Refined large N duality for torus knots

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ABSTRACT: We formulate large N duality of $U(N)$ refined Chern-Simons theory with a torus knot/link in S^3. By studying refined BPS states in M-theory, we provide the explicit form of low-energy effective actions of Type IIA string theory with D4-branes on the Ω-background. This form enables us to relate refined Chern-Simons invariants of a torus knot/link in S^3 to refined BPS invariants in the resolved conifold. Assuming that the extra $U(1)$ global symmetry acts on BPS states trivially, the duality predicts graded dimensions of cohomology groups of moduli spaces of M2-M5 bound states associated to a torus knot/link in the resolved conifold.

Dedicated to John H. Schwarz on his 75th birthday

Contents

1 Introduction

Large N duality is an equivalence between a $\mathrm{U}(N)$ gauge theory at large N and a string theory. The original idea was due to 't Hooft [tH74] and we have witnessed various successful incarnations of this idea in string theory. Among them, Gopakumar and Vafa proposed in the celebrated paper [GV99] that the large N limit of $\mathrm{U}(N)$ Chern-Simons theory on S^3 is equivalent to topological string theory on the resolved conifold. Since Chern-Simons theory is realized on topological branes wrapped on S^3 in the deformed conifold T^*S^3 [Wit95], this duality can be also interpreted as *geometric transition* at large N in string theory. This proposal has far-reaching consequences both in physics and in mathematics. One of significant consequences is a striking connection between two seemingly different theories of invariants. On the one hand, Chern-Simons theory provides a natural framework of quantum invariants of three-manifolds and knots [Wit89]. On the other hand, topological string theory on a Calabi-Yau three-fold is mathematically formulated as theories of enumerative invariants involving some moduli spaces of curves in the three-fold. In particular, Labastida, Mariño, Ooguri and Vafa (LMOV) [OV00, LMn01, LMnV00, LMn02] have put forth a remarkable relationship between quantum knot invariants and enumerative integral invariants in the resolved conifold by incorporating a knot in the duality.

As a parallel development, a theory of knot homology, which categorifies quantum invariants, has been constructed by Khovanov and Rozansky [Kho00, KR08a, KR08b], which

enriches the large N duality to a great extent. In fact, Gukov, Schwarz and Vafa [GSV05] proposed that $\mathfrak{sl}(N)$ knot homology [KR08a] at large N can be identified with the space of BPS states in the resolved conifold, which can be understood as triply-graded HOMFLY-PT homology [KR08b]. Furthermore, Aganagic and Shakirov have formulated refined Chern-Simons theory by using U(1) symmetry of a Seifert manifold or a torus knot [AS15, AS12b]. Remarkably, the large N limits of refined Chern-Simons invariants of a torus knot are conjectured to be equal to Poincaré polynomials of the corresponding HOMFLY-PT homology when colors are labelled by rectangular Young diagrams. In addition, refined Chern-Simons theory on S^3 at large N is dual to refined topological string theory on the resolved conifold. Based on these results, refined topological vertex formulations are reproduced by using refined Chern-Simons theory at large N [AS12c].

The goal of this paper is to formulate large N duality of refined Chern-Simons theory with a torus knot. In other words, we shall find a relation between refined Chern-Simons invariants of a torus knot and refined BPS states in the resolved conifold, which is the generalization of the work of LMOV in refined topological string theory. This formulation also extends the proposal of [GSV05] to any colors in the case of torus knots. Since the form of partition functions on the deformed conifold has been determined in [AS15, AS12b, AS12c], we shall investigate the form of partition functions on the resolved conifold and will find implications of the large N duality on refined BPS spectra.

To this end, let us first recall relevant setups of M-theory in which refined topological string theories are embedded. On the deformed conifold side, we will consider the following brane configurations:

$$
\begin{array}{llll}
\text{space-time:} & S^1 \times TN_4 \times T^*S^3 & \text{space-time:} & S^1 \times TN_4 \times T^*S^3 \\
N \text{ M5-branes:} & S^1 \times D_q \times S^3 & N \text{ M5-branes:} & S^1 \times D_q \times S^3 \\
M \text{ M5'-branes:} & S^1 \times D_q \times \mathcal{N}_K & M \text{ M5'-branes:} & S^1 \times D_{\bar{t}} \times \mathcal{N}_K
\end{array} \quad (1.1)
$$

Here, D_q is the two-dimensional cigar and $D_{\bar{t}}$ is the two-dimensional base of the Taub-NUT space TN_4. Moreover, writing the local complex coordinate z_1 for D_q and z_2 for $D_{\bar{t}}$, we turn on the Ω-background by the action

$$
(z_1, z_2) \to (q z_1, t^{-1} z_2) . \tag{1.2}
$$

The N M5-branes wrap the zero section (a special Lagrangian submanifold) of the cotangent bundle T^*S^3 of a three-sphere, realizing Chern-Simons theory with U(N) gauge group on S^3 [Wit95]. The M probe M5'-branes are located at another Lagrangian submanifold, which is the co-normal bundle $\mathcal{N}_K \subset T^*S^3$ to a knot $K \subset S^3$ where the knot K is realized as the intersection of the two stacks of the M5-branes $K = S^3 \cap \mathcal{N}_K$ [OV00]. As in [AS12c], we consider two distinct probe M5'-branes; one spanning on D_q and the other on $D_{\bar{t}}$. Although both the configurations preserve four supercharges, the q and $\bar{t}$ branes become topological branes and anti-branes, respectively, in topological string theory.

The groundbreaking proposal in [GV99] is that, at large N, the geometry undergoes the transition where S^3 shrinks and $S^2 = \mathbb{CP}^1$ is blown up. As S^3 shrinks, in order for the probe M5'-branes to avoid the singularity, the conormal bundle $\mathcal{N}_K$ is lifted to the fiber direction and it no longer touches S^3. We refer the reader to [DSV13] for detailed treatment. As a

result, the Calabi-Yau three-fold becomes the resolved conifold $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$, the N M5-branes turn into B-field flux supporting $\mathbb{CP}^1$ and the M M5'-branes are situated on a Lagrangian submanifold $\mathcal{L}_K \subset X$ encoding the information of the knot K:

$$
\begin{array}{llll}
\text{space-time:} & S^1 \times TN_4 \times X & \text{space-time:} & S^1 \times TN_4 \times X \\
M \text{ M5'-branes:} & S^1 \times D_q \times \mathcal{L}_K & M \text{ M5'-branes:} & S^1 \times D_{\bar{t}} \times \mathcal{L}_K
\end{array} \quad (1.3)
$$

Indeed the large N duality can be summarized as an equivalence of partition functions. On the deformed conifold, the partition function is a generating function of refined Chern-Simons invariants at large N. On the other hand, the form of a partition function on the resolved conifold can be determined by adapting Schwinger calculation and it is expressed as a sum of contributions of all the BPS states arising from M2-branes attached to the M5'-branes [GV98b, GV98a, OV00, LMnV00]. Hence, the large N duality relates refined Chern-Simons theory to M2-M5' bound BPS states. Moreover, in the refined context, we will conjecture a positivity property for refined Chern-Simons invariants.

The organization of this paper is as follows. In §2, we review refined Chern-Simons theory and generating functions of refined Chern-Simons invariants of torus knots. In §3, we shall perform Schwinger's computation for BPS states in M-theory to determine low-energy effective actions of Type IIA string theory in the refined context. We first consider M-theory without M5'-branes for illustrative purpose and then proceed to the case with M5'-branes. In §4, the large N duality is proposed for torus knots at the refined level. This relates refined Chern-Simons invariants to graded dimensions of BPS states in the resolved conifold, which results in a positivity conjecture for refined Chern-Simons invariants. In §5, the large N duality is generalized to multi-component torus links. In §6, we discuss related open problems. In particular, we comment on property of BPS states for non-torus knots. In Appendix A, we present a brief summary of symmetric functions used in this paper. Appendix B provides explicit formulas for reformulated invariants in terms of refined Chern-Simons invariants by using the large N duality. Appendix C contains tables for BPS degeneracies associated to torus knots/links in the resolved conifold.

2 Deformed conifold side

On both the deformed conifold and resolved conifold side, low-energy effective actions are determined by taking account of small fluctuations around trajectory of a BPS particle arising from an M2-brane. On the deformed conifold side, partition functions of refined topological string theory have been analyzed in [AS15, AS12b, AS12c]. In this section, we briefly review generating functions of refined Chern-Simons invariants and we refer the reader to the original papers [AS15, AS12b, AS12c] for more details.

The work of Witten [Wit95] provides the way to interpret Chern-Simons partition functions as an index counting BPS states in M-theory. From the viewpoint of M-theory, BPS states arise from M2-branes bridging between the M5-branes and the M5'-branes, and they propagate only along the M5-branes. In low energy, one can suppresses $S^3 \subset T^*S^3$, and we can view that these states propagate along $\mathbb{R} \times D_q \subset \mathbb{R} \times TN_4$ when S^1 is large. Therefore, Chern-Simons partition functions can be identified with an index $\text{Tr}(-1)^F q^{J_1 - J_2}$ of the three-

dimensional theory on $S^1 \times D_q$ with $\mathcal{N} = 2$ supersymmetry where J_1 is the Cartan generator of the SO(3) Lorentz symmetry of $S^1 \times D_q$ and J_2 is the generator of the $U(1)_2$ $\mathcal{R}$-symmetry in $\mathcal{N} = 2$ supersymmetry. In fact, J_1 and J_2 can be considered as generators of the rotations around the z_1 and z_2 plane, respectively.

In [AS15], it is argued that the extra $U(1)_R$ symmetry exists when K is a torus knot $T_{m,n}$ (more generally Seifert three-manifolds). Then, using the charge S_R of this $U(1)_R$ symmetry, a refined Chern-Simons invariant of a torus knot is defined by a refined index [1]

$$\overline{\mathrm{rCS}}_{\mathrm{SU}(N),\lambda}(T_{m,n}) \ \text{“} = \text{”} \ \mathrm{Tr}_{\,\mathcal{H}_\lambda(T_{m,n})}\,(-1)^F\,q^{J_1-S_R}\,t^{S_R-J_2}\ .$$

where " $=$ " means the equality up to a suitable normalization. Therefore, refined Chern-Simons theory is defined via the 3d/3d correspondence. In §3.2, we shall provide more explanation about the refined index. These invariants can be computed by using the modular S and T matrices associated to Macdonald functions [AS15] or double affine Hecke algebra [Che13]. Moreover, it is proven in [GN15] that there exists a unique rational function $\overline{\mathrm{rCS}}_\lambda(T_{m,n}; a, q, t)$ which is the stable large N limit of refined Chern-Simons invariants with $\mathrm{SU}(N)$ gauge group [2] of a torus knot in the following sense:

$$\overline{\mathrm{rCS}}_\lambda(T_{m,n}; a = t^N, q, t) = \overline{\mathrm{rCS}}_{\mathrm{SU}(N),\lambda}(T_{m,n}; q, t)\ .$$

The stable limit $\overline{\mathrm{rCS}}_\lambda(T_{m,n}; a, q, t)$ is sometimes called *DAHA-superpolynomial*. For instance, the refined Chern-Simons invariants of the unknot

$$\overline{\mathrm{rCS}}_\lambda(\bigcirc\,; a, q, t) = \prod_{x \in \lambda} \frac{t^{l'(x)} - aq^{a'(x)}}{1 - q^{a(x)}t^{l(x)+1}}$$

so that, at the $a = t^N$ specialization, it becomes the principal specialization of the Macdonald function

$$\overline{\mathrm{rCS}}_\lambda(\bigcirc\,; a = t^N, q, t) = P_\lambda(t^\rho; q, t) = \overline{\mathrm{rCS}}_{\mathrm{SU}(N),\lambda}(\bigcirc\,; q, t)\ ,$$

where ρ is the Weyl vector of $\mathfrak{sl}(N)$. (See Figure 1 for the notation.) When the color λ is labelled by a rectangular Young diagram, it is conjectured that the invariant $\overline{\mathrm{rCS}}_\lambda(T_{m,n}; a, q, t)$ with change of variables (4.3) coincides with the Poincaré polynomial of the corresponding HOMFLY-PT homology.

Moreover, a partition function of refined topological string theory on the deformed conifold can be determined by taking account of BPS particles arising from annulus M2-branes stretched between S^3 and $\mathcal{N}_{T_{m,n}}$. For the q-brane setting in (1.1), the M2-brane stretching between S^3 and $\mathcal{N}_{T_{m,n}}$ gives rise to two bifundamental $\mathcal{N} = 2$ chiral multiplets, Φ and $\widetilde{\Phi}$, on $S^1 \times D_q$ charged as $(\mathbf{N}, \overline{\mathbf{M}})$ and $(\overline{\mathbf{N}}, \mathbf{M})$, respectively, under $U(N) \times U(M)$. The chiral multiplet Φ has charges $(0, 0; -\frac{1}{2})$ under $(J_1, J_2; S_R)$ and $\widetilde{\Phi}$ is neutral. The three-dimensional $\mathcal{N} = 2$ index can be evaluated by counting "single-letter index" [IY11, KW11] where the chiral multiplet Φ contributes to the single-letter index by $(q^{1/2}t^{-1/2} - 1)/(1 - q)$ and the

---

[1]Charges of J_1, J_2 and S_R are normalized to be half-integers.

