Supplementary materials for “Beating the standard sensitivity-bandwidth limit of cavity-enhanced interferometers with internal squeezed-light generation”

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Derivation of the noise spectrum and signal transfer function

In this section we derive the theoretical model for our interferometric system. It corresponds to a Fabry-Perot cavity with nonlinear media inside, that parametrically amplifies one quadrature of the light (see Fig. 1). Using the perturbation theory, we decompose the light field into a steady-state amplitude with amplitude A_0 and laser carrier frequency ω_0 and a slowly varying noise amplitude $a(t)$ (see details in [1]):

$$A(t) = \sqrt{\frac{2\pi\hbar\omega_0}{\mathcal{A}c}}\left[A_0 e^{-i\omega_0 t} + a(t)e^{-i\omega_0 t}\right] + \text{h.c.} \tag{1}$$

$$a(t) = \int_{-\infty}^{\infty} a(\omega_0+\Omega)e^{-i\Omega t}\frac{d\Omega}{2\pi}, \tag{2}$$

where $\mathcal{A}$ is the laser beam cross-section area, $\hbar$ is the reduced Plank constant. Note that we omit the hats on the operator for brevity, although all the fields are quantised. Below we consider only the noise fields in the frequency domain. Furthermore, we define the two-photon amplitude and phase quadratures at a sideband frequency Ω correspondingly as

$$a_x(\Omega) = \frac{a(\omega_0+\Omega)+a^\dagger(\omega_0-\Omega)}{\sqrt{2}}, \quad a_y(\Omega) = \frac{a(\omega_0+\Omega)-a^\dagger(\omega_0-\Omega)}{i\sqrt{2}}. \tag{3}$$

These operators obey the commutation relation

$$[a_x(\Omega), a_x(\Omega')] = [a_y(\Omega), a_y(\Omega')] = 0, \tag{4}$$

$$[a_x(\Omega), a_y(\Omega')] = [a_x(\Omega), a_y(\Omega')] = 2\pi i\delta(\Omega+\Omega'). \tag{5}$$

Using these two-photon quadratures, we can apply the input-output formalism [2, 3] and find the steady-state fields in the system. The signal we consider is a phase modulation on the light field induced by motion of the mirror with infinite mass caused by an external force. This modulation adds a phase shift on the light reflected off the movable mirror: $E_{\text{refl}} = E_{\text{in}}e^{2ikx(\Omega)} \approx E_{\text{in}}(1+2ik_p x(\Omega))$, where k_p is the light's wave vector, $E_{\text{refl,in}}$ are the amplitudes of the reflected and incident light fields, and $x(\Omega)$ is a small mirror displacement. The signal appears only in the equations for the phase quadrature of the light field. We model the optical loss by a beamsplitter reflecting some part of the light

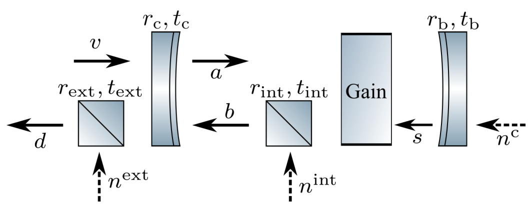

FIG. 1. The schematic representation of our interferometric system. The signal $s = 2ik_p x(\Omega)$ originates from the displacement $x(\Omega)$ of the back mirror which is assumed to be of infinite mass. Two sources of loss are assumed. Internal loss and detection loss, and both are modelled with a beam splitter. The optical parametrical amplification process creates a gain in one of the quadratures, and an attenuation in the orthogonal one.

fields to the environment and mixing in some vacuum from the environment. The optical parametrical amplification process is not included in the model. We treat crystal as a gain medium that linearly amplifies with gain e^q a certain quadrature (amplitude in our case) and deamplifies the orthogonal one. We call q a squeezing factor in a single pass through the crystal.

The system of input-output equations for the amplitude (denoted by x) and phase (denoted by y) quadratures reads

$$\begin{cases} a_x(\Omega) = t_\mathrm{c} v_x(\Omega) + r_\mathrm{c} b_x(\Omega) \\ b_x(\Omega) = a_x(\Omega) t_\mathrm{int} r_\mathrm{b} e^{2i\Omega\tau} e^{2q} + n_x^\mathrm{c}(\Omega) t_\mathrm{int} t_\mathrm{b} e^{i\Omega\tau} e^q + r_\mathrm{int} n_x^\mathrm{int}(\Omega) \\ d_x(\Omega) = t_\mathrm{det}\left(-r_\mathrm{c} v_x(\Omega) + t_\mathrm{c} b_x(\Omega)\right) + r_\mathrm{det} n_x^\mathrm{ext}(\Omega) \end{cases} \tag{6}$$

