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p-brane Dyons, θ-terms and Dimensional Reduction

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Abstract

We present two novel derivations of the recently established $(-)^p$ factor in the charge quantization condition for p-brane dyon sources in spacetime dimension D=$2p$+2. The first requires consistency of the condition under the charge shifts produced by (generalized) θ-terms. The second traces the sign difference between adjoining dimensions to compactification effects.

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It was recently established [1] that the generalized dyon quantization condition, for (p–1)-brane dyons coupled to Abelian p-forms in spacetime dimension D=2p+2, involves a p-dependent sign:

$$e\bar{g} + (-)^p \bar{e} g = 2\pi n\hbar, \; n \in Z \; . \tag{1}$$

The $-$ sign is of course that familiar in D=4 electrodynamics [2, 3, 4]. This sign dependence was actually anticipated [5] through analysis of chiral sources coupled to chiral 2p-forms. It was particularly stressed in [6], where it was related to supergravity duality groups in higher dimensions [7]. Another approach is based on dyon-dyon scattering; the relation between D=10 and D=4 is also discussed there [8].

Our aim is to illuminate this phenomenon through two new arguments. The first is based on the study of the shift in the "electric" charges induced by a (p-generalized) θ-term. The other uses dimensional reduction, or rather enhancement, to relate the conditions (1) in adjoining dimensions. For concreteness, we shall work primarily with D=4 one-forms and D=6 two-forms to illustrate the generic situations.

1. θ-terms.

a) In D=4 it is well known that adding a θ-term, $(\theta/2)F_{\mu\nu}\,{}^*F^{\mu\nu}$ (* always represents dualization) to the Lagrangian has the effect of shifting the electric charge of an (e, g) dyon according to [9]

$$e' = e - 2g\theta \; . \tag{2}$$

A remarkable feature of this shift is its compatibility with the usual Dirac quantization condition for electric and magnetic charges. Namely, if one simultaneously shifts all dyon electric charges according to (2) starting from values (e_a, g_a) that obey (1), then the charges

(e'_a, g_a) also do, because

$$e'_a g_b - e'_b g_a = (e_a - g_a\theta)g_b - (e_b - g_b\theta)g_a = e_a g_b - e_b g_a \ . \qquad (3)$$

The $-$ sign is crucial in this result. Indeed, it is the answer to the converse question: what sign in the quantization condition (1) leaves it invariant under the shift (2)?

b) In D=6, there is no θ-term for a single 2-form since $F_{ABC}\,{}^*F^{ABC}$ vanishes identically. However, a θ-term *is* possible with two 2-forms $A^{(i)}$, $i = 1, 2$. The sources here are strings characterized by four strengths ("charges") $(e_a^{(i)},\ g_a^{(i)})$, the respective electric (magnetic) charges of string a coupled to $A^{(i)}$. We use a uniform convention for the signs of the couplings $(e_a^{(i)},\ g_a^{(i)})$ to the 2-forms: the electric couplings enter with the same sign in the minimal coupling term $\sum_{i=1,a}^{2} e_a^{(i)} \int A^{(i)}$, for example. Single-valuedness of the wave function leads to the quantization condition[1]

$$(e_a^{(1)} g_b^{(1)} \pm g_a^{(1)} e_b^{(1)}) + (e_a^{(2)} g_b^{(2)} \pm g_a^{(2)} e_b^{(2)}) = 2\pi n\hbar,\ n \in Z \qquad (4)$$

with a relative $+$ sign between the contributions associated with the two 2-forms because of our identical coupling conventions. We have left the $\pm$ sign open in (4) to show next how the θ-angle argument selects the $+$ sign. [Of course, for sources that couple to only one of the fields – say $A^{(1)}$ –, the second term on the left is absent in (4).] Now add the extended

[1] One canonical way to derive this condition consists of attaching Dirac membranes – the higher dimensional analogs of the Dirac strings [2] – to the sources [10, 11]. Requiring the membranes to remain unobservable quantum-mechanically then implies (4). Indeed, the phase picked up by the wave-function when the Dirac membrane attached to string a performs a complete turn around string b, while the Dirac membrane attached to string b simultaneously performs a complete turn around string a (the "double-pass" of [1]) is $(1/\hbar)$ times the left-hand side of (4).

