HOPs and COPs: Frames with partitionable transversals

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Abstract

In this paper, we construct Room frames with partitionable transversals. Direct and recursive constructions are used to find sets of disjoint complete ordered partitionable (COP) transversals and sets of disjoint holey ordered partitionable (HOP) transversals for Room frames. Our main results include upper and lower bounds on the number of disjoint COP transversals and the number of disjoint HOP transversals for Room frames of type 2^n. This work is motivated by the large number of applications of these designs.

1 Introduction and Definitions

Let S be a set, and let $\{S_1, \ldots, S_n\}$ be a partition of S. An $\{S_1, \ldots, S_n\}-$*Room frame* is an $|S| \times |S|$ array, F, indexed by S, which satisfies the following properties:

1. every cell of F either is empty or contains an unordered pair of symbols of S,
2. the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq n$ (these subarrays are referred to as *holes*),
3. each symbol $x \notin S_i$ occurs once in row (or column) s, for any $s \in S_i$,
4. the pairs occurring in F are $\{s, t\}$, where $(s, t) \in (S \times S) \setminus \bigcup_{i=1}^{n} (S_i \times S_i)$.

An $\{S_1, S_2, \ldots, S_n\}$-Room frame F is said to be *skew* if at most one of the cells (i, j) and (j, i) $(i \neq j)$ is nonempty. The *type* of a Room frame F is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. We usually use an "exponential" notation to describe types: a type ${t_1}^{u_1}{t_2}^{u_2} \ldots {t_k}^{u_k}$ denotes that there are u_i holes of size t_i, $1 \leq i \leq k$. The *order* of the Room frame is $|S|$. As is done in the literature, we often refer to a Room frame simply as a *frame*.

Let F be an $\{S_1, S_2, \ldots, S_n\}$-Room frame defined on S. A *complete transversal* is a set T of $|S|$ filled cells in F (one in each row and column) such that every element of S is contained in exactly two cells of T. If the pairs in the cells of T are ordered so that every element occurs once as a first coordinate and once as a second coordinate in a cell of T, then T is said to be an *ordered transversal*. If $|S|$ is even and the pairs in T can be partitioned into two subsets T_1 and T_2 ($|T_1| = |T_2| = \frac{|S|}{2}$) so that every element occurs precisely once in each of T_1 and T_2, then T is called *partitionable*. Note that a transversal is partitionable if and only if the 2-factor formed by the pairs in the cells is a union of *even* cycles. A complete ordered partitionable transversal is called a *COP transversal*.

Example 1.1 *A Room frame of type* 2^6 *with a COP transversal* T. *(The transversal* T *is in the boxed cells.)* $T = T_1 \cup T_2$ *where*

$$T_1 = \{(9,5),(11,7),(3,10),(12,1),(6,4),(8,2)\} \quad \textit{and}$$

$$T_2 = \{(5,11),(7,9),(2,3),(10,12),(4,8),(1,6)\}.$$

(See Lemma 3.6 for a description of the construction of this frame.)

			6 8	**3 10**	11 9	5 12					7 4
					3 7	4 9	**10 12**	8 5	6 11		
				12 9	10 8			2 11		7 5	**1 6**
9 5				11 8			2 6		12 7	1 10	
	7 10		1 9				4 11	12 3	**8 2**		
	12 4	11 10				**3 2**		7 1			9 8
3 11	9 6				**12 1**				5 4		10 2
10 4	**11 5**	12 6							1 3	2 9	
6 7			2 12			1 11				**4 8**	5 3
8 12		5 2	**7 11**	1 4						6 3	
	8 3	**9 7**			2 4	6 10	5 1				
		8 1	10 5	2 7			3 9	**6 4**			