$$\begin{cases} a_y(\Omega) = t_\mathrm{c} v_y(\Omega) + r_\mathrm{c} b_y(\Omega) \\ b_y(\Omega) = a_y(\Omega) t_\mathrm{int} r_\mathrm{b} e^{2i\Omega\tau} e^{-2q} + 2ik_p E x(\Omega) t_\mathrm{int} e^{i\Omega\tau} e^{-q} + n_y^\mathrm{c}(\Omega) t_\mathrm{int} t_\mathrm{b} e^{i\Omega\tau} e^{-q} + r_\mathrm{int} n_y^\mathrm{int}(\Omega) \\ d_y(\Omega) = t_\mathrm{det}\left(-r_\mathrm{c} v_y(\Omega) + t_\mathrm{c} b_y(\Omega)\right) + r_\mathrm{det} n_y^\mathrm{ext}(\Omega). \end{cases} \tag{7}$$

Here $\tau = L/c$ is the round trip propagation time, with L being the length of the cavity; c is the speed of light, q - single pass squeeze factor, $x(\Omega)$ - mirror displacement induced by a signal, k_p - wave vector of the carrier field, E is the mean amplitude of the light field inside the detector, $t_\mathrm{c,b}, r_\mathrm{c,b}$ - amplitude transmissivity and reflectivity of coupling and back mirrors, such that $r_\mathrm{c,b}^2 + t_\mathrm{c,b}^2 = 1$, r_det - detection loss, r_int - intra-cavity loss without the coupling and the back mirrors; $r_\mathrm{det,int}^2 + t_\mathrm{det,int}^2 = 1$.

This set of equations can be solved for the detected fields $d_{x,y}$:

$$\begin{aligned} d_x(\Omega) = {} & \frac{t_\mathrm{det}}{e^{-2q} - e^{2i\Omega\tau} r_\mathrm{c} r_\mathrm{b} t_\mathrm{int}} \left(v_x(\Omega)\left(-r_\mathrm{c} e^{-2q} + r_\mathrm{b} r_\mathrm{int}^2 e^{2i\Omega\tau}\right) + n_x^\mathrm{c}(\Omega) t_\mathrm{c} t_\mathrm{b} t_\mathrm{int} e^{-q} e^{i\Omega\tau} + \right. \\ & \left. + n_x^\mathrm{int}(\Omega) t_\mathrm{c} r_\mathrm{int} e^{-2q})\right) + r_\mathrm{det} n_x^\mathrm{ext}(\Omega), \end{aligned} \tag{8}$$

$$\begin{aligned} d_y(\Omega) = {} & \frac{t_\mathrm{det}}{e^{2q} - e^{2i\Omega\tau} r_\mathrm{c} r_\mathrm{b} t_\mathrm{int}} \left(2ik_p E x(\Omega) t_\mathrm{c} t_\mathrm{int} e^{q} e^{i\Omega\tau} + v_y(\Omega)\left(-r_\mathrm{c} e^{2q} + r_\mathrm{b} t_\mathrm{int} e^{2i\Omega\tau}\right) + n_y^\mathrm{c}(\Omega) t_\mathrm{c} t_\mathrm{b} t_\mathrm{int} e^{q} e^{i\Omega\tau} + \right. \\ & \left. + n_y^\mathrm{int}(\Omega) t_\mathrm{c} r_\mathrm{int} e^{2q})\right) + r_\mathrm{det} n_y^\mathrm{ext}(\Omega). \end{aligned} \tag{9}$$

The spectrum of the noise $a(\Omega)$ is defined:

$$S_a(\Omega)\delta(\Omega - \Omega') = \frac{1}{2}\left\langle a(\Omega)a(\Omega') + a(\Omega')a(\Omega)\right\rangle. \tag{10}$$

Then, assuming that all noises in the system are uncorrelated, we find the spectral density of the detected noise

$$S_n(\Omega) = 1 - \frac{t_\mathrm{c}^2 t_\mathrm{det}^2 t_\mathrm{int}^2 (1 - e^{-2q})(1 + e^{-2q} r_\mathrm{b}^2)}{1 + r_\mathrm{c}^2 r_\mathrm{b}^2 t_\mathrm{int}^2 e^{-4q} - 2 r_\mathrm{c} r_\mathrm{b} t_\mathrm{int} e^{-2q}\cos 2\Omega\tau}. \tag{11}$$