[2]The brane setting gives rise to $U(N)$ gauge group instead of $\mathrm{SU}(N)$ gauge group. However, the $U(1)$ part merely provides the correction due to the framing number as well as the linking number of a knot/link, which play no role in this paper. Therefore, the difference between $U(N)$ and $\mathrm{SU}(N)$ invariants will be ignored in this paper.

other $\widetilde{\Phi}$ yields $(1 - q^{1/2}t^{1/2})/(1-q)$. From the viewpoint of the three-dimensional theory on $S^1 \times D_q$, the M M5'-branes give rise to U(M) flavor symmetry, and a torus knot $T_{m,n}$ in S^3 contributes to the single-letter index via the 3d/3d correspondence. Therefore, putting the single-letter index invariant under U(N) $\times$ U(M) into the plethystic exponent, the refined index takes the form

$$Z^q_{\text{def,SU}(N)} = \exp\left(\sum_{d>0}\sum_\mu \frac{1}{d}\frac{t^{\frac{d}{2}} - t^{-\frac{d}{2}}}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}}\, \text{Ind}_{T_{m,n},\mu}(q^d, t^d)\, s_\mu(x^d)\right) ,$$

where $\text{Ind}_{T_{m,n},\mu}(q,t)$ is a contribution of the knot $T_{m,n}$ to the single-particle index charged under the flavor representation μ. This expression can be expanded by the basis of holonomies of the U(M) group which are the Macdonald functions $P_\lambda(x; q, t)$ in refined Chern-Simons theory

$$Z^q_{\text{def,SU}(N)} = \sum_\lambda g_\lambda(q,t)\, \overline{\text{rCS}}_{\text{SU}(N),\lambda}(T_{m,n}; q, t)\, P_\lambda(x; q, t) , \tag{2.1}$$

where the function g_λ can be determined by using the unknot invariants and it turns out to be the Macdonald norm defined in Appendix A. At the unrefined limit $q = t$, it reduces to the generating function of SU(N) quantum invariants $\overline{J}_{\text{SU}(N),\lambda}(T_{m,n}; q)$ first obtained in [OV00]

$$Z_{\text{def,SU}(N)} = \sum_\lambda \overline{J}_{\text{SU}(N),\lambda}(T_{m,n}; q)\, s_\lambda(x) ,$$

where $s_\lambda(x)$ are the Schur functions.

On the other hand, the $\bar{t}$-branes intersect with the q-branes at a point in D_q in (1.1) so that annulus M2-branes stretched between S^3 and $\mathcal{N}_{T_{m,n}}$ bring about only a single bifundamental fermionic particle. Since it contributes to the refined index by -1, the index is of the form

$$Z^{\bar{t}}_{\text{def,SU}(N)} = \exp\left(\sum_{d>0}\sum_\mu \frac{(-1)}{d}\, \text{Ind}_{T_{m,n},\mu}(q^d, t^d)\, s_\mu(x^d)\right) .$$

For the $\bar{t}$-brane setting, the expansion in terms of the basis of holonomies of the U(M) group is more subtle. Since the $\bar{t}$-branes are topological anti-branes, colors for the holonomy need to be transposed to the one for the ordinary branes. In addition, the role of the equivariant parameters should be exchanged $(q, t) \leftrightarrow (t^{-1}, q^{-1})$ due to (1.2) for the holonomy of the $\bar{t}$-branes. Therefore, the partition function for the $\bar{t}$-branes takes the form

$$Z^{\bar{t}}_{\text{def,SU}(N)} = \sum_\lambda \overline{\text{rCS}}_{\text{SU}(N),\lambda}(T_{m,n}; q, t)\, P_{\lambda^T}(-x; t, q) , \tag{2.2}$$

where we use the property $P_{\lambda^T}(-x; t^{-1}, q^{-1}) = P_{\lambda^T}(-x; t, q)$ of the Macdonald functions.

3 Resolved conifold side

In the seminal papers [GV98b, GV98a], Gopakumar and Vafa (GV) proposed that an effective action of Type IIA string theory compactified on a Calabi-Yau manifold can be determined by considering effects of BPS particles arising M2-branes in the anti-selfdual graviphoton

background. Moreover, the form of an effective action has been explicitly evaluated by applying Schwinger computations. Subsequently, Labastida, Mariño, Ooguri and Vafa (LMOV) [OV00, LMn01, LMnV00, LMn02] have carried out similar analyses in the presence of D4-branes, which can be regarded as an open-string analogue of the GV formula. At the unrefined level, thorough analysis and elaborate explanation for these formulas have been presented in [Mn05, DW16]. In this section, we will find the explicit form of an effective action of Type IIA string theory in the presence of D4-branes in the *refined* case.

3.1 Without M5-branes

To this end, let us first review an effective action of Type IIA string theory compactified on a Calabi-Yau manifold X without D4-branes. In this subsection, we assume that a Calabi-Yau three-fold X is general and we do not necessarily restrict ourselves to the resolved conifold. The effective action of Type IIA string theory has F-terms that admit genus expansion

$$F^{\rm cl} = \log Z_{\rm res}^{\rm cl} = -i \sum_{g \geq 0} \int_{\mathbb{R}^4} {\rm d}^4 x \, {\rm d}^4 \theta \, \mathcal{F}_g(\mathcal{X}_\Lambda)(\mathcal{W}_{AB}\mathcal{W}^{AB})^g \; ,$$

where $\mathcal{F}_g$ are holomorphic functions of chiral superfields $\mathcal{X}_\Lambda$ ($\Lambda = 0, \cdots, b_2(X)$) associated to vector multiplets and $\mathcal{W}_{AB}$ is a chiral superfield whose bottom component is anti-selfdual graviphoton field strength

$$\mathcal{W}_{AB} = \frac{1}{2}T_{AB} - R_{ABCD}\theta\sigma^{CD}\theta + \cdots \; .$$

Since the graviphoton field strength takes the value $T = g_{\rm st}({\rm d}x_1 \wedge {\rm d}x_2 - {\rm d}x_3 \wedge {\rm d}x_4)$, the genus g contribution is proportional to $g_{\rm st}^{2g-2}$. Gopakumar and Vafa proposed that the free energy F can be evaluated by summing up contributions of BPS states in M-theory. They arise from M2-branes wrapped on a holomorphic curve Σ in the Calabi-Yau manifold X.

To see the spin content of BPS states, let us recall the five-dimensional $\mathcal{N} = 1$ supersymmetry algebra consisting of eight supercharges with an $\mathrm{Sp}(1)_R = \mathrm{SU}(2)_R$ $\mathcal{R}$-symmetry. Since the graviphoton field breaks the five-dimensional Lorentz group $\mathrm{SO}(1,4)$, it is convenient to rewrite the algebra in terms of four-dimensional notation, so that the supercharges can be organized as $Q_\alpha^I, Q_{\dot\alpha}^I$ where $I = 1, 2$ are the $\mathrm{SU}(2)_R$ indices, and α and $\dot\alpha$ are negative and positive chirality of the rotational group $\mathrm{SO}(4) = \mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$. Then, the supersymmetry algebra is given by

$$\{Q_\alpha^I, Q_\beta^J\} = \varepsilon_{\alpha\beta}\varepsilon^{IJ}(H + \zeta) \; ,$$
$$\{Q_{\dot\alpha}^I, Q_{\dot\beta}^J\} = \varepsilon_{\dot\alpha\dot\beta}\varepsilon^{IJ}(H - \zeta) \; ,$$
$$\{Q_\alpha^I, Q_{\dot\beta}^J\} = -i\Gamma^\mu_{\alpha\dot\beta}\varepsilon^{IJ}P_\mu \; .$$

where ζ is the *real* five-dimensional central charge. Then, short left-handed BPS multiplets take the form,

$$\left((0,0;\tfrac{1}{2}) \oplus (\tfrac{1}{2},0;0)\right) \otimes (J_\ell, J_r; S_R) \; , \tag{3.1}$$

which represent BPS particles of mass $m = \zeta$ at rest. The first spin content is indeed the half-hypermultiplet representation and the second is an arbitrary finite dimensional representation

of $SU(2)_\ell \times SU(2)_r \times SU(2)_R$. It is easy to see that the unrefined index

$$\text{Tr}(-1)^{2(J_\ell+J_r)} q^{2J_\ell} e^{-\beta H}$$

receives contributions only from the short left-handed BPS multiplets. Therefore, a one-loop calculation involving small fluctuations around a BPS particle trajectory takes the form

$$
\begin{aligned}
F^{\text{cl}} &= -\int_0^\infty \frac{ds}{s} \frac{\text{Tr}_{\mathcal{BPS}}(-1)^{2(J_\ell+J_r)} q^{2sJ_\ell} e^{-sm}}{(q^{\frac{s}{2}} - q^{-\frac{s}{2}})^2} \ , \\
&= -\sum_{d>0} \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{d} \frac{\text{Tr}_{\mathcal{H}(\beta)}(-1)^{2(J_\ell+J_r)} q^{2dJ_\ell}}{(q^{\frac{d}{2}} - q^{-\frac{d}{2}})^2} e^{-d\beta\cdot\tau} \ ,
\end{aligned}
\tag{3.2}
$$

where we identify the parameters by $q = e^{ig_{st}}$ and m is the central charge of BPS particles. Let us closely look at the meaning of the GV formula (3.2).

- the denominator is obtained by Schwinger computation for the one-loop determinant of BPS particle with the half-hypermultiplet representation in the anti-selfdual graviphoton background of the form

$$
T^- = \frac{1}{2}
\begin{pmatrix}
0 & g_{st} & & \\
-g_{st} & 0 & & \\
& & 0 & -g_{st} \\
& & g_{st} & 0
\end{pmatrix} .
\tag{3.3}
$$

  The two dimensional contribution (the upper block) can be evaluated by summing up all the Landau levels $(\frac{1}{2} + n)g_{st}$ for $n \in \mathbb{Z}_{\geq 0}$:

$$\sum_{n\geq 0} e^{-ig_{st}(1+2n)/2} = \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \ .$$

  Including an identical factor for the lower block, the Schwinger calculation provides the denominator.

- the unrefined index in the numerator receives contributions from other massive BPS states (3.1), which can be understood as fermion zero modes on M2-branes. The detail analysis for the BPS spectra will be given below.

- the central charge of a BPS state is given by the area of a holomorphic curve Σ the M2-brane wraps and the Kaluza-Klein momentum. If we take a basis C_I ($I = 1, \cdots, b_2(X)$) of $H_2(X,\mathbb{Z})$, then the homology class of the curve is expressed by $[\Sigma] = \sum_I \beta_I C_I$ with $\beta_I \in \mathbb{Z}$. Denoting the complexified Kahler parameter of the 2-cycle C_I by τ^I, the area of the M2-brane is equal to $\tau \cdot \beta = \sum_I \tau^I \beta_I$. Then, the central charge of the BPS state is given by $m = \tau \cdot \beta + 2\pi in$ where n is the Kaluza-Klein momentum of the M2-brane along the M-theory circle.

- from the first to the second line, we perform a Poisson resummation $\sum_{n\in\mathbb{Z}} e^{-2\pi ins} = \sum_{d\in\mathbb{Z}} \delta(s-d)$, which re-expresses the sum over the Kaluza-Klein momenta as a sum over winding numbers.

In refined topological string theory, the graviphoton field is no longer anti-selfdual, and it rather takes the form

$$T = \epsilon_1 dx_1 \wedge dx_2 - \epsilon_2 dx_3 \wedge dx_4 \ , \tag{3.4}$$

which introduces the Ω-background (1.2). Furthermore, by using the $\mathrm{SU}(2)_R$ $\mathcal{R}$-symmetry, we define a refined index [Nek04] as

$$\mathrm{Tr}(-1)^{2(J_\ell+J_r)} q_\ell^{2J_\ell} q_r^{2(J_r-S_R)} e^{-\beta H} \ .$$

It is easy to see that only left-handed multiplets (3.1) again contribute to the index while the long multiplets and the right-handed multiplets do not. To write the refined index in terms of the equivariant parameters $q = e^{i\epsilon_1}$ and $t = e^{i\epsilon_2}$, we introduce $J_1 = J_\ell + J_r$ and $J_2 = J_r - J_\ell$ so that it takes the form

$$\mathrm{Tr}(-1)^F q^{J_1-S_R} t^{S_R-J_2} e^{-\beta H} \ ,$$

where we define $q_\ell = (qt)^{1/2}$ and $q_r = (q/t)^{1/2}$. Then, the free energy at the refined level can be written as

$$\begin{aligned}
F_{\mathrm{ref}}^{\mathrm{cl}} &= \int_0^\infty \frac{ds}{s} \frac{\mathrm{Tr}_{\mathcal{BPS}}(-1)^{2(J_\ell+J_r)} q^{s(J_1-S_R)} t^{s(S_R-J_2)} e^{-sm}}{(q^{\frac{s}{2}}-q^{-\frac{s}{2}})(t^{-\frac{s}{2}}-t^{\frac{s}{2}})} \ , \\
&= \sum_{d>0} \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{d} \frac{\mathrm{Tr}_{\mathcal{H}(\beta)}(-1)^{2(J_\ell+J_r)} q^{d(J_1-S_R)} t^{d(S_R-J_2)}}{(q^{\frac{d}{2}}-q^{-\frac{d}{2}})(t^{-\frac{d}{2}}-t^{\frac{d}{2}})} e^{-d\beta \cdot \tau} \ .
\end{aligned}$$

Since we now have different equivariant parameters the upper and lower block in (3.3), the denominator is resolved to $(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2 \to (q^{\frac{1}{2}}-q^{-\frac{1}{2}})(t^{\frac{1}{2}}-t^{-\frac{1}{2}})$.

