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A NOTE ON SPIN TWO FIELDS IN CURVED BACKGROUNDS

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Abstract: We reconsider the consistency constraints on a free massless symmetric, rank 2, tensor field in a background and confirm that they uniquely require it to be the linear deviation about (cosmological) Einstein gravity. Neither adding non-minimal higher derivative terms nor changing the gauge transformations by allowing terms non-analytic in the cosmological constant alters this fact.

Higher ($s > 1$) spin fields are well-known to encounter consistency problems in curved backgrounds. This is especially manifest for massless systems (at least in finite numbers). The borderline case is spin 2, where the consistency constraints involve only the Ricci – rather than the full Riemann – tensor [1]. Our note intends to fill a minor gap in the (correct) belief that a free spin-2 field in a background describes small excitations off Einstein gravity. We show that the constraints found in previous treatments cannot be alleviated even by adding nonminimal terms or by exploiting an apparent additional freedom in gauge transformations involving terms non-analytic in the cosmological constant.

We follow the notation of [2], where details and conventions may be found. The action describing a (for notational convenience only) contravariant tensor density field $h^{\mu\nu}$ in a metric background is

$$I_2[h] = \int d^4x h^{\mu\nu} \theta_{\mu\nu\alpha\beta} h^{\alpha\beta} \tag{1}$$

where θ is the appropriate second order hermitian operator (generalizing that in a flat background) that yields the field equation

$$2G^L_{\mu\nu}(h) \equiv \Box h_{\mu\nu} - (D^\lambda D_\nu h_{\mu\lambda} + D^\lambda D_\mu h_{\nu\lambda}) + g_{\mu\nu} D^\alpha D^\beta h_{\alpha\beta} = 0 \tag{2}$$

Here all operators, including covariant derivatives D_μ, are with respect to the background metric $g_{\mu\nu}$. Choosing a different ordering of derivatives in θ would lead to non minimal coupling terms $\sim Rh$ in $G^L_{\mu\nu}$. There is in any case no ordering that preserves the Bianchi identity $D^\mu G^L_{\mu\nu} = 0$ of flat space. In terms of the action (1), the deviation from the Bianchi identity is

$$\begin{aligned} \delta I_2[h] &= -2 \int \xi^\mu D^\nu G^L_{\mu\nu}(h) \\ &\equiv \int d^4x \xi^\mu \left[R_{\mu\sigma} D_\nu h^{\sigma\nu} + \frac{1}{2} \left(D_\alpha R_{\mu\beta} + D_\beta R_{\mu\alpha} - D_\mu R_{\alpha\beta} \right) h^{\alpha\beta} \right] \end{aligned} \tag{3}$$

under the gauge transformation

$$(-g)^{-\frac{1}{2}} \delta_0 h_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu - g_{\mu\nu} D^\alpha \xi_\alpha . \tag{4}$$

In other words, consistency – i.e., vanishing of (3) for arbitrary ξ^μ and $h^{\alpha\beta}$ – requires that the background be Ricci-flat ($R_{\alpha\beta} = 0$). This "obstruction" may be restated as the fact that the system is the linearized deviation of the dynamical Einstein contravariant metric density, defined by

$$\tilde{\mathfrak{g}}^{\mu\nu} = \mathfrak{g}^{\mu\nu} + h^{\mu\nu} \tag{5}$$

as further explained in [2].

Adding nonminimal terms $\sim \int d^4xhRh$ - as would also result from a different ordering - does not cure the difficulty. Indeed, the most general quadratic terms[1] that can be added to the action (with proper background covariance) are

$$\begin{aligned} I_{NM} &= a \int d^4x h^{\mu\nu} R_{\mu\alpha\nu\beta} h^{\alpha\beta} (-g)^{-\frac{1}{2}} + b \int d^4x h^{\mu\alpha} R_{\alpha\beta} h^{\beta}{}_{\mu} (-g)^{-\frac{1}{2}} \\ &+ c \int d^4x h^{\mu\nu} R_{\mu\nu} h (-g)^{-\frac{1}{2}} + \int d^4x \left(d\ Rh^2 + e\ Rh_{\mu\nu} h^{\mu\nu} \right) (-g)^{-\frac{1}{2}} \end{aligned} \tag{6}$$