The transfer function of the signal $x(\Omega)$ through the optical cavity to the detector:

$$T(\Omega) = 2ik_p E \frac{t_\mathrm{c} t_\mathrm{det} t_\mathrm{int} e^q e^{i\Omega\tau}}{e^{2q} - e^{2i\Omega\tau} r_\mathrm{c} r_\mathrm{b} t_\mathrm{int}}, \tag{12}$$

and it's spectral shape:

$$|T(\Omega)|^2 = \frac{8\pi P_c}{\hbar\lambda L} \frac{e^{-2q} t_\mathrm{c}^2 t_\mathrm{det}^2 t_\mathrm{int}^2}{1 + r_\mathrm{c}^2 r_\mathrm{b}^2 t_\mathrm{int}^2 e^{-4q} - 2 r_\mathrm{c} r_\mathrm{b} t_\mathrm{int} e^{-2q}\cos 2\Omega\tau}, \tag{13}$$

where $P_c = \hbar k_p c|E|^2$ is the light power inside the cavity, and λ is the carrier wavelength.

Approximate description

In this section we simplify the expressions for the spectral density (11) and signal transfer function (13) by making several assumptions. The amplitude transmissivities of the coupling and the back mirrors, as well as the internal loss, are much smaller than unity, and we can approximate correspondingly $r_\mathrm{c,b} \approx 1 - t_\mathrm{c,b}^2/2$ and $t_\mathrm{int} \approx 1 - r_\mathrm{int}^2/2$; the squeezing factor q is much smaller than unity, so we can approximate $e^q \approx 1 + q$; the frequency of interest is much

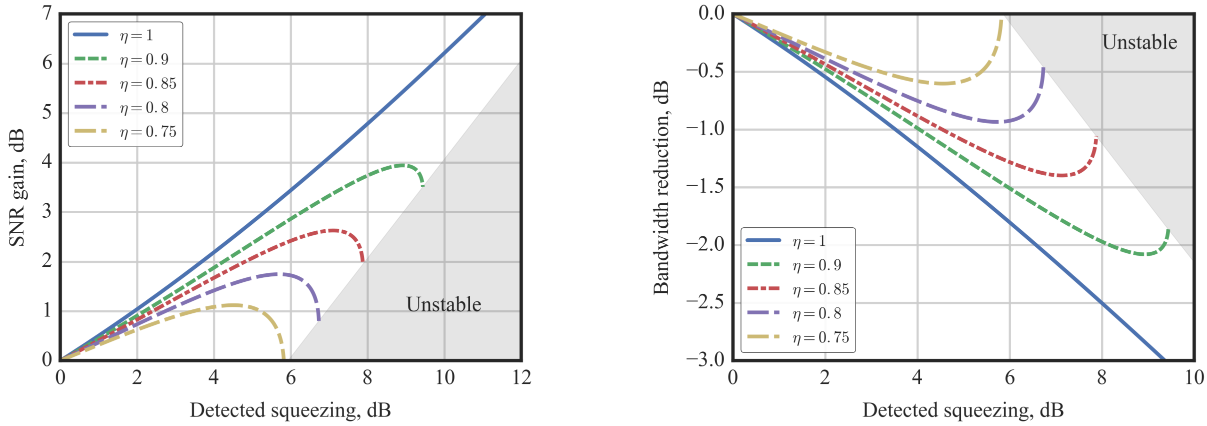

FIG. 2 (color online). Dependence of the gain in the signal-to-noise ratio (left) and the reduction in the bandwidth (right) depending on the detected squeeze factor. Different plots represent the influence of the detection loss on the enhancement. The existence of an optimal squeezing is demonstrated. The shaded region represents the parametric gains for which the intra-cavity field becomes unstable.

smaller than the FSR of the cavity $\Omega \ll 1/2\tau$, which enables us to make a Taylor expansion: $\cos\Omega\tau \approx 1 - \Omega^2\tau^2/2$. With these assumptions we introduce the variables

$$\gamma_c = \frac{ct_c^2}{4L}, \quad \gamma_s = \frac{qc}{L}, \quad \gamma_l = \frac{cl^2}{4L}, \quad l^2 = r_{\rm int}^2 + t_{\rm b}^2, \quad \Gamma = \gamma_c + \gamma_s + \gamma_l. \tag{14}$$