Now let us study other massive BPS states that contribute to the refined index. A propagating M2-brane wrapped on a holomorphic curve $\Sigma_g \subset X$ of genus g generates a 5d particle that preserves a half of superysymmetry [BBS95]. For the low energy description, we need to take into account fermion zero-modes on Σ_g where half of them transform under $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ as $(\frac{1}{2},0)$ and the other half transform as $(0,\frac{1}{2})$. The $(\frac{1}{2},0)$ fermionic zero modes can be interpreted as differential forms on Σ_g and therefore there are zero-modes consisting of $2g$ copies of the $(\frac{1}{2},0)$ representation on the curve Σ_g of genus g. In other words, the $(\frac{1}{2},0)$ zero modes on Σ_g can be interpreted as the cohomology of the Jacobian of Σ_g where the $\mathrm{SU}(2)_\ell$ action can be understood as the natural Lefschetz $\mathrm{SU}(2)$ action on the cohomology $\mathcal{H}_g := H^*(\mathrm{Jac}(\Sigma_g))$ of the Jacobian. Quantization of this system gives g copies of the spin content $(0,0;\frac{1}{2}) \oplus (\frac{1}{2},0;0)$ under $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r \times \mathrm{SU}(2)_R$ so that the contribution to the index is $(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^g (t^{-\frac{1}{2}}-t^{\frac{1}{2}})^g$.

On the other hand, the $(0,\frac{1}{2})$ fermion zero modes are related by supersymmetry to infinitesimal deformations of Σ_g as a holomorphic curve in X. Let us denote a moduli space $\mathcal{M}_{g,\beta}$ that parametrizes the bosonic deformations of Σ_g under the homology class $\beta \in H_2(X,\mathbb{Z})$ inside the Calabi-Yau three-fold X. Then, if $\mathcal{M}_{g,\beta}$ and Σ_g are both smooth, the total moduli space $\widehat{\mathcal{M}}_{\mathrm{cl}}$ for the BPS states in this configuration is

$$\begin{array}{ccc}
\mathrm{Jac}(\Sigma_g) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{cl}} \\
 & & \downarrow{\scriptstyle \pi} \\
 & & \mathcal{M}_{g,\beta}
\end{array} \qquad . \tag{3.5}$$

Since the zero-modes of $(0, \frac{1}{2})$ fermions are differential forms on $\mathcal{M}_{g,\beta}$, the space of total BPS states in this situation is the de Rham cohomology $H^*(\mathcal{M}_{g,\beta}; \mathcal{H}_g)$ of $\mathcal{M}_{g,\beta}$ with values in $\mathcal{H}_g$ where the $SU(2)_r$ action is the natural Lefschetz $SU(2)$ action on $H^*(\mathcal{M}_{g,\beta})$. However, the assumption that Σ_g is always smooth is almost never satisfied and there are usually singular fibers in (3.5), which makes it difficult to give a rigorous mathematical definition of GV invariants. (However, important progress has been made recently in mathematics [PT10, MT16].)

In physics, one can formally count the number of BPS states. For this purpose, we decompose the zero-modes of $(0, \frac{1}{2})$ fermions into the spectrum $H^*(\mathcal{M}_{g,\beta}) \cong \bigoplus A_{g,\beta,J_r,S_R}$ with respect to J_r and S_R spins so that the total BPS spectrum takes the form

$$\mathcal{H}(\beta) \cong \bigoplus_{g,J_r,S_R} \mathcal{H}_g \otimes A_{g,\beta,J_r,S_R} .$$

As a result, denoting the number of states with fixed charges by

$$\widehat{N}_{g,\beta,J_r,S_R} := \dim A_{g,\beta,J_r,S_R} ,$$

the refined free energy takes the form

$$F_{\mathrm{ref}}^{\mathrm{cl}} = \sum_{d>0} \sum_{\mathrm{charges}} \frac{1}{d} (-1)^{2J_r} \widehat{N}_{g,\beta,J_r,S_R} (q^{\frac{d}{2}} - q^{-\frac{d}{2}})^{g-1} (t^{-\frac{d}{2}} - t^{\frac{d}{2}})^{g-1} e^{-d\beta\cdot\tau} \sum_{j_r=-J_r}^{J_r} \sum_{s_R=-S_R}^{S_R} \left(\frac{q}{t}\right)^{d(j_r - s_R)} ,$$

where charges are summed over $g \geq 0$, $\beta \in H_2(X, \mathbb{Z})$, and $J_r, S_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. In particular, the integral numbers for unrefined BPS states

$$n_{g,\beta} = \sum_{J_r,S_R \in \frac{1}{2}\mathbb{Z}_{\geq 0}} (-1)^{2J_r}(2J_r + 1)(2S_R + 1)\widehat{N}_{g,\beta,J_r,S_R}$$

are called GV invariants, and $n_{g,\beta} = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_{g,\beta}} \chi(\mathcal{M}_{g,\beta})$ if $\mathcal{M}_{g,\beta}$ is smooth. We refer the reader to [AS12c, CKK14, CDDP15, NO14, GHKPK17, references therein] for recent developments on refined closed BPS invariants.

3.2 With M5-branes

Now let us include M5'-branes on $S^1 \times \mathbb{R}^2 \times \mathcal{L} \subset S^1 \times \mathbb{R}^4 \times X$ like (1.3) where $\mathcal{L}$ is a special Lagrangian submanifold of a Calabi-Yau three-fold X. In general, a half of supersymmetry is preserved even if we include a number of M5'-branes unless their supports $\mathcal{L}_i \subset X$ are special Lagrangian [BBS95]. By reducing on the M-theory circle, we have Type IIA string theory with D4-branes. Then, its low energy effective action has terms that are supported on the world-volume $\mathbb{R}^2 \subset \mathbb{R}^4$ of the D4-branes so that it takes form

$$F^{\mathrm{op}} = \sum_{g,h\geq 0} \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \mathrm{d}^4\theta \, \delta(x^2)\delta(\theta^2) \, \mathcal{F}_{g,h}(\mathcal{X}_\Lambda; \mathcal{V}_\sigma)(\mathcal{W}^2)^g \mathcal{W}_\parallel^{h-1} , \tag{3.6}$$

where $\mathcal{V}_\sigma$ ($\sigma = 1, \ldots, b_1(\mathcal{L})$) are chiral superfields associated to the moduli of $\mathcal{L}$. Here $\mathcal{W}_\parallel$ is the "parallel" component of the graviphoton superfield $\mathcal{W}_{AB}$. More precisely, for the anti-selfdual graviphoton background (3.3), $T_{12}^- = T_\parallel/2$ if the D4-branes are located on the D_q

plane and $T_{34}^- = T_\parallel/2$ if they are put on the $D_{\bar t}$ plane where $T_\parallel$ is the bottom component of the superfield $\mathcal{W}_\parallel$. At the unrefined level, each integral in (3.6) is proportional to $g_s^{2g+h-2} = g_s^{-\chi}$, which is natural from the viewpoint of string perturbation theory since the Euler characteristic of a Riemann surface of genus g with h holes is equal to $\chi = 2 - 2g - h$.

Even in the presence of M5'-branes, one can apply the same idea that F^{op} can be determined by analyzing contributions of BPS states in M-theory [OV00, LMn01, LMnV00, LMn02]. To obtain a contribution to the Type IIA effective action supported on $\mathbb{R}^2$, a relevant BPS state arises from an M2-brane wrapped on $\Sigma \subset X$ and generally attached to the M5'-branes on $\mathcal{L}$ so that they propagate only along the M5'-branes. From a low energy point of view, these states propagate along $S^1 \times \mathbb{R}^2 \subset S^1 \times \mathbb{R}^4$.

Therefore, let us investigate quantum numbers of BPS states in three-dimensional $\mathcal{N} = 2$ supersymmetric theory. The three-dimensional $\mathcal{N} = 2$ supersymmetry algebra is given by

$$\{Q_\alpha, \overline{Q}_\beta\} = -i\sigma^\mu_{\alpha\beta}P_\mu + i\epsilon_{\alpha\beta}\zeta \ ,$$
$$\{Q_\alpha, Q_\beta\} = 0 = \{\overline{Q}_\alpha, \overline{Q}_\beta\} \ .$$

For a theory on the q-branes, the supercharges Q are complex spinors in the spin-$\frac{1}{2}$ representation of the rotation group $\mathrm{SO}(3) \cong \mathrm{SU}(2)_1$, which is the diagonal subgroup of $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$. In addition, a $\mathrm{U}(1)_2$ $\mathcal{R}$-symmetry, which is actually the $\mathrm{U}(1)$ subgroup of the skew-diagonal subgroup $\mathrm{SU}(2)_2 = \{(x, x^{-1}) \in \mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r\}$ in five-dimension, rotates Q_α and $\overline{Q}_\alpha$. (See Table 1.) For a theory on the $\bar{t}$-branes, the roles of $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$ are exchanged so that the rotational group is identified with $\mathrm{SU}(2)_2$ and the $\mathcal{R}$-symmetry is $\mathrm{U}(1)_1 \subset \mathrm{SU}(2)_1$.

The three-dimensional unrefined index defined by

$$\mathrm{Tr}(-1)^F q^{J_1 - J_2} e^{-\beta H} \ . \tag{3.7}$$

counts states annihilated by the supercharges Q_+ and $\overline{Q}_-$. Then, only the left short multiplets, which represent BPS particles of mass $M = \zeta$ at rest,

$$\left((0; -\tfrac{1}{2}) \oplus (-\tfrac{1}{2}; 0)\right) \otimes (J_1; J_2) \ , \tag{3.8}$$

contribute to the index. The proposal of LMOV is that the free energy (3.6) in the anti-selfdual graviphoton background (3.3) takes the form

$$F^{\mathrm{op}} = \pm \int_0^\infty \frac{ds}{s} \frac{\mathrm{Tr}_{\mathcal{BPS}}(-1)^F q^{s(J_1 - J_2)} e^{-sm}}{(q^{\frac{s}{2}} - q^{-\frac{s}{2}})} \ ,$$
$$= \pm \sum_{d>0} \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{\vec{k}} \frac{1}{d} \frac{\mathrm{Tr}_{\mathcal{H}(\beta, \vec{k})}(-1)^F q^{d(J_1 - J_2)}}{(q^{\frac{d}{2}} - q^{-\frac{d}{2}})} e^{-d\beta \cdot \tau} p_{\vec{k}}(x) \ .$$

- since BPS particles propagate only along the M5'-branes, the Schwinger computation is here performed only on $\mathbb{R}^2 \subset \mathbb{R}^4$ spanned by the M5'-branes so that the denominator originates from either the upper (q-brane) or the lower block ($\bar{t}$-brane) of (3.3), depending on the M5'-brane configurations.

- as in the closed case §3.1, the unrefined index in the numerator receives contributions from other massive BPS states (3.8), which can be understood as fermion zero modes on M2-branes attached to the M5'-branes. The detail analysis for the BPS spectra will be given below.

- the central charge of a BPS state is expressed by the area of the M2-brane as well as the Kaluza-Klein modes. The area of a holomorphic curve $\Sigma \subset X$ whose boundary is on $\mathcal{L}$ is determined by its relative homology class in $H_2(X, \mathcal{L}; \mathbb{Z})$. A Lagrangian subvariety $\mathcal{L}$ in (1.3) for a knot is topologically homeomorphic to $S^1 \times \mathbb{R}^2$ [HLJ82, AV00], which simplifies the relative homology as

$$H_2(X, \mathcal{L}; \mathbb{Z}) \cong H_2(X; \mathbb{Z}) \oplus H_1(\mathcal{L}; \mathbb{Z}) \ .$$

Therefore, the homology class $[\Sigma] \in H_2(X, \mathcal{L}; \mathbb{Z})$ is expressed by $\beta = (\beta_1, \cdots, \beta_{b_2(X)}) \in H_2(X; \mathbb{Z})$ as in the closed string as well as the winding numbers $w = (w_1, \cdots, w_h) \in (H_1(\mathcal{L}; \mathbb{Z}))^h \cong \mathbb{Z}^h$ of boundary components $\partial \Sigma \cong (S^1)^h$. Since Σ is oriented, one can assume that w_i are all non-negative integers. To express the contribution from the boundary components, we define a vector $\vec{k}$ as follows: the i-th entry of $\vec{k}$ is the number of w_i's that take the value i. Then, when M M5'-branes wrap on $S^1 \times \mathbb{R}^2 \times \mathcal{L}$, we can write

$$e^{-m} = e^{-\tau \cdot \beta - 2\pi i n} \ p_{\vec{k}}(x) \ ,$$

where the fugacities x parametrize the Cartan subgroup of the U(M)-valued moduli of $\mathcal{L}$ and $p_{\vec{k}}(x)$ is defined in Appendix §A. Let us note that it is easy to transform from the winding basis $p_{\vec{k}}(x)$ to the representation basis $s_\mu(x)$ for the moduli of $\mathcal{L}$ by using (A.1).

	$2J_1$	$2J_2$	$2S_R$
Q_+	$+1$	$+1$	$+1$
Q_-	-1	$+1$	$+1$
$\overline{Q}_+$	$+1$	-1	-1
$\overline{Q}_-$	-1	-1	-1

Table 1. Charges for 3d $\mathcal{N} = 2$ supersymmetry on the q-brane.

