arbitrarily close to each other. If the entanglement entropy of A_1B is zero, however, this requires that there is no entanglement between a_1 and a_2 , i.e., that the mutual information $I(a_1 : a_2) = 0$ for all a_1 and a_2 . This follows from the fact that A_1B would be a pure state in this scenario and any reduced density matrix of a pure state has zero mutual information with any other density matrix that is not a different reduced density matrix of the original pure state. This situation would contradict the RT formula if a_1 and a_2 were sufficiently large and close together such that the minimal surface subtending them both would not, in a conventional holographic geometry as in Fig. 1, simply be the union of minimal surfaces subtending them separately. Because of this, such bulk configurations must be pathological from the perspective of the geometry, to match the pathology at the level of the RT formula.

Consider again the RT surface subtending A_1B in the geometry described above. This will correspond to the surface spanning the length of the spacetime region joining the two sides of the manifold depicted in Fig. 1, touching A at the intersection points of A_1 and A_2 and passing through O . This surface will in general have nonzero area in the bulk. Now consider the situation where

O is brought normally down to the CFT boundary A , as in Fig. 2. The RT surface as previously described still gives the entanglement entropy of A_1B via its bulk area. The bulk area of the RT surface, however, will now be zero, forcing there to be zero entanglement between A_1B and A_2C . In other words, for the geometry depicted in Fig. 2, there is no surface in the bulk that is homologous to both A_1B and its complement A_2C , so the RT surface cannot exist.

Motivated by this intuition, we conjecture that states of the form described, in which A_1B and A_2C are unentangled (i.e., in which the density matrix factorizes), for a CFT defined on the disjoint union of boundary manifolds A , B , and C , are forbidden from having a well-behaved holographic bulk geometry. In the next section, we will generalize this intuition into a precise statement about CFTs defined on an arbitrary number of boundaries and then prove the theorem.

3 A No-Go Theorem for a Classical Holographic Bulk

Let us now make some definitions. For a given CFT, we will denote as *the boundary* the collection of spacelike boundary manifolds on which the CFT is defined. We will call a surface A a *complete boundary* if it is a connected component of the boundary and if $\partial A = 0$. Then the boundary B is the disjoint union of complete boundary manifolds A_1, \dots, A_n . (We assume throughout that we work in a conformal frame such that $\partial A_i = 0$ on the slice of the spacetime we consider.) We will define a *bipartition* of B as some identification of two disjoint subsets B_1 and B_2 of B such that $B_1 \cup B_2 = B$. We will call such a bipartition *trivial* if B_1 and B_2 are each equal to the disjoint union of some subsets of the A_i , i.e., a trivial bipartition does not split any of the A_i in two. Hence, a nontrivial bipartition has the property that neither B_1 nor B_2 is clopen, i.e., ∂B_1 and ∂B_2 are both nonempty. By a *classical holographic bulk*, we will mean a continuous, classical spacetime manifold \mathcal{M} corresponding to a CFT state, where for each spacelike slice M of \mathcal{M} , ∂M corresponds to the boundary on which the CFT is defined and, crucially, \mathcal{M} is asymptotically anti-de Sitter (AdS), i.e., the metric on M near each complete boundary approaches a spacelike slice of the AdS metric and, for any point in M , there exists a path within that slice connecting the point with one of the complete boundaries. Hereafter, we will always restrict to some arbitrary spacelike slice M of \mathcal{M} and refer to M as the classical holographic bulk. Given a CFT state with a classical holographic bulk, we will call boundary regions A_1 and A_2 *bulk (dis)connected* if there (does not) exist a path, contained entirely within the bulk, i.e., within the interior of M , connecting a point in A_1 with a point in A_2 .

The discussion in Sec. 2 then leads us to the proposition that, given a holographic CFT defined on three complete boundaries A , B , and C , there does not exist any classical holographic bulk in which A is bulk connected with B and C but B and C are not bulk connected. Geometries violating this proposition will be pathological in a sense we will make precise. Accepting this proposition, we have a restriction on the CFT states themselves, in that for any state defined on ABC , any

nontrivial bipartition must be entangled, i.e., the regions of any nontrivial bipartition must have nonvanishing mutual information. In the case of a pure state $\rho = |\psi\rangle\langle\psi|$, this precludes ρ from being separable; in other words, for any nontrivial bipartition of ABC into regions 1 and 2, states in which $\rho = \rho_1 \otimes \rho_2$ are forbidden from having a classical holographic bulk.