Example 1.2 *A skew Room frame of type* 4^4 *[18] with 6 COP transversals.*

					11 32	30 31		21 33	03 13				01 12		10 23
				13 30			32 33	01 11	23 31			03 10		12 21	
				10 11			31 12			01 13	23 33		30 03		21 32
					12 13	33 10				21 31	03 11	32 01		23 30	
12 33			31 32					02 13		11 20		22 30	00 10		
	10 31	33 30							00 11		13 22	02 12	20 32		
	11 32	32 13						31 00		22 33				02 10	20 30
13 10			30 11						33 02		20 31			22 32	00 12
		31 03	13 23		20 33		11 22						21 02	00 01	
		11 21	33 01	22 31		13 20						23 00			02 03
11 23	33 03				31 02		00 13					20 21			01 22
31 01	13 21			33 00		02 11							22 23	03 20	
21 30		12 23				32 00	10 20	22 03			01 02				
	23 32		10 21			12 22	30 02		20 01	03 00					
32 03		01 10		12 20	30 00				21 22	02 23					
	30 01		03 12	32 02	10 22			23 20			00 21				

6 disjoint COP transversals for the skew Room frame of type 4^4

T_1^1	10,23	12,21	30,03	32,01	11,22	13,20	31,02	33,00
T_2^1	02,13	00,11	22,33	20,31	21,30	23,32	01,10	03,12
T_1^2	11,20	13,22	31,00	33,20	12,23	10,21	32,03	30,01
T_2^2	01,12	03,10	21,32	23,30	20,33	22,31	00,13	02,11
T_1^3	00,01	02,03	20,21	22,23	31,32	33,30	11,12	13,10
T_2^3	01,02	03,00	21,22	23,20	30,31	32,33	10,11	12,13
T_1^4	21,02	23,00	01,22	03,20	12,33	10,31	32,13	30,11
T_2^4	22,03	20,01	02,23	00,21	11,32	13,30	31,12	33,10
T_1^5	22,30	20,32	02,10	00,12	13,23	11,21	33,03	31,01
T_2^5	03,13	01,11	23,33	21,31	10,20	12,22	30,00	32,02
T_1^6	00,10	02,12	20,30	22,32	31,03	33,01	11,23	13,21
T_2^6	21,33	23,31	01,13	03,11	32,00	30,02	12,20	10,22

Example 1.3 *A skew Room frame of type* 2^5 *with four disjoint COP transversals. The transversals are* T^i *and the ordered pairs marked with* $*$ *give the partitions* T_1^i, $1 \leq i \leq 4$. *(See Lemma 3.5 for a description of the construction of this frame.)*

Room frame number 63

			79	36			58		24
		68			37	49		25	
	69			18			04		57
78					09	15		46	
	27	19				38			06
26			08				39	17	
59			14		28			03	
	48	05		29					13
	35	47			16	02			
34			56	07			12		

T^1	T^2	T^3	T^4
7,9	3,6	5,8*	2,4*
8,6*	7,3*	9,4	5,2
0,4*	5,7	8,1	6,9
5,1	6,4*	0,9*	8,7*
6,0	1,9	7,2	3,8
1,7*	8,0*	2,6*	9,3*
2,8	9,5*	3,0	4,1
9,2*	4,8	1,3*	0,5*
3,5*	0,2	4,7*	1,6*
4,3	2,1*	6,5	7,0

Let F be an $\{S_1, S_2, \ldots, S_n\}$-Room frame defined on S. A *holey transversal with respect to the hole* S_i is a set T of $|S - S_i|$ filled cells in F, one in each row and column of the array indexed by $S - S_i$, such that every element of $S - S_i$ is contained in exactly two cells of T. If the pairs in the cells of T are ordered so that every element occurs once as a first coordinate and once as a second coordinate in a cell of T, then T is said to be a *holey ordered transversal*. If $|S - S_i|$ is even and the pairs in T can be partitioned into two subsets T_1 and T_2 so that every element in $S - S_i$ occurs precisely once in each of T_1 and T_2, then T is called *partitionable.* A holey ordered partitionable transversal is called a *HOP transversal.*

In this paper, we are only interested in the existence of sets of disjoint HOP transversals with respect to the same hole, say W, in a frame. For convenience, we often abbreviate our notation and simply refer to a set of disjoint HOP transversals with respect to hole W in a frame F as a set of disjoint HOP transversals in F. (For applications and information on constructions of sets of disjoint HOP transversals with respect to different holes, see [14, 11].)