Now, let us consider the case where X is the resolved conifold and $\mathcal{L}$ is the configuration $\mathcal{L}_{T_{m,n}}$ for a torus knot as in (1.3). As we have discussed in §2, this configuration preserves the extra U$(1)_R$ global symmetry, which can be actually interpreted as the U(1) subgroup of the SU$(2)_R$ $\mathcal{R}$-symmetry in five dimension with eight supercharges. In this case, the three-dimensional $\mathcal{N} = 2$ supercharges have the quantum numbers under this symmetry shown in Table 1. Since $J_1 - S_R$ and $S_R - J_2$ commute with the supercharges Q_+ and $\overline{Q}_-$, one can refine the index by

$$\mathrm{Tr} \, (-1)^F \, q^{J_1 - S_R} \, t^{S_R - J_2} e^{-\beta H} \ .$$

Therefore, one can consider refined topological string theory in the presence of M5'-branes supported on $S^1 \times \mathbb{R}^2 \times \mathcal{L}_{T_{m,n}}$ by using the refined index. First, we shall consider the q-brane setting in (1.3) at the refined level. In the refined graviphoton background (3.4), the Schwinger computation provides the form of the free energy

$$F^q_{\mathrm{ref}} = \sum_{d>0} \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_\mu \frac{1}{d} \frac{\mathrm{Tr}_{\mathcal{H}(\beta, \mu)} (-1)^F q^{d(J_1 - S_R)} t^{d(S_R - J_2)}}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} e^{-d\beta \cdot \tau} s_\mu(x^d) \ .$$

The BPS states that contribute to the refined index are fermion zero modes on an M2-brane wrapped on a holomorphic curve $\Sigma_{g,h} \subset X$ whose boundary is on $\mathcal{L}$. Since the presence

of the M5'-branes breaks $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r \times \mathrm{SU}(2)_R$ to $\mathrm{SU}(2)_1 \times \mathrm{U}(1)_2 \times \mathrm{U}(1)_R$, it is not so straightforward to study quantum numbers of BPS states as in §3.1. However, as before, fermion zero modes on an M2-brane can be associated to cohomology groups of the moduli space

$$
\begin{array}{ccc}
\mathrm{Jac}(\Sigma_{g,h}) & \longrightarrow & \widehat{\mathcal{M}}_{\mathrm{op}} \\
& & \downarrow{\scriptstyle \pi} \\
& & \mathcal{M}_{g,h,\beta}
\end{array}
\quad ,
$$

where $\mathcal{M}_{g,h,\beta}$ parametrizes holomorphic deformations of $\Sigma_{g,h} \subset X$. More precisely, as emphasized in [LMnV00], the fermion zero modes are the cohomology groups $H^*(\widehat{\mathcal{M}}_{\mathrm{op}}) \cong H^*(\mathrm{Jac}(\Sigma_{g,h})) \otimes H^*(\mathcal{M}_{g,h,\beta})$ of the moduli space *mod out by the action (Sprecht module) of the permutation group* $\mathfrak{S}_h$ where the group $\mathfrak{S}_h$ exchanges h distinguished holes of $\Sigma_{g,h}$. The Jacobian $\mathrm{Jac}(\Sigma_{g,h})$ of a curve of genus g with h holes is topologically $(T^2)^g \times (S^1)^{h-1}$ where the permutation group $\mathfrak{S}_h$ does not act on the cohomology $H^*((T^2)^g)$ of the Jacobian of the "bulk" Riemann surface. Therefore, the contribution to the refined index from $H^*((T^2)^g)$ is $(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g(t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g$ as in §3.1.

The projection of $H^*((S^1)^{h-1}) \otimes H^*(\mathcal{M}_{g,h,\beta})$ onto the invariant subspace can be done by using the Schur functor $\mathbf{S}_\mu$. More explicitly, the invariant subspace in BPS states $\mathcal{H}(\beta, \mu)$ with charge $\beta \in H_2(X, \mathbb{Z})$ and μ (a representation of $\mathrm{U}(M)$) can be written as

$$
\mathrm{Inv}\Big(H^*((S^1)^{h-1}) \otimes H^*(\mathcal{M}_{g,h,\beta})\Big)_{\mathcal{H}(\beta,\mu)} =
$$
$$
\bigoplus_{\mu' \, \mu''} C_{\mu \, \mu' \, \mu''} \mathbf{S}_{\mu'}(H^*((S^1)^{h-1})) \otimes \mathbf{S}_{\mu''}(H^*(\mathcal{M}_{g,h,\beta})) \ , \quad (3.9)
$$

where the Clebsch-Gordon coefficients $C_{\mu \, \mu' \, \mu''}$ of the permutation group $\mathfrak{S}_h$ is

$$
C_{\mu \, \mu' \, \mu''} = \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} \chi_\mu(C(\vec{k})) \chi_{\mu'}(C(\vec{k})) \chi_{\mu''}(C(\vec{k})) \ . \quad (3.10)
$$

It is easy to see that they are symmetric under the permutation of (μ, μ', μ'').

The cohomology of the Jacobian of $\Sigma_{g,h}$ can be interpreted as differential forms on $\Sigma_{g,h}$. In particular, the boundary part $H^*((S^1)^{h-1})$ are spanned by one-forms $d\theta_i$, $(i = 1, \cdots, h)$, which are Poincaré dual to the holes in the curve $\Sigma_{g,h}$, and they are subject to the linear constraint $\sum_i d\theta_i = 0$. Moreover, one can consider the differential form $d\theta_i$ as the fermion zero modes ψ_i on a rigid curve with charges

$$
(J_1; J_2, S_R) = (\tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2}) \ . \quad (3.11)
$$

As explained in [LMnV00, Mn05], the BPS spectra $\mathbf{S}_\mu(H^*((S^1)^{h-1}))$ can be obtained by acting the fermion zero modes ψ_i on the vacuum $|0\rangle$. The defining representation V of $\mathfrak{S}_h$ can be constructed by acting one fermion ψ_i on the vacuum $|0\rangle$, and its dimension is $h - 1$ due to $\sum_i \psi_i = 0$. The rest of the spectra are generated by taking the wedge products $\wedge^d V$ of the defining representation. Assigning the Young diagram ⊞⊞⊞⊞⊞ with h boxes of one row

to the trivial representation $|0\rangle$, the irreducible representations $\wedge^d V$ of $\mathfrak{S}_h$ are called *hook representations* since their Young tableau are of the form with $(h-d)$-boxes in the first row:

Assuming that the vacuum $|0\rangle$ is neutral, the state $\wedge^d V$ has charges

$$(J_1; J_2, S_R) = (\tfrac{d}{2}; -\tfrac{d}{2}, \tfrac{d}{2}) \ ,$$

because it is essentially obtained by acting d-wedge products of the fermions charged by (3.11). Hence, the contribution from the state $\wedge^d V$ to the refined index is $(-t)^d$. As a result, the refined index only over the BPS states $\mathbf{S}_\mu(H^*((S^1)^{h-1}))$

$$B_\mu := \mathrm{Tr}_{\mathbf{S}_\mu(H^*((S^1)^{h-1}))}(-1)^F q^{J_1-S_R} t^{S_R-J_2}$$

is summarized as

$$B_\mu(t) = \begin{cases} (-t)^d t^{-\frac{|\mu|-1}{2}} & \mu : \text{hook rep for } \wedge^d V \\ 0 & \mu : \text{otherwise} \end{cases} \ .$$

In fact, we normalize B_μ by $t^{-\frac{|\mu|-1}{2}}$ so that they satisfy

$$B_\mu(t^{-1}) = (-1)^{|\mu|-1} B_{\mu^T}(t) \ .$$

It is easy to see that the BPS states $\mathcal{H}_{\mu,g} \cong H^*((T^2)^g) \otimes \mathbf{S}_\mu(H^*((S^1)^{h-1}))$ obey $J_1 + J_2 = 0$, corresponding to the $(\frac{1}{2}, 0)$ fermions under $\mathrm{SU}(2)_\ell \times \mathrm{SU}(2)_r$ in five dimension. On the other hand, the BPS states $\mathbf{S}_\mu(H^*(\mathcal{M}_{g,h,\beta}))$ satisfy $J_1 - J_2 = 0$, analogous to the $(0, \frac{1}{2})$ fermions in five dimension. Although they can contribute to the unrefined index (3.7) only by signs, the refined index receives non-trivial contributions. Defining $J_r := \frac{1}{2}(J_1 + J_2)$, one can decompose $\mathbf{S}_\mu(H^*(\mathcal{M}_{g,h,\beta}))$ into the spectrum $\bigoplus A_{\mu,g,\beta,J_r,S_R}$ with respect to J_r and S_R charges so that the total BPS states are

$$\mathcal{H}(\beta,\mu) \cong \bigoplus_{\mu',\mu'',g,J_r,S_R} C_{\mu,\mu',\mu''} \, \mathcal{H}_{\mu',g} \otimes A_{\mu'',g,\beta,J_r,S_R} \ .$$

As a result, writing the number of BPS states with fixed charges by

$$\widehat{N}_{\mu,g,\beta,J_r,S_R} := \dim A_{\mu,g,\beta,J_r,S_R} \ , \tag{3.12}$$

the refined free energy takes the form

$$F_{\mathrm{ref}}^q = \sum_{d>0} \sum_\mu \frac{1}{d} \frac{f_\mu^q(a^d, q^d, t^d)}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} s_\mu(x^d) \ , \tag{3.13}$$

$$f_\mu^q(a,q,t) = \sum_{\text{charges}} (-1)^{2J_r} C_{\mu\,\mu'\,\mu''} B_{\mu'}(t) \widehat{N}_{\mu'',g,\beta,J_r,S_R}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g \left(\frac{q}{t}\right)^{J_r-S_R-\frac{\beta}{2}} a^\beta \ ,$$

where charges are summed over $g \geq 0$, $\beta \in H_2(X,\mathbb{Z})$, $J_r, S_R \in \frac{1}{2}\mathbb{Z}$, and all representations μ', μ'' of U(M). Note that, to see relation to refined Chern-Simons invariants in the next section, here we define the parameter by

$$a := e^{-\tau} \sqrt{\frac{q}{t}} \ .$$

In fact, the integral numbers for unrefined BPS states

$$\widehat{N}_{\mu,g,\beta} = \sum_{J_r, S_R \in \frac{1}{2}\mathbb{Z}} (-1)^{2J_r} \widehat{N}_{\mu,g,\beta,J_r,S_R} \tag{3.14}$$

are called LMOV invariants. Moreover, Mariño and Vafa proposed the multi-covering formula that relates the LMOV invariants $\widehat{N}_{\mu,g,\beta}$ to open Gromov-Witten invariants in the presence of D4-branes supported on $\mathbb{R}^2 \times \mathcal{L}$ [MnV02]. In the case of the framed unknot, the multi-covering formula have been proven based on localization method and combinatorics [LLZ03].

Next let us consider the $\bar{t}$-brane setting in (1.3). The form of the free energy is

$$F_{\text{ref}}^{\bar{t}} = \sum_{d>0} \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{\mu} \frac{1}{d} \frac{\text{Tr}_{\mathcal{H}(\beta,\mu)}(-1)^F q^{d(J_1 - S_R)} t^{d(S_R - J_2)}}{t^{-\frac{d}{2}} - t^{\frac{d}{2}}} e^{-d\beta \cdot \tau} s_\mu(x^d) \ ,$$

where the Schwinger computation on the $D_{\bar{t}}$ plane yields the denominator. Although the BPS spectra in the $\bar{t}$-brane setting are essentially the same as those in the q-brane setting, their J_1 and J_2 charges are exchanged. Hence, the fermion zero modes ψ_i for $H^*((S^1)^{h-1})$ have charges $(J_2; J_1, S_R) = (\frac{1}{2}; -\frac{1}{2}, \frac{1}{2})$ so that the refined index over the BPS states $\mathbf{S}_\mu(H^*((S^1)^{h-1}))$ is given by $B_\mu(q^{-1})$. The other part is exactly same as the q-brane setting so that the free energy for the $\bar{t}$-branes is

$$F_{\text{ref}}^{\bar{t}} = \sum_{d>0} \sum_{\mu} \frac{1}{d} \frac{f_\mu^{\bar{t}}(a^d, q^d, t^d)}{t^{-\frac{d}{2}} - t^{\frac{d}{2}}} s_\mu(x^d) \ , \tag{3.15}$$

$$f_\mu^{\bar{t}}(a,q,t) = \sum_{\text{charges}} (-1)^{2J_r} C_{\mu \, \mu' \, \mu''} B_{\mu'}(q^{-1}) \widehat{N}_{\mu'',g,\beta,J_r,S_R} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g \left(\frac{q}{t}\right)^{J_r - S_R - \frac{\beta}{2}} a^\beta \ .$$

It is easy to see that the free energy for the q-branes and that for the $\bar{t}$-branes are related by

$$F_{\text{ref}}^{\bar{t}}(a,q,t) = F_{\text{ref}}^q(a,t^{-1},q^{-1}) \ , \tag{3.16}$$

which can be expected from the equivariant action (1.2) on $\mathbb{C}^2$.

Finally, let us extract the common part of $f_\mu^q(a,q,t)$ and $f_\mu^{\bar{t}}(a,q,t)$ as

$$\widehat{f}_\mu(a,q,t) = \sum_{\text{charges}} (-1)^{2J_r} \widehat{N}_{\mu,g,\beta,J_r,S_R} (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g \left(\frac{q}{t}\right)^{J_r - S_R - \frac{\beta}{2}} a^\beta \ , \tag{3.17}$$

which is invariant under the exchange $(q,t) \leftrightarrow (t^{-1}, q^{-1})$. If we can define an invertible symmetric matrix

$$M_{\mu \, \mu'}(t) := \sum_{\mu''} C_{\mu \, \mu' \, \mu''} B_{\mu''}(t) \ ,$$

then we obtain concise expressions

$$f_\mu^q(a,q,t) = \sum_\rho M_{\mu\rho}(t)\widehat{f}_\rho(a,q,t) \ ,$$

$$f_\mu^{\bar{t}}(a,q,t) = \sum_\rho M_{\mu\rho}(q^{-1})\widehat{f}_\rho(a,q,t) \ . \tag{3.18}$$

4 Large N duality for torus knots

The geometric transition implies that a generating function of Chern-Simons invariants with a knot at large N is equal to a topological string amplitude with M5'-branes associated to the knot in the resolved conifold

$$\lim_{\substack{N\to\infty \\ q^N=a}} Z_{\mathrm{def},\mathrm{U}(N)}(q) = Z_{\mathrm{res}}(a,q) \ .$$

The work of LMOV not only determines the form of low-energy effective actions of Type IIA string theory with D4-branes on the resolved conifold but also provides its connection to Chern-Simons invariants of a knot at large N. In other words, colored HOMFLY-PT polynomials are related to LMOV invariants. In this section, we shall put forth the large N duality for torus knots in the refined context.