θ-term

$$\frac{1}{2}\,\theta\,\epsilon_{ij}F^{(i)}_{ABC}\,{}^*F^{(j)ABC} \tag{5}$$

to the free Lagrangian F^2_{ABC}. As in D=4, the effect of this term is to shift the electric charges, but this time by

$$e_a^{(i)\prime} \;=\; e_a^{(i)} - (3!)\,\theta\,\epsilon^{ij}g_a^{(j)}\ . \tag{6}$$

The antisymmetry of this shift is traceable to that of the θ-term; more explicitly, the θ-term involves only mixed couplings, with opposite signs: $\theta A^{(1)}_{0n}\partial_m B^{(2)nm}$, $-\theta A^{(2)}_{0n}\partial_m B^{(1)nm}$. [Had we taken opposite conventions for the couplings, there would be a relative minus sign in (4) between the contributions of the two fields, and the same sign in (6).] For the quantization condition (4) to be invariant under the shift (6) then requires the + sign there. Thus, also in D=6 a (generalized) θ-angle argument determines the sign in the quantization condition.

From these two examples, it is clear that the $(-)^p$ factor is a reflection of the opposite symmetries of the θ-terms $F{}^*F$ and $\epsilon_{ij}F^{(i)}\,{}^*F^{(j)}$ in alternating dimensions.

2. Adjoining dimensions.

We now turn to the argument from dimensional reduction (actually, "enhancement"). Since higher dimension is clearly more restrictive, our logic will be to show that the quantization rule in D=$2p$–2 for those specific configurations obtained by reduction from $2p$ imposes the form of the D=$2p$ rule as well. Specifically we shall show that if the quantization condition holds with one sign in D=$2p-2$, then it must hold with the opposite one in D=$2p$; in particular, the + sign in D=6 follows from the $-$ sign in D=4.

We relate D=6 to D=4 by toroidal compactification, $M^6 = R^4\times T^2$. The spacetime coordinates x^A $(A = 0, 1, \ldots, 5)$ split into $x^A = (x^\mu; x^4, x^5)$ where $(x^4,\ x^5)$ parametrize the torus which, for our purposes, may be assumed to be the standard $(dx^4)^2 + (dx^5)^2$, with $(x^4,\ x^5)$ having respective ranges $[0, L_4]$ and $[0, L_5]$. [For fields independent of $(x^4,\ x^5)$ as considered here, one may always diagonalize the internal metric, but we chose *not* to also rescale the ranges to unity.] The full spatial $5D$ rotational symmetry is broken by the compactification of course. However, there is a useful residual "Y-symmetry," under simultaneous interchange of x_4/L_4 with x_5/L_5 together with a $4D$ parity (P) transformation. Indeed, we will conclude generally that the quantization condition in D=$2p$–2, together with Y-symmetry, implies the corresponding one at D=$2p$.

A non-chiral 2-form A_{AB} in D=6 induces two D=4 U(1) gauge fields $A^{(i)}_\mu$. The reduction proceeds by assuming A_{AB} to be constant along the internal tori and to have only $A_{4\mu}$ and $A_{5\mu}$ as non-zero components. [The other, $A_{\mu\nu}$ components and the higher modes induce further four-dimensional fields which are not relevant to our discussion.] The correspondence is

$$A^{(1)}_\mu = \sqrt{L_4 L_5} A_{4\mu},\ A^{(2)}_\mu = \sqrt{L_4 L_5} A_{5\mu} \tag{7}$$

as follows from reduction of the 2-form action $\int d^6x F^2_{ABC}$. In D=4 terms, (besides the P) Y-transformations interchange the two $A^{(i)}$.