Example 1.4 *A Room frame of type 2^5 with a HOP transversal. The Room frame displayed in Example 1.3 contains a HOP transversal T with respect to the hole $H = \{0, 1\}$. $T = T_1 \cup T_2$ where $T_1 = \{(4,7),(5,6),(2,9),(3,8)\}$ and $T_2 = \{(8,2),(9,3),(6,4),(7,5)\}$.*

Our study of Room frames with partitionable transversals is motivated by the large number of applications of these designs. Recursive constructions which use Room frames with partitionable transversals have been used to help settle the existence of several different types of designs including skew Howell designs [13], three orthogonal partitioned incomplete Latin squares ($OPILS$) of type t^n [11], partitioned generalized balanced tournament designs with block size 3 [12], and Howell designs with sub-designs [4]. They have also been used to help settle two problems which have been of interest for several years: the existence of incomplete Room squares (or Room frames of type $1^{n-s}s$) [9] and the existence of uniform Room frames (type t^n) [5, 10, 9].

The purpose of this paper is to construct Room frames with sets of disjoint COP transversals and with sets of disjoint HOP transversals. In the next section, we describe bounds for the maximum number of COP and HOP transversals for frames of type $(2t)^n$. Section 2 also contains some direct constructions for Room frames with partitionable transversals. In Section 3, we discuss the computer algorithms which were used to generate sets of COP and HOP transversals for several small orders. Recursive constructions are contained in Section 4. In Section 5, we combine our results for small orders with the recursive constructions to obtain general results about sets of COP and HOP transversals for Room frames of type 2^n. Finally, in Section 6, we apply our existence results and show how frames with partitionable transversals can be used to construct almost uniform Room frames.

2 Bounds and Direct Constructions

We first find bounds for the maximum number of disjoint partitionable transversals for Room frames of type $(2t)^n$. Let $C(T)$ denote the maximum number of disjoint COP transversals in a frame of type T, and let $H(T)$ denote the maximum number of disjoint HOP transversals with respect to some hole W in a frame of type T.

Lemma 2.1 $C((2t)^n) \leq t(n-1)$.

Proof: There are $2tn(t(n-1))$ distinct pairs in a frame of type $(2t)^n$. Since each COP transversal contains $2tn$ pairs, $C((2t)^n) \leq ((2tn)t(n-1)/2tn) = t(n-1)$. □

Corollary 2.2 $C(2^n) \leq n-1$.

These bounds can be met. The frame of type 4^4 in Example 1.2 has a set of six COP transversals ($t=2, n=4$), and the frame of type 2^5 in Example 1.3 has a set of four disjoint COP transversals. Recursive constructions can be used to find infinite classes which meet this bound (see Section 4).

Lemma 2.3 $H((2t)^n) \leq t(n-4)$.

Proof: Let F be a frame of type $(2t)^n$ defined on V with the first hole defined on a set W where $|W| = 2t$. A HOP transversal with respect to W consists of $2t(n-1)$ pairs, one in each row and column of the array indexed by $V-W$. In addition, the elements in W do not occur in a HOP transversal with respect to W. Thus, there are $2tn(t(n-1))-4t^2(n-1)-4t^2(n-1) = 2t(n-1)t(n-4)$ pairs which can be used for HOP transversals with respect to W. This gives $H((2t)^n) \leq (2t(n-1)t(n-4))/(2t(n-1)) = t(n-4)$ □

Corollary 2.4 $H(2^n) \leq n-4$.

This bound can also be met. The frame of type 2^5 in Example 1.4 has a single HOP transversal with respect to the hole $\{0,1\}$.

Our first direct construction uses starters and adders. Let G be an additive abelian group of order g, and let H be a subgroup of G of order h, where $g-h$ is even. A frame starter in $G-H$ is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq \frac{g-h}{2}\}$ which satisfies the following two properties.

(1) $\{s_i : 1 \le i \le \frac{g-h}{2}\} \cup \{t_i : 1 \le i \le \frac{g-h}{2}\} = G - H$.

(2) $\{\pm(s_i - t_i) : 1 \le i \le \frac{g-h}{2}\} = G - H$.