On the deformed conifold side, we have reviewed generating functions of refined Chern-Simons invariants in §2 . At large N, we substitute the stable limit $\overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)$ [AS15, GN15] for SU(N) invariants $\overline{\mathrm{rCS}}_{\mathrm{SU}(N),\lambda}(T_{m,n};q,t)$ in (2.1) and (2.2). Then, using the forms of the refined free energy on the resolved conifold determined in §3.2, the equivalences of the partition functions for both the q-brane and $\bar{t}$-brane setting can be recapitulated as

$$\sum_\lambda \overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)\, g_\lambda(q,t)P_\lambda(x;q,t) = \exp\left(\sum_{d=1}^\infty \sum_\mu \frac{1}{d}\frac{f_\mu^q(T_{m,n};a^d,q^d,t^d)}{q^{\frac{d}{2}}-q^{-\frac{d}{2}}}s_\mu(x^d)\right) \ , \tag{4.1}$$

$$\sum_\lambda \overline{\mathrm{rCS}}_\lambda(T_{m,n};a,q,t)\, P_{\lambda^T}(-x;t,q) = \exp\left(\sum_{d=1}^\infty \sum_\mu \frac{1}{d}\frac{f_\mu^{\bar{t}}(T_{m,n};a^d,q^d,t^d)}{t^{-\frac{d}{2}}-t^{\frac{d}{2}}}s_\mu(x^d)\right) . \tag{4.2}$$

These identities determine the refined indices f_μ^q, $f_\mu^{\bar{t}}$ and $\widehat{f}_\mu$ on the resolved conifold in terms of refined Chern-Simons invariants $\overline{\mathrm{rCS}}_\lambda$. Therefore, we call f_μ^q, $f_\mu^{\bar{t}}$ and $\widehat{f}_\mu$ refined reformulated invariants. We shall present general formulas in Appendix B.

In the case of the unknot $K = \bigcirc$, the formulas above become the Cauchy formulas (A.2) so that only two BPS numbers are non-vanishing as in the unrefined case, *i.e.* the refined reformulated invariants $f_\lambda^q(\bigcirc)$, $f_\lambda^{\bar{t}}(\bigcirc)$ colored by non-trivial representations ($\lambda \neq \square$) of the unknot vanish. Geometric picture has been drawn in [OV00, Figure 3] where $\mathcal{L}$ is $S^1 \times \mathbb{R}^2$ with S^1 the equator of $\mathbb{CP}^1$ in the resolved conifold and the two BPS states correspond to the M2-branes covering the upper and lower hemisphere of $\mathbb{CP}^1$.

The refined reformulated invariants can be explicitly evaluated by using refined Chern-Simons invariants of torus knots obtained in [AS15, DBMM$^+$13, Che13, FGSA12, Sha13]. In all the examples we have checked, the refined reformulated invariants $f_\mu^q(T_{m,n})$ and $f_\mu^{\bar{t}}(T_{m,n})$ obey the relation (3.18). Moreover, after making change of variables

$$a = -\mathbf{a}^2\mathbf{t} \ , \quad q^{\frac{1}{2}} = -\mathbf{q}\mathbf{t} \ , \quad t^{\frac{1}{2}} = \mathbf{q} \ , \tag{4.3}$$

$\widehat{f}_\rho(T_{m,n})$ can be written in the form

$$\widehat{f}_\rho(T_{m,n}) = \sum_{g \geq 0} \sum_{\beta, F \in \mathbb{Z}} \widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n})(\mathbf{q}\mathbf{t} - \mathbf{q}^{-1}\mathbf{t}^{-1})^g(\mathbf{q} - \mathbf{q}^{-1})^g \mathbf{a}^{2\beta}\mathbf{t}^F \ . \qquad (4.4)$$

Surprisingly, we observe that the numbers $\widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n})$ are always *non-negative* integers for any ρ, g, β, F. Let us emphasize that this is not obvious. Even if we assume that $\widehat{f}_\rho(T_{m,n})$ takes the form (3.17), we have

$$\widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n}) = \sum_{2(J_r-S_R)=F} (-1)^{2S_R}\widehat{N}_{\rho,g,\beta,J_r,S_R}(T_{m,n}) \ ,$$

which could be negative. Since $\widehat{N}_{\rho,g,\beta,J_r,S_R}(T_{m,n})$ are non-negative integrals by definition (3.12), the positivity of $\widehat{\mathbf{N}}_{\rho,g,\beta,F}(T_{m,n})$ strongly suggests that the extra $U(1)_R$ global symmetry S_R acts trivially on the BPS states $\mathbf{S}_\rho(H^*(\mathcal{M}_{g,h,\beta}))$ so that F is indeed equal to $2J_r$. The same phenomena has been found for refinement of analytically continued WRT invariants of Lens spaces $L(p,1)$ defined by the 3d/3d correspondence [GPV16, GPPV17].[3]

Now let us formulate the conjecture of refined large N duality for torus knots, which is the main claim of this paper.

The extra $U(1)_R$ global symmetry S_R acts trivially on the BPS states $\mathbf{S}_\mu(H^*(\mathcal{M}_{g,h,\beta}))$ in the resolved conifold. Hence, the refined reformulated invariants $f_\mu^q(T_{m,n})$ and $f_\mu^{\bar{t}}(T_{m,n})$, expressed in terms of refined Chern-Simons invariants of a torus knot $T_{m,n}$ via the geometric transition (4.1) and (4.2) (or more explicitly (B.1) and (B.2)), can be written

$$f_\mu^q(T_{m,n}; a, q, t) = \sum_\rho M_{\mu\rho}(t)\widehat{f}_\rho(T_{m,n}; a, q, t) \ ,$$
$$f_\mu^{\bar{t}}(T_{m,n}; a, q, t) = \sum_\rho M_{\mu\rho}(q^{-1})\widehat{f}_\rho(T_{m,n}; a, q, t) \ , \qquad (4.5)$$

where $\widehat{f}_\rho(T_{m,n})$ takes the form

$$\widehat{f}_\rho(T_{m,n}; a, q, t) = \sum_{\text{charges}} (-1)^{2J_r}\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g(t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g \left(\frac{q}{t}\right)^{J_r - \frac{\beta}{2}} a^\beta \ , \qquad (4.6)$$

with *non-negative* integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$. Furthermore, for ρ, g, β fixed, the $2J_r$ charges of non-zero (hence positive) integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ are either all even or all odd so that no cancellation occurs in the unrefined limit (3.14) and therefore the LMOV invariant is

$$\widehat{N}_{\rho,g,\beta}(T_{m,n}) = \pm \sum_{J_r \in \frac{1}{2}\mathbb{Z}} \widehat{N}_{\rho,g,\beta,J_r}(T_{m,n}) \ . \qquad (4.7)$$

The conjecture on the trivial action of S_R implies that the numbers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ indeed yield complete information about BPS degeneracies in M-theory on the resolved conifold with

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the M5'-branes associated to a torus knot $T_{m,n}$. From geometric point of view, they are graded dimensions of the cohomology groups $\mathbf{S}_\mu(H^*(\mathcal{M}_{g,h,\beta}))$ of the moduli space of M2-M5' bound states. In Appendix C, we present some examples of $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ for the trefoil $T_{2,3}$ and the $T_{2,5}$ knot. In addition, a `Mathematica` file attached to arXiv page contains more information about reformulated invariants. It is easy to see that, for ρ, g, β fixed, $2J_r$ charges of non-trivial $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ are either all even or all odd. In addition, the property (4.7) is manifest if we compare them with tables given in [LMnV00].

It is known that refined Chern-Simons invariants $\overline{\mathrm{rCS}}_\lambda$ of a torus knot colored by non-rectangular Young diagrams generally contain both positive and negative coefficients even after the change of variables (4.3). (For instance, see [Che13, §3.4].) However, this formulation lends itself to the natural interpretation of refined Chern-Simons invariants as a generating function of BPS states, providing non-negative integers $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ for any color ρ. Thus, this can be interpreted as *positivity conjecture* of refined Chern-Simons invariants of a torus knot.

Furthermore, we notice several interesting features of refined reformulated invariants. First, instead of taking the genus expansion of M2-branes (4.6), we find that the naive change of variables (4.3) for $\widehat{f}_\mu(T_{m,n})$ always yields a Laurent polynomial with non-negative integral coefficients

$$\widehat{f}_\mu(T_{m,n}; a = -\mathbf{a}^2\mathbf{t}, q = \mathbf{q}^2\mathbf{t}^2, t = \mathbf{q}^2) = \pm\mathbf{a}^\bullet\mathbf{q}^\bullet\mathbf{t}^\bullet\sum_{i,j,k}\widehat{\mathbf{N}}_{\mu;i,j,k}(T_{m,n})\mathbf{a}^{2i}\mathbf{q}^{2j}\mathbf{t}^k \ ,$$

with $\widehat{\mathbf{N}}_{\mu;i,j,k} \in \mathbb{Z}_{\geq 0}$. Therefore, this evidence also indicates that there exists underlying cohomology groups of some moduli spaces for $\widehat{\mathbf{N}}_{\mu;i,j,k}$. Below some examples are given:

$$\widehat{f}_\square(T_{2,3}) = \frac{\mathbf{a}}{\mathbf{q}^2}(\mathbf{a}^2\mathbf{t} + 1)(\mathbf{a}^2\mathbf{q}^2\mathbf{t}^3 + \mathbf{q}^4\mathbf{t}^2 + 1)$$

$$\widehat{f}_{\square\square}(T_{2,3}) = \frac{\mathbf{a}^2}{\mathbf{q}^4}\left(\mathbf{a}^2\mathbf{t} + 1\right)\left(\mathbf{a}^2\mathbf{t}^3 + 1\right)\left(\mathbf{q}^4\mathbf{t}^2 + 1\right)\left(\mathbf{a}^2\mathbf{t} + \mathbf{q}^2\right)\left(\mathbf{a}^2\mathbf{q}^2\mathbf{t}^3 + 1\right)$$

$$\widehat{f}_{\boxminus}(T_{2,3})$$
$$= \frac{\mathbf{a}^2}{\mathbf{q}^6}\left(\mathbf{a}^2\mathbf{t} + 1\right)\left(\mathbf{a}^2\mathbf{t} + \mathbf{q}^2\right)\left(\mathbf{a}^2\mathbf{q}^2\mathbf{t}^3 + 1\right)\left(\mathbf{q}^8\left(\mathbf{a}^2\mathbf{t}^7 + \mathbf{t}^4\right) + \mathbf{q}^4\mathbf{t}^2\left(\mathbf{a}^2\left(\mathbf{t}^3 + \mathbf{t}\right) + 2\right) + \mathbf{a}^2\mathbf{t}^3 + 1\right) \ .$$

Second, we also observe a positivity property when we make the same substitution (4.3) for $f^q_{[r]}(T_{m,n})$ colored by symmetric representations $\lambda = [r]$ as well as $f^{\bar{t}}_{[1^r]}(T_{m,n})$ colored by anti-symmetric representations $\lambda = [1^r]$:

$$f^q_{[r]}(T_{m,n}; -\mathbf{a}^2\mathbf{t}, \mathbf{q}^2\mathbf{t}^2, \mathbf{q}^2) = \pm\mathbf{a}^\bullet\mathbf{q}^\bullet\mathbf{t}^\bullet\sum_{i,j,k}\mathbf{N}^q_{[r];i,j,k}(T_{m,n})\mathbf{a}^{2i}\mathbf{q}^{2j}\mathbf{t}^k \ ,$$

$$f^{\bar{t}}_{[1^r]}(T_{m,n}; -\mathbf{a}^2\mathbf{t}, \mathbf{q}^2\mathbf{t}^2, \mathbf{q}^2) = \pm\mathbf{a}^\bullet\mathbf{q}^\bullet\mathbf{t}^\bullet\sum_{i,j,k}\mathbf{N}^{\bar{t}}_{[1^r];i,j,k}(T_{m,n})\mathbf{a}^{2i}\mathbf{q}^{2j}\mathbf{t}^k \ ,$$

where $\mathbf{N}^q_{[r];i,j,k}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$ and $\mathbf{N}^{\bar{t}}_{[1^r];i,j,k}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$. For instance, we have

$$f^q_{\square\square}(T_{2,3}) = \frac{\mathbf{a}^2\mathbf{t}^2}{\mathbf{q}}(\mathbf{a}^2\mathbf{t} + 1)(\mathbf{q}^8(\mathbf{a}^4\mathbf{t}^8 + \mathbf{a}^2\mathbf{t}^5) + \mathbf{q}^6(\mathbf{a}^6\mathbf{t}^9 + \mathbf{a}^4\mathbf{t}^6 + \mathbf{a}^2\mathbf{t}^5 + \mathbf{t}^2) + \mathbf{a}^2\mathbf{q}^4\mathbf{t}^3(\mathbf{a}^2(\mathbf{t}^3 + \mathbf{t}) + 2) + \mathbf{q}^2(\mathbf{a}^6\mathbf{t}^5 + \mathbf{a}^4\mathbf{t}^4 + \mathbf{a}^2\mathbf{t} + 1) + \mathbf{a}^2\mathbf{t}(\mathbf{a}^2\mathbf{t} + 1)) \ ,$$

$$f_{\square}^{\bar{t}}(T_{2,3}) = \frac{\mathbf{a}^2}{\mathbf{q}^7\mathbf{t}}(\mathbf{a}^2\mathbf{t}+1)(\mathbf{a}^2\mathbf{q}^8\mathbf{t}^5(\mathbf{a}^2\mathbf{t}+1)+\mathbf{q}^6\mathbf{t}^2(\mathbf{a}^6\mathbf{t}^5+\mathbf{a}^4\mathbf{t}^4+\mathbf{a}^2\mathbf{t}+1)+\mathbf{a}^2\mathbf{q}^4\mathbf{t}^3(\mathbf{a}^2(\mathbf{t}^3+\mathbf{t})+$$
$$2)+\mathbf{q}^2(\mathbf{a}^6\mathbf{t}^7+\mathbf{a}^4\mathbf{t}^4+\mathbf{a}^2\mathbf{t}^3+1)+\mathbf{a}^4\mathbf{t}^4+\mathbf{a}^2\mathbf{t}) \ .$$

Let us conclude this section by mentioning the implication of the symmetry (3.16) for the free energy in the resolved conifold. As the right hand sides of (4.1) and (4.2) are interchanged by $(q,t) \leftrightarrow (t^{-1}, q^{-1})$, so are the left hand sides. Using the property of Macdonald functions $P_\lambda(-x; t, q) = (-1)^{|\lambda|} P_\lambda(x; t^{-1}, q^{-1})$, this implies

$$g_\lambda(q,t)\ \overline{\mathrm{rCS}}_\lambda(T_{m,n}; a, q, t) = (-1)^{|\lambda|}\ \overline{\mathrm{rCS}}_{\lambda^T}(T_{m,n}; a, t^{-1}, q^{-1}) \ ,$$

which is called *mirror/transposition symmetry* of refined Chern-Simons invariants. This explanation is first presented in [AS12c, §3.1].