We will now generalize these propositions to CFT boundaries with an arbitrary number of connected components, proving the results in two parts.

Theorem 1. *Given a CFT on a boundary consisting of an arbitrary number of complete boundaries A_1, \dots, A_n with a classical holographic bulk, for any two distinct points x_1 and x_2 in the same A_i , x_1 and x_2 must be bulk connected.*

Proof. Suppose that the theorem is false, i.e., that there exists a classical holographic bulk M in which, for some $x_1, x_2 \in A_i$, we have that x_1 and x_2 are not bulk connected. Let us write as M_1 and M_2 the regions that are bulk connected to x_1 and x_2 , respectively. That is, M_1 consists of all $x \in M$ such that x is bulk connected to x_1 ; a similar definition holds for M_2 . We write as M_1^c the complement of M_1 in the bulk, so by construction M_1 and M_1^c are bulk disconnected. Note that M_1 and M_2 must be disjoint in the bulk, since if there were a point $x \in M_1 \cap M_2$, one could find a bulk path from x_1 to x_2 as the union of path p_1 from x_1 to x and p_2 from x_2 to x , which is forbidden by the hypothesis that x_1 and x_2 are bulk disconnected. Thus, $M_2 \subset M_1^c$, so since M_2 by hypothesis exists, M_1^c is nonempty.

Now, given a point $x_0 \in A_i$, we can consider a (spacelike) geodesic $p(\lambda) \subset M$ extending into the bulk, where $x_0 = p(0)$. We can enforce that the geodesic extends into the bulk by requiring that $|g_{\mu\nu}(dp)/d\lambda)^\mu(dp/d\lambda)^\nu|_{\lambda=0}^{1/2} > |\gamma_{ab}(dp)/d\lambda)^a(dp/d\lambda)^b|_{\lambda=0}^{1/2}$, where $g_{\mu\nu}$ is the bulk metric and γ_{ab} is the induced metric on the boundary. Then it becomes well defined to say that $p(\lambda)$ enters M_1 or M_1^c , but not both, depending on whether $p(\lambda) \in M_1$ or M_1^c for infinitesimal $\lambda > 0$.

Let us now denote as U the space of initial data for spacelike geodesics in M starting on A_i and extending into the bulk. Importantly, since M is a classical holographic bulk, it is continuous and asymptotically AdS. Geodesics in M therefore asymptotically approach the geodesics of a spacelike slice of the AdS metric near each complete boundary. AdS geodesics do not bifurcate, since AdS satisfies the Hausdorff condition. By continuity of M , U is connected. (In particular, the topology of U is just $A_i \otimes \mathbb{R}^{D-2}$ for a bulk of spacetime dimension D .) For a given point $y \in U$, we can uniquely specify whether the geodesic p_y to which it corresponds enters M_1 or M_1^c ; let $U_1 = \{y \in U | p_y \text{ enters } M_1\}$ and $U_2 = \{y \in U | p_y \text{ enters } M_1^c\}$. By construction, $U = U_1 \cup U_2$ and, since a geodesic can only enter M_1 or M_1^c , U_1 and U_2 are disjoint. Note that U_1 and U_2 are both nonempty, since there exist geodesics entering M_1 from $x_1 \in A_i$ and entering $M_2 \subset M_1^c$ from $x_2 \in A_i$. Define V as the boundary of U_1 , which is also the boundary of U_2 ; we have $V = \bar{U}_1 \cap \bar{U}_2$. Since U is connected and U_1 and U_2 are proper subsets, neither U_1 nor U_2 are clopen and hence V is nonempty. By continuity of M , geodesics defined by initial data in V must enter both M_1 and

M_1^c . Hence, for any $y \in V$, the geodesic p_y bifurcates, in contradiction with the requirement that M be Hausdorff. This contradiction completes the proof. \square

We note an immediate consequence of this result for the connectivity structure of the classical holographic bulk.