The D=6 sources that correspond to point particles in D=4 are strings winding around the internal torus directions. For a single string along x^4 at $\mathbf{x} = 0$, $x^5 = a$, the current has as its only non-vanishing components

$$J^{04}_e = e\delta^{(3)}(\mathbf{x})\delta(x^5 - a), \quad J^{04}_m = g\delta^{(3)}(\mathbf{x})\delta(x^5 - a) \tag{8}$$

where (e, g) are the respective electric and magnetic strengths of the string. The zero modes of the 2-form field couple only to the zero modes of the source. Thus, from the point of view of the zero modes, one can replace the source by a continuous distribution of parallel strings aligned along x^4, with constant electric and magnetic strengths per unit length, (ρ_5, σ_5), along the transverse (x^5) direction. Such a distribution yields a membrane wrapping around the torus and does not excite the higher modes ("vertical reduction" of [12]). This alternative description preserves translation invariance along x^5. Replacing the above source by a stack of strings at $\mathbf{x} = 0$ aligned along x^4 amounts to replacing the currents of (8) by

$$J_e^{04} = \rho_5 \delta^{(3)}(\mathbf{x}), \quad J_m^{04} = \sigma_5 \delta^{(3)}(\mathbf{x}) \ . \tag{9}$$

These currents are obtained by summing the currents of the individual strings, *e.g.*, $J_e^{04}(\mathbf{x}, x^5) = \rho_5 da \delta^{(3)}(\mathbf{x}) \delta(x^5 - a)$ for the string located at $x^5 = a$. The corresponding D=6 charges are

$$e = \rho_5 L_5, \quad g = \sigma_5 L_5. \tag{10}$$

From the $4D$ point of view, the stack appears to have the $U(1)$ charges

$$\left(e^{(1)},\, g^{(1)},\, e^{(2)},\, g^{(2)}\right) = \left(e\sqrt{\frac{L_4}{L_5}}, 0, 0, g\sqrt{\frac{L_4}{L_5}}\,\right) \tag{11}$$

as shown by the analysis of the equations of motion given below. Again, we adopt the same sign conventions for the two $U(1)$'s and define electric and magnetic charges in $4D$ in such a way that $\boldsymbol{\nabla} \cdot \mathbf{E}^{(i)} \sim +e^{(i)}$, $\boldsymbol{\nabla} \cdot \mathbf{B}^{(i)} \sim +g^{(i)}$ (with same $+$ sign for both i). Similarly, the current of a single dyonic string (e', g') lying on the x^5 axis at $(\mathbf{x} = \mathbf{b}, x^4 = c)$ is given by

$$J_e^{05} = e' \delta^{(3)}(\mathbf{x} - \mathbf{b}) \delta(x^4 - c), \quad J_m^{05} = g' \delta^{(3)}(\mathbf{x} - \mathbf{b}) \delta(x^4 - c) \ . \tag{12}$$

Again, from the zero mode point of view, this can be replaced by a suitable stack of strings whose $6D$ currents are

$$J_e^{05} = \rho_4 \delta^{(3)}(\mathbf{x} - \mathbf{b}), \quad J_m^{05} = \sigma_4 \delta^{(3)}(\mathbf{x} - \mathbf{b}) \ . \tag{13}$$

Here the D=6 charges are

$$e' = \rho_4 L_4, \quad g' = \sigma_4 L_4 \ , \tag{14}$$

while the $4D$ charges are

$$\left(e^{(1)},\, g^{(1)},\, e^{(2)},\, g^{(2)}\right) = \left(0, -g'\sqrt{\frac{L_5}{L_4}}, e'\sqrt{\frac{L_5}{L_4}}, 0\right) \ . \tag{15}$$

These charges have two properties: First, a dyonic string in D=6 along x^4 or x^5 does not appear as a dyon in D=4. Rather, it is *electrically* charged for one $U(1)$ and *magnetically* charged for the other $U(1)$. To get dyons for the same $U(1)$ in D=4, one needs to superpose strings along both x^4 and x^5. The same remark applies to the chiral case (in D=6), for which the two D=4 $U(1)$'s are related by the duality rotation $\mathbf{B}^{(2)} = \mathbf{E}^{(1)}$, $\mathbf{E}^{(2)} = -\mathbf{B}^{(1)}$. The above strings would appear as either purely electric (first string) or purely magnetic (second string) but do not carry both types of charges. Second, there is a crucial flip of sign in the magnetic charges for the two $U(1)$'s. The simultaneous existence of the "dual" configurations (15) and (21) reflects the Y symmetry. Indeed, a Y-transformation, to a D=4 observer, just induces this dual exchange. To understand how the D=4 assignments arise, consider the field equations, $\partial_A F^{ABC} = J_e^{BC}$, $\partial_A {}^*F^{ABC} = J_m^{BC}$, for the given sources in terms of the D=4 fields. For the source (10) along x^4, the equations reduce to