5 Large N duality for torus links

In this section, we generalize refined large N duality to torus links with L components. For each component of a torus link, we introduce M_i M5'-branes supported on the conormal bundle $\mathcal{L}_i$ of the component in the deformed conifold T^*S^3. These M5'-branes still remain after the geometric transition. Since a half of supersymmetry is preserved if the supports of M5'-branes are special Lagrangian submanifolds in a Calabi-Yau as explained in §3.2, it is straightforward to extend the analysis in the previous sections to torus links. To avoid repetitive explanation, we shall present only essential results in this section.

For a torus link $T_{m,n}$, we need to introduce the fugacities x_i ($i = 1, \cdots, L$) that parametrize the Cartan subgroup of the $\mathrm{U}(M_i)$-valued moduli of $\mathcal{L}_i$ both on the deformed conifold and on the resolved conifold. In the resolved conifold, we consider a holomorphic curve $\Sigma_{g,h}$ with $h = \sum_{i=1}^L h_i$ boundaries where h_i boundaries end on $\mathcal{L}_i$. Therefore, we have to project the space $H^*((S^1)^{h-1}) \otimes H^*(\mathcal{M}_{g,h,\beta})$ on the invariant subspace of the relevant symmetry $\mathfrak{S}_{h_1} \times \cdots \times \mathfrak{S}_{h_L}$. Eventually, the BPS spectrum for a torus link analogous to (3.9) is

$$\mathrm{Inv}\Big(H^*((S^1)^{h-1}) \otimes H^*(\mathcal{M}_{g,h,\beta})\Big)_{\mathcal{H}(\beta,\mu_1,\cdots,\mu_L)} =$$
$$\bigoplus_{\{\mu_i'\}\{\mu_i''\}} C_{\mu_1\,\mu_1'\mu_1''} \cdots C_{\mu_L\,\mu_L'\mu_L''} \mathbf{S}_{\mu_1',\cdots,\mu_L'}(H^*((S^1)^{h-1})) \otimes \mathbf{S}_{\mu_1'',\cdots,\mu_L''}(H^*(\mathcal{M}_{g,h,\beta})) \ .$$

For the q-branes, the fermion zero modes $\psi_1^{(i)}, \cdots, \psi_{h_i}^{(i)}$ coming from the boundaries ending on $\mathcal{L}_i$ contribute to the refined index by $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})B_{\mu_i'}(t)$. The factor $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, which is absent in the case of a torus knot, stems from the fact that we do not impose the linear constraint $\sum_{j=1}^{h_i} \psi_j^{(i)} = 0$ on each boundary. Therefore, we have

$$\mathrm{Tr}_{\mathbf{S}_{\mu_1',\cdots,\mu_L'}(H^*((S^1)^{h-1}))} (-1)^F q^{J_1-S_R} t^{S_R-J_2} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{L-1} \prod_{i=1}^{L} B_{\mu_i'}(t) \tag{5.1}$$

where the linear constraint $\sum_{i=1}^L \sum_{j=1}^{h_i} \psi_j^{(i)} = 0$ on the total boundary fermion zero modes yields the factor $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{-1}$.

As in the case of torus knots, we conjecture that *the extra* $\mathrm{U}(1)_R$ *global symmetry* S_R *acts trivially on the BPS states* $\mathbf{S}_{\mu_1'', \cdots, \mu_L''}(H^*(\mathcal{M}_{g,h,\beta}))$. Therefore, we decompose the BPS states

$$\mathbf{S}_{\mu_1'', \cdots, \mu_L''}(H^*(\mathcal{M}_{g,h,\beta})) \cong \bigoplus_{J_r} A_{\rho_1, \cdots, \rho_L, g, \beta, J_r}$$

with respect to only J_r charges but not S_R charges and define

$$\widehat{N}_{\rho_1, \cdots, \rho_L, g, \beta, J_r} := \dim A_{\rho_1, \cdots, \rho_L, g, \beta, J_r} \ .$$

The free energy of the $\bar{t}$-branes can be obtained from that of the q-branes by exchanging $(q, t) \leftrightarrow (t^{-1}, q^{-1})$. Thus, we can write them in the forms

$$F_{\mathrm{ref}}^q = \sum_{d>0} \sum_{\{\mu_i\}} \frac{1}{d} \frac{(t^{\frac{d}{2}} - t^{-\frac{d}{2}})^{L-1}}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} f_{\mu_1, \cdots, \mu_L}^q(T_{m,n}; a^d, q^d, t^d) \prod_{i=1}^L s_{\mu_i}(x_i^d) \ ,$$

$$F_{\mathrm{ref}}^{\bar{t}} = \sum_{d>0} \sum_{\{\mu_i\}} \frac{1}{d} \frac{(q^{-\frac{d}{2}} - q^{\frac{d}{2}})^{L-1}}{t^{-\frac{d}{2}} - t^{\frac{d}{2}}} f_{\mu_1, \cdots, \mu_L}^{\bar{t}}(T_{m,n}; a^d, q^d, t^d) \prod_{i=1}^L s_{\mu_i}(x_i^d) \ ,$$

where the refined reformulated invariants have the forms

$$f_{\mu_1, \cdots, \mu_L}^q(T_{m,n}; a, q, t) = \sum_{\rho_1, \cdots, \rho_L} M_{\mu_1 \rho_1}(t) \cdots M_{\mu_L \rho_L}(t) \widehat{f}_{\rho_1, \cdots, \rho_L}(T_{m,n}; a, q, t) \ ,$$

$$f_{\mu_1, \cdots, \mu_L}^{\bar{t}}(T_{m,n}; a, q, t) = \sum_{\rho_1, \cdots, \rho_L} M_{\mu_1 \rho_1}(q^{-1}) \cdots M_{\mu_L \rho_L}(q^{-1}) \widehat{f}_{\rho_1, \cdots, \rho_L}(T_{m,n}; a, q, t) \quad (5.2)$$

and $\widehat{f}_{\rho_1, \cdots, \rho_L}(T_{m,n})$ are of the form

$$\widehat{f}_{\rho_1, \cdots, \rho_L}(T_{m,n}; a, q, t) = \sum_{\mathrm{charges}} (-1)^{2J_r} \widehat{N}_{\rho_1, \cdots, \rho_L, g, \beta, J_r}(T_{m,n}) (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^g (t^{-\frac{1}{2}} - t^{\frac{1}{2}})^g \left(\frac{q}{t}\right)^{J_r - \frac{\beta}{2}} a^\beta$$

$$(5.3)$$

with *non-negative* integers $\widehat{N}_{\rho_1, \cdots, \rho_L, g, \beta, J_r}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$. Here we factor out $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{L-1}$ in (5.1) from the definition of refined reformulated invariants since it depends only on the number L of link components.

As a result, the large N duality of refined Chern-Simons theory with a torus link $T_{m,n}$ with L components can be summarized as

$$\sum_{\lambda_i} \overline{\mathrm{rCS}}_{\lambda_1, \cdots, \lambda_L}(T_{m,n}; a, q, t) \prod_{i=1}^L g_{\lambda_i}(q, t) P_{\lambda_i}(x_i; q, t) =$$

$$\exp\left(\sum_{d>0} \sum_{\{\mu_i\}} \frac{1}{d} \frac{(t^{\frac{d}{2}} - t^{-\frac{d}{2}})^{L-1}}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}} f_{\mu_1, \cdots, \mu_L}^q(T_{m,n}; a^d, q^d, t^d) \prod_{i=1}^L s_{\mu_i}(x_i^d)\right) \ , \quad (5.4)$$

$$\sum_{\lambda_i} \overline{\mathrm{rCS}}_{\lambda_1, \cdots, \lambda_L}(T_{m,n}; a, q, t) \prod_{i=1}^L P_{\lambda_i^T}(-x_i; t, q) =$$

$$\exp\left(\sum_{d>0} \sum_{\{\mu_i\}} \frac{1}{d} \frac{(q^{-\frac{d}{2}} - q^{\frac{d}{2}})^{L-1}}{t^{-\frac{d}{2}} - t^{\frac{d}{2}}} f_{\mu_1, \cdots, \mu_L}^{\bar{t}}(T_{m,n}; a^d, q^d, t^d) \prod_{i=1}^L s_{\mu_i}(x_i^d)\right) \ . \quad (5.5)$$

where the reformulated invariants are of the form (5.2) with (5.3). These identities enable us to express the reformulated invariants in terms of refined Chern-Simons invariants of a torus link, which are presented in (B.1) and (B.2). Thus, the large N duality provides a rather non-trivial connection of refined Chern-Simons invariants $\overline{\mathrm{rCS}}_{\lambda_1,\cdots,\lambda_L}(T_{m,n}; a, q, t)$ of a torus link to enumerative invariants $\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}(T_{m,n}) \in \mathbb{Z}_{\geq 0}$ in the resolved conifold.

In Appendix C, we present some examples of $\widehat{N}_{\rho_1,\rho_2,g,\beta,J_r}$ for the Hopf link $T_{2,2}$ and the $T_{2,4}$ link by using the results in [DBMM+13, GNS+16]. As in the case of torus knots, it is easy to see that, for ρ_1, ρ_2, g, β fixed, $2J_r$ charges of non-trivial $\widehat{N}_{\rho_1,\rho_2,g,\beta,J_r}$ are either all even or all odd. Therefore, we conjecture that this is true for any torus link with L components so that the LMOV invariant is

$$\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta}(T_{m,n}) = \pm \sum_{J_r \in \frac{1}{2}\mathbb{Z}} \widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}(T_{m,n}) \ .$$

6   Discussion

In this paper, we have formulated large N duality of refined Chern-Simons theory with a torus link. Assuming that the extra $\mathrm{U}(1)_R$ global symmetry acts trivially on the BPS states coming from holomorphic deformations of M2-branes, this formulation gives a remarkable relation between refined Chern-Simons invariants of a torus link and graded dimensions of cohomology groups of moduli spaces of M2-M5 bound states in the resolved conifold. Therefore, this leads to *positivity conjecture* of refined Chern-Simons invariants of a torus link. Conversely, one can obtain complete information about BPS spectra in M-theory on the resolved conifold with M5'-branes supported on $\mathbb{R}^3 \times \mathcal{L}_{T_{m,n}}$ by using the geometric transition. It is also worth mentioning that, for M-theory on any non-compact toric Calabi-Yau three-fold with M5-branes, its free energy on the Ω-background takes the forms (3.13) and (3.15) if the extra $\mathrm{U}(1)_R$ global symmetry is preserved. It is important to understand in which situations the extra $\mathrm{U}(1)_R$ global symmetry acts on the space of BPS states trivially as in this paper.

It would be interesting to categorify large N duality for non-torus knots. Recently, conjectural formulas for Poincaré polynomials of colored HOMFLY-PT homology have been obtained for a certain class of non-torus knots. (For instance, [GNS+16, references therein].) However, the formulas for non-torus knots are expressed in terms of the variables $(\mathbf{a}, \mathbf{q}, \mathbf{t})$ instead of the variables (a, q, t) and the naive change of variables (4.3) does not work for non-torus knots. This is the reason why we cannot simply extend the formulation in this paper to the case of non-torus knots.

Nevertheless, we can make observation about BPS spectra in the resolved conifold for a non-torus knot. As proposed in [GSV05], in the fundamental representation, a reformulated invariant is equivalent to the Poincaré polynomial of *unreduced* HOMFLY-PT homology for any knot even at the categorified level:

$$\frac{\widehat{f}_\square(K; \mathbf{a}, \mathbf{q}, \mathbf{t})}{\mathbf{q} - \mathbf{q}^{-1}} = \overline{\mathcal{P}}_\square(K; \mathbf{a}, \mathbf{q}, \mathbf{t}) \ .$$

For instance, the Poincaré polynomial of unreduced HOMFLY-PT homology of the simplest non-torus knot, a figure-eight, is

$$\overline{\mathcal{P}}_\square(\mathbf{4_1}; \mathbf{a}, \mathbf{q}, \mathbf{t}) = \frac{\left(\mathbf{a}^2\mathbf{t} + 1\right)\left(\mathbf{a}^2\mathbf{q}^4\mathbf{t}^3 + \mathbf{q}^2\left(\mathbf{a}^4\mathbf{t}^4 + \mathbf{a}^2\mathbf{t}^2 + 1\right) + \mathbf{a}^2\mathbf{t}\right)}{\mathbf{a}^3\mathbf{q}^2\mathbf{t}^3(\mathbf{q} - \mathbf{q}^{-1})} \ .$$

This formula can be brought into the form (4.4) and the resulting BPS degeneracies can be tabulated as follows:

β	-3/2	-1/2			1/2			3/2
$g\backslash F$	-2	-1	0	1	0	1	2	3
0	1	2	1	1	1	1	2	1
1	0	0	1	0	0	1	0	0

Table 2. $\widehat{\mathbf{N}}_{[1],g,\beta,F}$ for the figure-eight knot

At the de-categorification limit $\mathbf{t} = -1$, it is easy to see that there are (boson-fermion) cancellations by sign although they reduce to the corresponding LMOV invariants [LMnV00, Figure 7].