Corollary 1.1. *Given a CFT on a boundary consisting of an arbitrary number of complete boundaries A_1, \dots, A_n , there does not exist a classical holographic bulk in which A_i is bulk connected with A_j and A_k , but A_j and A_k are bulk disconnected.*

Proof. We again proceed by contradiction, assuming that there exists a classical holographic bulk M with boundary consisting of some complete boundary manifolds A_1, \dots, A_n such that, for some $A_{i,j,k}$, A_i is bulk connected to A_j and A_k , but A_j and A_k are bulk disconnected.

We choose points x_1 and x_2 in A_i such that x_1 is bulk connected to A_j and x_2 is bulk connected to A_k . If $x_1 = x_2$ (hereafter, x), then we have chosen a point that is bulk connected with both A_j and A_k . The space U_x of initial data for (spacelike) geodesics originating on x and extending into the bulk (within M) is just a subset of U defined in Theorem 1, for which $U_1 \cap U_x$ and $U_2 \cap U_x$ are both nonempty. Since the topology of U_x in an asymptotically AdS spacetime is \mathbb{R}^{D-2} and hence U_x is connected, it follows that neither $U_1 \cap U_x$ nor $U_2 \cap U_x$ are clopen. Thus, just as in Theorem 1, the set $V_x = V \cap U_x$ is nonempty and, by continuity of the spacetime, geodesics corresponding to initial data in V_x must originate on x and bifurcate, entering both A_j and A_k . This situation violates the Hausdorff condition, in contradiction with the requirement that M be Hausdorff.

Similarly, if $x_1 \neq x_2$, then we have distinct points in the same boundary manifold that are not bulk connected, in which case Theorem 1 immediately applies, forbidding this setup as a classical holographic bulk. \square

Let us now interpret Theorem 1 in terms of the entanglement structure of the CFT state.

Theorem 2. *Given a pure CFT state defined on a boundary that possesses a classical holographic bulk, the boundary regions B_1 and B_2 defined by any nontrivial bipartition must be entangled with each other. That is, $\rho \neq \rho_{B_1} \otimes \rho_{B_2}$.*

Proof. Suppose the theorem is false, so that for some CFT state with a classical holographic bulk there exists a nontrivial bipartition of the boundary into B_1 and B_2 such that B_1 and B_2 are unentangled. Since the bipartition is nontrivial, there must exist some complete boundary A such that $A \cap B_1$ and $A \cap B_2$ are both nonempty.

In general, the RT surface associated with the entanglement between B_1 and B_2 is given by a bulk surface that is codimension-one within M (codimension-two within \mathcal{M}) and is a disjoint union of complete codimension-one surfaces within each connected component of the bulk. By *complete* codimension-one surfaces, we mean surfaces Σ such that, within a given connected component of M ,

any bulk path from B_1 to B_2 must pass through Σ . Among all complete codimension-one surfaces separating B_1 and B_2 , the RT surface is the one with minimal area in M .¹ The fact that B_1 and B_2 are unentangled implies that the RT surface associated with B_1 and B_2 has zero area and hence the surface does not exist in the bulk. Therefore, there must exist no bulk path from any point in B_1 to any point in B_2 , since such a path would be required to pass through the nonexistent RT surface.

We therefore choose some $x_1 \in B_1 \cap A$ and $x_2 \in B_2 \cap A$. Since B_1 and B_2 are disjoint, x_1 and x_2 are distinct and, by the argument above, must be bulk disconnected. Theorem 1 then applies, which implies that the geometry is not a classical holographic bulk. This contradiction completes the proof. \square

We can actually generalize this result to partitions of the boundary into multiple parts. Let us define a *nontrivial partition* of the boundary B as a partition of B into B_1, \dots, B_n , $B_i \cap B_j = \emptyset$, $\cup_i B_i = B$, such that $\partial B_i \neq 0$, i.e., B_i is not clopen for all i . We then have the following result.