where $h \equiv h^{\alpha}{}_{\alpha}$. The term involving the full Riemann tensor only makes matters worse, leaving Riemann dependence in the field equations and Bianchi "identities", which would pick up the term $-2a\xi^{\nu} R_{\mu\alpha\nu\beta} D^{\mu} h^{\alpha\beta}$, thereby forcing flatness. Hence one must take $a = 0$. Similarly, the terms involving the Ricci tensors leave uncancelled $\xi^{\alpha} R_{\alpha\beta} D^{\beta} h$ unless $c = 0$ or $\xi^{\mu} R_{\alpha\beta} D_{\mu} h^{\alpha\beta}$ unless b also vanishes. Finally, the Ricci scalar terms clearly cannot eliminate the Ricci tensor in the violation of the Bianchi identities. Accordingly, the terms in (3) cannot be cancelled by variation of (6), in agreement with the arguments given in [1].

The above procedure can be generalized slightly – but significantly – by addition of a "cosmological deviation" term

$$I_C = -\frac{\Lambda}{4} \int d^4x \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) (-g)^{-\frac{1}{2}} , \tag{7}$$

whose variation under (4) is

$$\delta I_C = \Lambda \int d^4x \xi_{\mu} D_{\nu} h^{\mu\nu} , \tag{8}$$

which in turn shifts the Ricci tensor term in (3) by the cosmological addition $R_{\mu\nu} \to R_{\mu\nu} + \Lambda g_{\mu\nu}$. Correspondingly, the h-field of (5) is now interpreted as

[1]The terms allowed for improving the action must be quadratic in h: allowing terms linear in h would merely amount to reconstructing the Einstein action expanded about a metric which is not solution of the Einstein equations, and hence would not help.

the perturbation off Einstein gravity with a cosmological constant: This is what consistency now requires of the background. Note that our methodology differs slightly from that of [2], adding a dynamical term to the spin 2 field and finding that a change is induced on the background, rather than embedding it in a cosmological background ab initio.

The remaining hope then is whether the cosmological term (7) can be used to alter the background constraints. That is, can we alter the gauge transformations such that the constraint terms requiring $(R_{\mu\nu} + \Lambda g_{\mu\nu}) = 0$ are removed? This route will now be seen to be ineffective as well.

Consider a modification of the gauge change non-analytic in Λ, permitted by the cosmological term[2],

$$\delta h_{\mu\nu} = \delta_0 h_{\mu\nu} + \delta_1 h_{\mu\nu}, \;\; \delta_1 h_{\mu\nu} = (1/\Lambda)\Theta_{\mu\nu}{}^{\alpha}\xi_{\alpha} \tag{9}$$

where $\Theta_{\mu\nu}{}^{\alpha}$ is an operator of the form $R_{\mu\nu}D^{\alpha}$ or $D^{\alpha}(R_{\mu\nu} \quad)$. The idea is to take advantage of the $\Lambda h^2_{\mu\nu}$ term (7) in the action by adjusting the $1/\Lambda$ term in (9) to cancel the constraint (3). This can indeed be done, but leaves two residues: the first are $(1/\Lambda)O(R^2)$ terms from $\delta_1 I_2$, which can (perhaps) be removed in turn by an iterative procedure, $\delta_2 h_{\mu\nu} \sim (1/\Lambda^2)O(R^2)$, etc. However, it fails for the very simple reason that nothing removes the variation $\delta_0 I_C \sim \int d^4x \xi_\nu (D_\mu h^{\mu\nu})$ of the cosmological term, which is the first term in the expansion of the action. A Λ^2-term in the action would be needed, but (purely on dimensional grounds) there is no local candidate.

One may reformulate the above results in cohomological terms. The BRST structure for a free spin-2 field in Minkowski space has been investigated in [4]. It can then be shown that the deformation of the model corresponding to a change in the background is consistent, i.e., defines a cohomological class of the BRST differential at ghost number zero, only if the modified background is also a solution of the Einstein equations.

We conclude that there is indeed but one consistent spin-2 model, and so incidentally only one graviton.

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[2]This use of non-analytic terms to restore consistency is borrowed from the Russian school (e.g., [3] and subsequent work) where it is used for infinite towers of spins.

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