It was observed in [LMnV00, LMn02] that, for any knot K, the LMOV invariants $\widehat{N}_{\mu,g,\beta}(K)$ have the same parity of their a-gradings β, *i.e.* $(-1)^\beta\widehat{N}_{\mu,g,\beta}$ become all non-negative integers for any μ, g, β up to appropriate grading shifts. However, as we see in the example of the figure-eight, there are cancellations behind for non-torus knots, which can be seen only at the categorical level. Presumably, one of the reasons why the formulation of large N duality for non-torus knots at the categorical level is not straightforward originates from the fact that the property of underlying BPS states is different from that (4.7) for torus knots.

Giving a mathematical definition of refined LMOV invariants $\widehat{N}_{\rho,g,\beta,J_r}(T_{m,n})$ discussed in this paper is a challenging, but important open problem. Refined GV invariants have been discussed in the literature [CKK14, CDDP15, NO14, GHKPK17, references therein] as refined Pandharipande-Thomas (Donaldson-Thomas) invariants for non-compact toric Calabi-Yau threefolds. However, mathematical understanding of their open analogues are still immature although the Poincaré polynomials of uncolored HOMFLY-PT homology of torus knots have been related to motivic Donaldson-Thomas invariants in [DHS12]. Actually, upon the reduction on the cigar of the Taub-NUT in (1.3), BPS states can be understood as D6-D4-D2-D0 bound states. Therefore, it is an important task to give a mathematical definition of D6-D4-D2-D0 bound states for refine Chern-Simons invariants discussed in this paper. It would be also intriguing to find a connection to (a certain variant of) "P=W conjecture" [CDDP15, references therein].

Another direction to pursue is to find large N duality of refined Chern-Simons theory with different gauge groups. In fact, $SO(2N)$ refined Chern-Simons theory has been proposed [AS12a], generalizing Kauffman polynomials. In addition, large N duality for Kauffman polynomials has been put forward by incorporating orientifolds in the resolved conifold [Mn10]. Consequently, this leads to an integrality conjecture involving both colored Kauffman and

HOMFLY-PT polynomials. It is natural to ask whether a positivity property can be seen when the conjecture of [Mn10] is refined.

Note added: After the completion of the paper, we were informed of an independent work [DDP] on the same topic. Since their approach is different from ours, it would be great to find a connection to their work.

Acknowledgements

A Symmetric functions

In this appendix, we review basics of symmetric functions relevant to this paper. For more detail, we refer the reader to [Mac98].

Let $x = (x_1, x_2, \cdots)$ be an infinite number of the variables, $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a Young diagram (i.e. non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ and $|\lambda| = \sum_i \lambda_i < \infty$) and $\vec{k} = (k_1, k_2, \cdots)$ be a vector with a infinite number of entries, almost zero, and whose nonzero entries are positive integers. The Young diagram λ and the vector $\vec{k}$ are in one-to-one correspondence with the relation $k_i = m_i(\lambda)$, where $m_i(\lambda)$ is a multiplicity of i in λ.

Figure 1. Arm, leg, co-arm and co-leg

First, we define the power-sum symmetric functions by $p_d(x) = \sum_{i=1}^{\infty} x_i^d$. It is convenient to denote their products by $p_\lambda(x) = \prod_i p_{\lambda_i}(x)$ and $p_{\vec{k}} = \prod_i p_i^{k_i}$. These are bases of the ring of symmetric functions. Schur and Macdonald functions can be defined by introducing an inner product on the ring of symmetric functions.

Schur functions

The Schur functions $s_\lambda(x)$ are uniquely defined by orthogonality and normalization conditions:

$$\langle s_\lambda, s_\mu \rangle = 0 , \qquad\qquad \text{if} \ \ \lambda \neq \mu ,$$
$$s_\lambda(x) = w_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu} w_\mu(x) , \qquad\qquad u_{\lambda\mu} \in \mathbb{Q} ,$$

where $w_\lambda(x)$ is the monomial symmetric function, $<$ is dominance partial ordering ($\lambda \geq \mu \Leftrightarrow$ $|\lambda| = |\mu|$ and $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$ for all i), and the inner product is defined by

$$\langle p_\lambda, p_\lambda \rangle = \delta_{\lambda\mu} z_\lambda , \qquad z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! ,$$

where $m_i = m_i(\lambda)$ is a multiplicity of i in λ.

The relation between Schur and power sum symmetric functions is known as Frobenius formula:

$$s_\lambda(x) = \sum_{\vec{k}} \frac{\chi_\lambda(C(\vec{k}))}{z_{\vec{k}}} p_{\vec{k}}(x) , \qquad\qquad p_{\vec{k}}(x) = \sum_\lambda \chi_\lambda(C(\vec{k})) s_\lambda(x) , \qquad (A.1)$$

where $\chi_\lambda(C(\vec{k}))$ is the character of the representation λ of the permutation group $\mathfrak{S}_h$ evaluated at the conjugacy class $C(\vec{k})$ where $h = \sum_j j k_j$. The Frobenius formula is used for the computation of Clebsch-Gordon coefficients (3.10) of the permutation group $\mathfrak{S}_h$.

Macdonald functions

The Macdonald functions $P_\lambda(x; q, t)$ are uniquely defined by orthogonality and normalization conditions:

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 , \qquad\qquad \text{if} \ \ \lambda \neq \mu,$$
$$P_\lambda(x; q, t) = w_\lambda(x) + \sum_{\mu < \lambda} u_{\lambda\mu}(q, t) w_\mu(x) , \qquad u_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t) ,$$

where the inner product is defined by

$$\langle p_\lambda, p_\lambda \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i \geq 1} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} , \qquad z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! .$$

At the $q = t$ specialization, the Macdonald functions reduce to the Schur functions. From the definition one can show

$$\frac{(q/t)^{|\lambda|}}{g_\lambda(q, t)} := \langle P_\lambda, P_\lambda \rangle_{q,t} = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1} t^{l(s)}}{1 - q^{a(s)} t^{l(s)+1}} ,$$

where an arm length $a(s) = \lambda_i - j$ and a leg length $l(s) = \lambda_j^T - i$ for each box $s = (i, j)$ in λ are depicted in Figure 1.

We denote by $\mathfrak{X}_\lambda(\vec{k}; q, t)$ coefficients in the expansion of a Macdonald function $P_\lambda(x; q, t)$ with respect to $p_{\vec{k}}(x)$:

$$P_\lambda(x; q, t) = \sum_{\vec{k}} \mathfrak{X}_\lambda(\vec{k}; q, t) p_{\vec{k}}(x) .$$

Note that at the $q = t$ specialization

$$\mathfrak{X}_\lambda(\vec{k}; q, q) = \frac{\chi_\lambda(C(\vec{k}))}{z_{\vec{k}}} .$$

Let us express some Macdonald functions in terms of the power-sum functions:

$$P_{\square} = p_1$$
$$P_{\square\square} = \frac{(1-t)(1+q)}{(1-tq)}\frac{p_1^2}{2} + \frac{(1+t)(1-q)}{(1-tq)}\frac{p_2}{2},$$
$$P_{\boxminus} = \frac{p_1^2}{2} - \frac{p_2}{2}$$
$$P_{\square\square\square} = \frac{(1+q)(1-q^3)(1-t)^2}{(1-q)(1-tq)(1-tq^2)}\frac{p_1^3}{6} + \frac{(1-q)(1-t^2)(1-q^3)}{(1-q)(1-tq)(1-tq^2)}\frac{p_1 p_2}{2}$$
$$\qquad + \frac{(1-q)(1-q^2)(1-t^3)}{(1-t)(1-tq)(1-tq^2)}\frac{p_3}{3}$$
$$P_{\boxplus} = \frac{(1-t)(2qt+q+t+2)}{1-qt^2}\frac{p_1^3}{6} + \frac{(1+t)(t-q)}{1-qt^2}\frac{p_1 p_2}{2} - \frac{(1-q)(1-t^3)}{(1-t)(1-qt^2)}\frac{p_3}{3}$$
$$P_{\boxminus} = \frac{p_1^3}{6} - \frac{p_2 p_1}{2} + \frac{p_3}{3}.$$

For instance, we have $\mathfrak{X}_{\boxplus}(\vec{k} = (3,0,0); q, t) = \frac{(1-t)(2qt+q+t+2)}{6(1-qt^2)}$ because of $p_{\vec{k}=(3,0,0)} = p_1^3$.

Cauchy formulas

The Cauchy formulas play a very important role in this paper. The Cauchy formulas for Schur functions read off:

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \exp\left(\sum_{d>0}\frac{1}{d}p_d(x)p_d(y)\right) , \quad \sum_\lambda s_\lambda(x) s_{\lambda^T}(y) = \exp\left(\sum_{d>0}\frac{(-1)^{d-1}}{d}p_d(x)p_d(y)\right) .$$

The analogues of Macdonald functions are

$$\sum_\lambda g_\lambda(q,t) P_\lambda(x; q, t) P_\lambda(y; q, t) = \exp\left(\sum_{d>0}\frac{1}{d}\frac{t^{\frac{d}{2}} - t^{-\frac{d}{2}}}{q^{\frac{d}{2}} - q^{-\frac{d}{2}}}p_d(x)p_d(y)\right) ,$$

$$\sum_\lambda P_\lambda(x; q, t) P_{\lambda^T}(y; t, q) = \exp\left(\sum_{d>0}\frac{(-1)^{d-1}}{d}p_d(x)p_d(y)\right) . \qquad (A.2)$$

B Explicit formulas of refined reformulated invariants

In this appendix, we will derive explicit formulas for refined reformulated invariants of a torus link $T_{m,n}$ with L components in terms of its refined Chern-Simons invariants from (5.4) and (5.5) by following [LMn02]. To this end, we define the plethystic exponential and its inverse

$$\mathrm{Exp}(F) = \exp\left(\sum_{d=1}^\infty \frac{\psi_d}{d}\right) \circ F , \qquad \mathrm{Log}(F) = \sum_{d=1}^\infty \frac{\mu(d)}{d}\log(\psi_d \circ F) ,$$

where an operator ψ_d is defined by $\psi_d \circ F(a,q,t;x) := F(a^d, q^d, t^d; x^d)$ and $\mu(d)$ is the Möbius function. If one set

$$F := \frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{L-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{\{\mu_i\}} f^q_{\mu_1, \cdots, \mu_L}(T_{m,n}; a, q, t) \prod_{i=1}^{L} s_{\mu_i}(x_i) \ ,$$

then the right hand side of (5.4) can be written as $\mathrm{Exp}(F)$. Hence, one can manipulate the identity (5.4) as

$$F = \mathrm{Log} \left(\sum_{\{\lambda_i\}} \overline{\mathrm{rCS}}_{\lambda_1 \cdots \lambda_L}(T_{m,n}; a, q, t) \prod_{i=1}^{L} g_{\lambda_i}(q,t) P_{\lambda_i}(x_i; q, t) \right)$$

$$= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \log \left(\sum_{\{\lambda_i\}} \overline{\mathrm{rCS}}^{(d)}_{\lambda_1 \cdots \lambda_L} \prod_{i=1}^{L} g_{\lambda_i}(q^d, t^d) P_{\lambda_i}(x_i^d; q^d, t^d) \right)$$

$$= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \prod_{\alpha=1}^{m} \sum_{\{\lambda_i^{(\alpha)}\}} \overline{\mathrm{rCS}}^{(d)}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}} \prod_{i=1}^{L} g_{\lambda_i^{(\alpha)}}(q^d, t^d) P_{\lambda_i^{(\alpha)}}(x_i^d; q^d, t^d)$$

$$= \sum_{d=1}^{\infty} \frac{\mu(d)}{d} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \prod_{\alpha=1}^{m} \sum_{\{\lambda_i^{(\alpha)}\}} \overline{\mathrm{rCS}}^{(d)}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}} \prod_{i=1}^{L} g_{\lambda_i^{(\alpha)}}(q^d, t^d) \sum_{\vec{k}_i^{(\alpha)}} \mathfrak{X}_{\lambda_i^{(\alpha)}}(\vec{k}_i^{(\alpha)}; q^d, t^d) p_{\vec{k}_i^{(\alpha)}}(x_i^d) \ ,$$

where $\overline{\mathrm{rCS}}^{(d)}_{\lambda} = \overline{\mathrm{rCS}}_\lambda(T_{m,n}; a^d, q^d, t^d)$ and other notations are given in Appendix A. To compare with the coefficient of $\prod_{i=1}^{L} s_{\mu_i}(x_i)$, we introduce $\vec{k}_d$ for $\vec{k} = (k_1, k_2, \cdots)$ as $(\vec{k}_d)_{di} = (\vec{k})_i$, i.e.