$$\partial_i E^{i(1)} \equiv +\sqrt{L_4 L_5}\partial_i F^{i04} = +e\sqrt{\frac{L_4}{L_5}}\delta^{(3)}(\mathbf{x}) \tag{16}$$

and

$$\partial_i B^{i(2)} \equiv +\sqrt{L_4 L_5}\partial_i((1/2!)\epsilon^{imn045} F_{mn5}) \equiv +\sqrt{L_4 L_5}\partial_i {}^*F^{i04} = +g\sqrt{\frac{L_4}{L_5}}\delta^{(3)}(\mathbf{x}), \tag{17}$$

where $i, m, n = 1, 2, 3$. For the source (14), one finds

$$\partial_i E^{i(2)} \equiv \sqrt{L_4 L_5}\partial_i F^{i05} = +e'\sqrt{\frac{L_5}{L_4}}\delta^{(3)}(\mathbf{x} - \mathbf{b}) \tag{18}$$

and

$$\partial_i B^{i(1)} \equiv +\sqrt{L_4 L_5}\partial_i((1/2!)\epsilon^{imn054} F_{mn4}) \equiv -\sqrt{L_4 L_5}\partial_i {}^*F^{i05} = -g'\sqrt{\frac{L_5}{L_4}}\delta^{(3)}(\mathbf{x} - \mathbf{b}), \tag{19}$$

with a minus sign because $\epsilon^{imn054} = -\epsilon^{imn045}$. This leads to the assignments (11) and (15).

We now deduce the quantization condition in D=6 from that in D=4. For the strings (11) and (15), the D=6 quantization condition is

$$eg' \pm e'g = 2\pi\hbar n, \quad n \in Z. \tag{20}$$

where we have again left the relative sign open. The quantization condition in D=4, on the other hand, is, in terms of D=4 charges,

$$(e_a^{(1)} g_b^{(1)} - e_b^{(1)} g_a^{(1)}) + (e_a^{(2)} g_b^{(2)} - e_b^{(2)} g_a^{(2)}) = 2\pi\hbar n, \quad n \in Z\ . \tag{21}$$

Recall that the relative + sign between the two $U(1)$ contributions is due to our identical coupling conventions for both. The only choice that makes (21) consistent with (20) is the + sign as is easy to verify by using the explicit values of the D=4 charges in terms of the D=6 ones. To show, finally, that the electric and magnetic charges of a single string in D=6 are constrained by $2eg = 2\pi n\hbar, \quad n \in Z$, we recall that this condition was obtained in [5] by exploiting the flexibility of Dirac membranes to perform motions that do not distinguish between the spatial directions. It comes as no surprise therefore, that one can recover this relation from the D=4 point of view by using Y symmetry. Indeed, together with the configuration $(e, 0, 0, g)$, Y implies that the configuration $(0, -g, e, 0)$ should also exist. Applying the D=4 dyon quantization condition to the two configuration appearing above, we recover this $e - g$ relation. We can imagine continuing this chain of arguments inductively: (1) consider the dyonic configuration in $2p$ dimensions; (2) list the $2p - 2$ dimensional configurations to which it gives rise, including those related by Y-symmetry; (3) apply the $2p - 2$ dimensional quantization rules which will therefore relate the $2p$ dimensional parameters, etc. So starting with say the D=4 quantization rules, those for higher dimensions will follow, and it is clear that there is a $(-)^p$ alternation.

In retrospect, it is not surprising that one can infer the D=6 quantization condition from that in D=4, together with the extra Y-symmetry it enjoys. Indeed, as was shown in [1], the respective quantization conditions with $+/-$ signs possess exactly the same general solutions (assuming existence of pure electric sources); hence (when $(C)P$ invariance is imposed in D=4) they are clearly equivalent [9].

To summarize, we have provided two independent derivations of the $(-)^p$ sign factor in the p-brane dyon quantization conditions. Both arguments are ultimately manifestations of the basic "double dual" identity $** = (-)^p$.

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