Equations (11), (13) then simplify to

$$|T(\Omega)|^2 \approx \frac{8\pi P_c}{\hbar\lambda L}\frac{\gamma_c\eta}{\Gamma^2+\Omega^2}, \tag{15}$$

$$S_n(\Omega) \approx 1 - \frac{4\gamma_c\gamma_s}{\Gamma^2+\Omega^2}\eta, \tag{16}$$

where $\eta = 1 - t_{\rm det}^2$ is the detection efficiency. Then the signal-to-noise ratio reads

$$\frac{|T(\Omega)|^2}{S_n(\Omega)} = \frac{8\pi P_c}{\hbar\lambda L}\frac{\gamma_c\eta}{\Gamma^2 - 4\gamma_c\gamma_s\eta + \Omega^2}, \tag{17}$$

with corresponding bandwidth

$$\mathcal{B} = \sqrt{\Gamma^2 - 4\gamma_c\gamma_s\eta}. \tag{18}$$

We define the integrated sensitivity, which connects to the sensitivity-bandwidth product

$$\rho = \int_0^{\omega_{\rm FSR}} \frac{|T(\Omega)|^2}{S_n(\Omega)} d\Omega = \mathcal{S}\times\mathcal{B}, \tag{19}$$

where $\omega_{\rm FSR}$ is a free spectral range of the cavity, and $\mathcal{S} = |T(0)|^2/S_n(0)$ is the peak sensitivity. The enhancement in the sensitivity-bandwidth product is given by

$$G = \rho/\rho_{q=0} = \frac{\gamma_c+\gamma_l}{\mathcal{B}}. \tag{20}$$

As we can see from the equations, the main source of reduction of the desired effect is the detection loss. It includes both optical loss in the path and quantum efficiency of the detector. We can see in Fig. 2 that there exists an optimal squeeze factor, at which the enhancement is maximal.

The treatment presented in this section is useful for understanding the concept and main properties of the internal squeezing, but in real cases some of the assumptions made here might be not valid. Thus one has to calculate the integrated sensitivity (19) directly from Eqs. (11), (13).

Calculation of the optical parametric oscillation threshold

In this section we derive the threshold value for the squeeze factor. The squeezing value cannot be arbitrary large inside the cavity, as at some pump power it will become unstable and initiate lasing. We find the stability criterion from the equation for the amplitude quadrature inside the cavity:

$$b_x = \frac{v_x(\Omega) r_\mathrm{b} t_\mathrm{c} t_\mathrm{int} e^{2q} e^{2i\Omega\tau} + n_x^\mathrm{c}(\Omega) t_\mathrm{b} e^{q} e^{i\Omega\tau} + n_x^\mathrm{int}(\Omega) r_\mathrm{int}}{1 - r_\mathrm{c} r_\mathrm{b} t_\mathrm{int} e^{2q} e^{2i\Omega\tau}}. \tag{21}$$

The threshold value represents the condition, at which the gain becomes larger than the overall loss through the coupler transmission and additional optical round trip loss. This condition is defined by setting the denominator equal to zero, and is reached when

$$e^{2q} = \frac{1}{r_\mathrm{c} r_\mathrm{b} t_\mathrm{int}}. \tag{22}$$

Maximal squeeze factor inside the cavity

Here we compare the maximally achievable squeezing inside the cavity in frequency and time domain, bringing the results of Ref. [4, 5] in accord with our notations.

Inside the cavity the squeezing spectrum of the phase quadrature is (in the assumption of frequency being much smaller than the cavity FSR):

$$S^\mathrm{in}(\Omega) = \frac{r_\mathrm{c}^2 r_\mathrm{int}^2 + t_\mathrm{c}^2 + r_\mathrm{c}^2 t_\mathrm{b}^2 t_\mathrm{int}^2 e^{-2q} + t_\mathrm{c}^2 r_\mathrm{b}^2 r_\mathrm{int}^2 t_\mathrm{int}^2 e^{-4q}}{(1 - r_\mathrm{c} r_\mathrm{b} t_\mathrm{int}^2 e^{-2q})^2 + 4 e^{-2q} r_\mathrm{c} r_\mathrm{b} t_\mathrm{int}^2 \Omega^2 \tau^2}. \tag{23}$$

At the threshold value in the limiting case $r_\mathrm{f,b} \to 1, r_\mathrm{int} \to 0$ the amount of squeezing approaches

$$S^\mathrm{in}(0)/S^\mathrm{in}(0)_{q=0} \to \frac{1}{4}, \tag{24}$$

which is what we call the 6 dB squeezing limit. On the other hand, sometimes in the literature one can find 3 dB as the intra-cavity limit for squeezing. This limit refers to the maximal reduction in the noise variance of the cavity mode. Indeed, by integrating the spectrum (23) over the full frequency range and applying the same limits one finds the value of 1/2 as a limit.

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