$$\vec{k}_d = (0, \cdots, 0, k_1, 0, \cdots, 0, k_2, 0, \cdots) \ ,$$

where k_1 is d-th entry, k_2 is $2d$-th entry and so on. Then, the properties $p_{\vec{k}}(x^d) = p_{\vec{k}_d}(x)$ and $p_{\vec{k}}(x) p_{\vec{k}'}(x) = p_{\vec{k}+\vec{k}'}(x)$ of the power sum functions tell us

$$\prod_{\alpha=1}^{m} p_{\vec{k}_i^{(\alpha)}}(x_i^d) = p_{\sum_{\alpha=1}^{m}(\vec{k}_i^{(\alpha)})_d}(x_i) = \sum_{\{\mu_i\}} \chi_{\mu_i}(C(\sum_{\alpha=1}^{m}(\vec{k}_i^{(\alpha)})_d)) s_{\mu_i}(x_i) \ .$$

Using these results, we obtain the explicit formula for f^q in terms of refined Chern-Simons invariants:

$$\frac{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{L-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} f^q_{\mu_1, \cdots, \mu_L}(T_{m,n}; a, q, t) = \tag{B.1}$$

$$\sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{d \cdot m} \sum_{\{\vec{k}_i^{(\alpha)}\}} \sum_{\{\lambda_i^{(\alpha)}\}} \prod_{i=1}^{L} \chi_{\mu_i}(C(\sum_{\alpha=1}^{m}(\vec{k}_i^{(\alpha)})_d)) \prod_{\alpha=1}^{m} g_{\lambda_i^{(\alpha)}}(q^d, t^d) \mathfrak{X}_{\lambda_i^{(\alpha)}}(\vec{k}_i^{(\alpha)}; q^d, t^d) \overline{\mathrm{rCS}}^{(d)}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}} \ .$$

Similarly, we can obtain the explicit formula for $f^{\bar{t}}$:

$$\frac{(q^{-\frac{d}{2}} - q^{\frac{d}{2}})^{L-1}}{t^{-\frac{d}{2}} - t^{\frac{d}{2}}} f^{\bar{t}}_{\mu_1, \cdots, \mu_L}(T_{m,n}; a, q, t) = \tag{B.2}$$

$$\sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{d \cdot m} \sum_{\{\vec{k}_i^{(\alpha)}\}} \sum_{\{\lambda_i^{(\alpha)}\}} \prod_{i=1}^{L} \chi_{\mu_i^T}(C(\sum_{\alpha=1}^{m} (\vec{k}_i^{(\alpha)})_d)) \prod_{\alpha=1}^{m} (-1)^{|\lambda_i^{(\alpha)}|} \mathfrak{X}_{(\lambda_i^{(\alpha)})^T}(\vec{k}_i^{(\alpha)}; t^d, q^d) \overline{\mathrm{rCS}}_{\lambda_1^{(\alpha)} \cdots \lambda_L^{(\alpha)}}^{(d)} .$$

In the following, we provide reformulated invariants of a torus knot colored by Young diagrams with a few boxes for the q-branes from (B.1):

$$\frac{f_\square^q}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \overline{\mathrm{rCS}}_\square,$$

$$\sqrt{\frac{t}{q}} \frac{f_{\square\square}^q}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \frac{qt-1}{q^2-1} \overline{\mathrm{rCS}}_{\square\square} - \frac{t-1}{2(q-1)} (\mathrm{rCS}_\square)^2 - \frac{t+1}{2(q+1)} \overline{\mathrm{rCS}}_\square^{(2)} ,$$

$$\sqrt{\frac{t}{q}} \frac{f_\boxtimes^q}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \frac{t-q}{q^2-1} \overline{\mathrm{rCS}}_{\square\square} + \frac{t^2-1}{qt-1} \overline{\mathrm{rCS}}_\boxtimes - \frac{t-1}{2(q-1)} (\mathrm{rCS}_\square)^2 + \frac{t+1}{2(q+1)} \overline{\mathrm{rCS}}_\square^{(2)} ,$$

$$\frac{t}{q} \frac{f_{\square\square\square}^q}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \frac{(qt-1)\left(q^2 t-1\right)}{\left(q^2-1\right)\left(q^3-1\right)} \overline{\mathrm{rCS}}_{\square\square\square} - \frac{(t-1)(qt-1)}{(q-1)^2(q+1)} \overline{\mathrm{rCS}}_{\square\square} \overline{\mathrm{rCS}}_\square$$
$$+ \frac{(t-1)^2}{3(q-1)^2} (\overline{\mathrm{rCS}}_\square)^3 - \frac{t^2+t+1}{3\left(q^2+q+1\right)} \overline{\mathrm{rCS}}_\square^{(3)} ,$$

$$\frac{t}{q} \frac{f_\boxtimes^q}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = - \frac{(q-t)(qt-1)}{(q-1)(q^3-1)} \overline{\mathrm{rCS}}_{\square\square\square} + \frac{(t-1)\left(qt^2-1\right)}{(q-1)\left(q^2 t-1\right)} \overline{\mathrm{rCS}}_\boxtimes$$
$$- \left[\frac{(t-1)^2}{(q-1)^2} \overline{\mathrm{rCS}}_{\square\square} + \frac{(t-1)^2(t+1)}{(q-1)(qt-1)} \overline{\mathrm{rCS}}_\boxtimes \right] \overline{\mathrm{rCS}}_\square$$
$$+ \frac{2(t-1)^2}{3(q-1)^2} (\overline{\mathrm{rCS}}_\square)^3 + \frac{t^2+t+1}{3\left(q^2+q+1\right)} \overline{\mathrm{rCS}}_\square^{(3)} ,$$

$$\frac{t}{q} \frac{f_\boxtimes^q}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \frac{(q-t)\left(q^2-t\right)}{\left(q^2-1\right)(q^3-1)} \overline{\mathrm{rCS}}_{\square\square\square} - \frac{(t^2-1)(q-t)}{(q-1)\left(q^2 t-1\right)} \overline{\mathrm{rCS}}_\boxtimes + \frac{(t^2-1)(t^3-1)}{(qt-1)\left(qt^2-1\right)} \overline{\mathrm{rCS}}_\boxtimes$$
$$+ \left[\frac{(t-1)(q-t)}{(q-1)^2(q+1)} \overline{\mathrm{rCS}}_{\square\square} - \frac{(t-1)^2(t+1)}{(q-1)(qt-1)} \overline{\mathrm{rCS}}_\boxtimes \right] \overline{\mathrm{rCS}}_\square$$
$$+ \frac{(t-1)^2}{3(q-1)^2} (\overline{\mathrm{rCS}}_\square)^3 - \frac{t^2+t+1}{3\left(q^2+q+1\right)} \overline{\mathrm{rCS}}_\square^{(3)} .$$

In addition, we present reformulated invariants of a torus knot colored by Young diagrams with a few boxes for the $\bar{t}$-branes from (B.2):

$$\frac{f_\square^{\bar{t}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \overline{\mathrm{rCS}}_\square,$$

$$\frac{-f_{\square\square}^{\bar{t}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \overline{\mathrm{rCS}}_\boxtimes + \frac{1}{2} \overline{\mathrm{rCS}}_\square^{(2)} - \frac{1}{2} (\overline{\mathrm{rCS}}_\square)^2 ,$$

$$\frac{-f_\boxtimes^{\bar{t}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \overline{\mathrm{rCS}}_{\square\square} + \frac{q-t}{qt-1} \overline{\mathrm{rCS}}_\boxtimes - \frac{1}{2} \overline{\mathrm{rCS}}_\square^2 - \frac{1}{2} \overline{\mathrm{rCS}}_\square^{(2)},$$

$$\frac{f_{\square\square\square}^{\bar{t}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \overline{\mathrm{rCS}}_\boxtimes - \overline{\mathrm{rCS}}_\square \overline{\mathrm{rCS}}_\boxtimes + \frac{1}{3} (\overline{\mathrm{rCS}}_\square)^3 - \frac{1}{3} \overline{\mathrm{rCS}}_\square^{(3)} ,$$

$$\frac{f_\boxtimes^{\bar{t}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \overline{\mathrm{rCS}}_\boxtimes + \frac{(t+1)(q-t)}{qt^2-1} \overline{\mathrm{rCS}}_\boxtimes - \left[\overline{\mathrm{rCS}}_{\square\square} + \frac{(q-1)(t+1)}{qt-1} \overline{\mathrm{rCS}}_\boxtimes \right] \overline{\mathrm{rCS}}_\square$$

$$+ \frac{2}{3}(\overline{\mathrm{rCS}}_\square)^3 + \frac{1}{3}\overline{\mathrm{rCS}}_\square^{(3)},$$

$$\frac{f_\square^{\bar{t}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \overline{\mathrm{rCS}}_{\square\square\square} + \frac{(q+1)(q-t)}{q^2 t - 1}\overline{\mathrm{rCS}}_{\square\square} + \frac{(q-t)\left(q-t^2\right)}{(qt-1)\left(qt^2-1\right)}\overline{\mathrm{rCS}}_{\square}$$

$$+ \left[-\overline{\mathrm{rCS}}_{\square\square} + \frac{(q-t)}{qt-1}\overline{\mathrm{rCS}}_{\square} \right]\overline{\mathrm{rCS}}_\square + \frac{1}{3}(\overline{\mathrm{rCS}}_\square)^3 - \frac{1}{3}\overline{\mathrm{rCS}}_\square^{(3)} \ .$$

C   Tables of BPS degeneracies

In this appendix, we shall list tables of non-negative integral invariants $\widehat{N}_{\rho_1,\cdots,\rho_L,g,\beta,J_r}$ of some torus knots/links. The formulas (B.1) and (B.2) enable us to evaluate reformulated invariants by using refined Chern-Simons invariants of torus knots/links obtained in [AS15, DBMM$^+$13, Che13, FGSA12, Sha13, GNS$^+$16]. Writing reformulated invariants into (5.3), we read off non-negative integral invariants.

β	1/2			3/2			5/2
$g\backslash 2J_r$	0	1	2	1	2	3	2
0	1	0	1	1	0	2	1
1	0	1	0	0	1	0	0

Table 3. $\widehat{N}_{[1],g,\beta,J_r}$ for the trefoil ($T_{2,3}$)

β	1			2					3							4					5		
$g\backslash 2J_r$	0	1	2	1	2	3	4	5	2	3	4	5	6	7	8	3	4	5	6	7	4	5	6
0	1	0	1	2	0	4	0	2	1	0	5	0	5	0	1	2	0	4	0	2	1	0	1
1	0	1	0	0	3	0	3	0	0	2	0	6	0	2	0	0	3	0	3	0	0	1	0
2	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0

Table 4. $\widehat{N}_{[2],g,\beta,J_r}$ for the trefoil ($T_{2,3}$)

β	1					2							3									4							5				
$g\backslash 2J_r$	0	1	2	3	4	1	2	3	4	5	6	7	2	3	4	5	6	7	8	9	10	3	4	5	6	7	8	9	4	5	6	7	8
0	1	0	2	0	1	2	0	7	0	5	0	2	1	0	8	0	9	0	5	0	1	3	0	7	0	4	0	2	2	0	1	0	1
1	0	2	0	2	0	0	5	0	10	0	5	0	0	3	0	14	0	12	0	3	0	0	6	0	9	0	5	0	0	2	0	2	0
2	0	0	1	0	0	0	0	4	0	4	0	0	0	0	3	0	8	0	3	0	0	0	0	4	0	4	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0

Table 5. $\widehat{N}_{[1,1],g,\beta,J_r}$ for the trefoil ($T_{2,3}$)

β	3/2					5/2					7/2		
$g\backslash 2J_r$	0	1	2	3	4	1	2	3	4	5	2	3	4
0	1	0	1	0	1	1	0	2	0	2	1	0	1
1	0	2	0	2	0	0	2	0	3	0	0	1	0
2	0	0	1	0	0	0	0	1	0	0	0	0	0

Table 6. $\widehat{N}_{[1],g,\beta,J_r}$ for the $T_{2,5}$

β	-1	0
$g\backslash 2J_r$	-2	-1
0	1	1

β	-3/2	-1/2
$g\backslash 2J_r$	-3	-2
0	1	1

Table 7. $\widehat{N}_{[1],[1],g,\beta,J_r}$ for the Hopf link $(T_{2,2})$ **Table 8**. $\widehat{N}_{[2],[1],g,\beta,J_r}$ for the Hopf link $(T_{2,2})$

β	-2			-1			0
$g\backslash 2J_r$	-7	-6	-5	-6	-5	-4	-3
0	1	0	2	1	0	3	1
1	0	1	0	0	1	0	0

Table 9. $\widehat{N}_{[2],[2],g,\beta,J_r}$ for the Hopf link $(T_{2,2})$

β	-2	-1
$g\backslash 2J_r$	-5	-4
0	1	1

Table 10. $\widehat{N}_{[2],[1,1],g,\beta,J_r}$ for the Hopf link $(T_{2,2})$

β	-1			0			1
$g\backslash 2J_r$	-4	-3	-2	-3	-2	-1	0
0	1	0	2	1	0	3	1
1	0	1	0	0	1	0	0

Table 11. $\widehat{N}_{[1],[1],g,\beta,J_r}$ for $T_{2,4}$

β	-3/2					-1/2					1/2		
$g\backslash 2J_r$	-7	-6	-5	-4	-3	-6	-5	-4	-3	-2	-3	-2	-1
0	1	0	2	0	3	1	0	3	0	5	1	0	2
1	0	2	0	3	0	0	2	0	4	0	0	1	0
2	0	0	1	0	0	0	0	1	0	0	0	0	0

Table 12. $\widehat{N}_{[2],[1],g,\beta,J_r}$ for $T_{2,4}$

β	-3/2			-1/2			1/2
$g\backslash 2J_r$	-5	-4	-3	-4	-3	-2	-1
0	1	0	1	1	0	2	1
1	0	1	0	0	1	0	0

Table 13. $\widehat{N}_{[1,1],[1],g,\beta,J_r}$ for $T_{2,4}$

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