Corollary 2.1. *For any nontrivial partition of B into B_1, \dots, B_n , define ρ_i to be the reduced density matrix on B_i . Then for a pure CFT state that corresponds to a classical holographic bulk, $\rho \neq \rho_1 \otimes \dots \otimes \rho_n$.*

Proof. This result follows from iteratively applying Theorem 2, in each successive step taking a nontrivial bipartition of a part of the previous bipartition. \square

4 Discussion

We have proven that there must be entanglement across any nontrivial bipartition of boundary CFT regions in any holographic theory via proof by contradiction with the Hausdorff condition in the dual spacetime. We will now recast our result in graph-theoretic language that can more succinctly encode this requisite entanglement structure, in particular showing that the only entanglement structures consistent with holography can all be represented as disjoint unions of complete graphs, as we will elaborate below.

Corollary 1.1 states that any pair of CFT boundaries must be connected to each other purely within the bulk if there exists a “chain” of bulk connections from one boundary to the next by which they can be linked. In other words, for a set of CFT boundaries A_1, \dots, A_n such that there is a bulk connection from A_i to A_{i+1} for all $i < n - 1$, A_1 and A_n must be bulk connected, i.e., there must exist a path from A_1 to A_n that does not pass through any boundary manifold; it

¹While we describe this using the RT formula, it is worth noting that because all of our proof techniques only involve statements about the existence of bulk surfaces and not detailed properties thereof, there is a priori no reason why these statements could not be trivially extended to the more general time-dependent Hubeny-Rangamani-Takayangi [9] formalism with extremal surfaces.

then follows that every pair of A_i and A_j must similarly be bulk connected. Let us represent a classical holographic bulk as a graph, with a vertex representing each boundary manifold and an edge between any two vertices that are bulk connected with each other. Consider a connected subgraph, containing vertices corresponding to bulk connected boundaries A_1 and A_2 . If A_2 is bulk connected with another boundary $A_3 \neq A_1$, then Corollary 1.1 forces A_1 to be bulk connected with A_3 as well. In the graph picture, this forces the existence of an edge between A_1 and A_3 ; continuing this reasoning enforces that the connected subgraph be a complete graph. As there can be CFT boundaries that are simply not associated by a chain of bulk connections, the bulk connectivity graph of the set of all CFT boundaries, for any state corresponding to a classical holographic bulk, must be a disconnected union of complete subgraphs.

Note that our bound does not imply that every subregion of the boundary is entangled with every other subregion. For example, consider a large bulk region (with boundary A) that contains two small black holes, each connected via a wormhole with a partner black hole in another asymptotically AdS region of spacetime (with boundaries B and C , respectively). Then our results permit a scenario in which some small subsets A_1 and A_2 of the degrees of freedom on A are in approximately thermofield double states with the degrees of freedom on B and C , respectively, with B and C being unentangled. This setup is not forbidden by our result, since $A_1 \cup B$ and $A_2 \cup C$ do not make up the entire boundary and hence do not constitute a bipartition. In order to rule out scenarios in which $A_1 \cup B$ and $A_2 \cup C$ are unentangled, we would be required to include all of A within either A_1 or A_2 .

An intuitive way to state our result from the CFT perspective is that, for the forbidden states, the spacetime would necessarily need access to scales at which classical general relativity does not apply, for the geometry to be precisely defined all the way to the UV. There is, then, no consistent way to impose a cutoff in this theory. From the AdS perspective, states of the forbidden form necessarily have gravity-side pathologies that extend all the way to the boundary and thus prevent the geometry from being asymptotically AdS.

Thus, we see that the entanglement entropy is strongly constrained to be nonzero for any nontrivial bipartition of the CFT state on the boundary for which a classical holographic bulk exists and, moreover, that the existence of a chain of bulk connections between a set of CFT boundaries requires the existence of a complete graph of bulk connections. This result complements the consequences for the known constraints on holographic entanglement entropy [5, 6, 10], in that the set of states that it allows or excludes is independent of and of a different character than those allowed or excluded by the previously known constraints. In the future, it would be interesting to study what fraction of quantum states under some appropriate measure satisfies our criterion, to assess just how constraining a requirement this is on the space of holographic theories. Moreover, it would be interesting to consider the extension of this statement to an ER=EPR perspective [11, 12].

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