An adder for S is an injection $A : S \to G - H$ such that

$$\bigcup_{i=1}^{(g-h/2)} \{s_i + A(s_i, t_i), t_i + A(s_i, t_i)\} = G - H.$$

It is well known that a frame starter and adder in $G - H$ can be used to construct a frame of type $h^{g/h}$, [7]. Any pair $\{x, y\}$ in the frame starter S can be used to construct a complete ordered partitionable transversal if the difference $x - y$ has even order in G. (This transversal is just the development of the pair $\{x, y\}$ under G.)

Frame starters and adders are used to construct the following infinite class.

Theorem 2.5 *[13, 7] If* $q \equiv 1 \pmod 4$ *is a prime power, then there exists a skew frame of type* (2^q) *with* $(q-1)/2$ *disjoint COP transversals.*

Corollary 2.6 *If* $q \equiv 1 \pmod 4$ *is a prime power, then* $C(2^q) \ge (q-1)/2$.

We use frame starters and adders for several small orders of n. Unfortunately, the following result tells us that we cannot easily use frame starters and adders to find sets of disjoint COP transversals in frames of type 2^n for all small orders.

Theorem 2.7 *[7] There does not exist a frame starter for a frame of type* 2^n *for* $n \equiv 2, 3 \pmod 4$.

Lemma 2.8 *There exists a frame of type* 2^n *with a set* T *of* ℓ *disjoint COP transversals for the following parameter sets* (n, ℓ)*: (8,4), (9,4), (12,6), (13,7), (16,8), (17,8), and (20,10).*

Proof: Each of these is constructed from a frame starter and adder pair. Table 2.1 contains a list of the references for the starter-adder pairs. □

Table 2.1
Starter-adder constructions for frames of type 2^n, $n \in \{8, 9, 12, 13, 16, 17\}$

frame	COPs	reference
2^8	4	[17]
2^9	4	Theorem 2.5
2^{12}	6	[17]
2^{13}	7	Theorem 2.5
2^{16}	8	[17]
2^{17}	8	Theorem 2.5
2^{20}	10	[15]

Frame starters and adders can sometimes be used to find HOP transversals. The starter and adder construction which was used for Theorem 2.5 can also be used to find a HOP for frames of type 2^q.

Theorem 2.9 *[13] If $q \equiv 1 \pmod 4$ is a prime power, then there exists a skew frame of type (2^q) with a HOP transversal.*

Corollary 2.10 *If $q \equiv 1 \pmod 4$ is a prime power, then $H(2^q) \geq 1$.*

We use intransitive frame starters and adders to construct frames with HOP transversals for several small orders. Let G be an abelian group of order g, and let H be a subgroup of order h where both g and h are even. Let k be a positive integer. A $2k$-intransitive frame starter-adder (IFSA) in $G - H$ is a quadruple (S, C, R, A) where

$$S = \{\{s_i, t_i\} : 1 \leq i \leq \tfrac{g-h}{2} - 2k\} \cup \{u_i : 1 \leq i \leq 2k\}$$

$$C = \{\{p_i, q_i\} : 1 \leq i \leq k\}$$

$$R = \{\{p'_i, q'_i\} : 1 \leq i \leq k\}.$$

$A : S \rightarrow G - H$ is an injection that satisfies the following properties:

(i) $S \cup C = G - H$

(ii) $\bigcup_i (\{s_i + A(s_i, t_i)\} \cup \{t_i + A(s_i, t_i)\} \cup \{u_i + A(u_i)\} \cup \{p'_i\} \cup \{q'_i\}) = G - H$

(iii) $\bigcup_i \{\pm(s_i - t_i), \pm(p_i - q_i), \pm(p'_i - q'_i)\} = G - H$

(iv) each element $p_i - q_i$ and $p_i' - q_i'$ has even order for $1 \le i \le k$.

A $2k$-IFSA in $G - H$ can be used to construct a frame of type $h^{g/h}(2k)$, see [18]. S is the starter and A is the adder. Note that each pair $\{s_i, t_i\}$ in S can be used to construct a holey ordered transversal with respect to the hole of size $2k$. If we can partition such a transversal, then we can construct sets of disjoint HOP transversals with respect to the hole of size $2k$. For example, the pair $\{8, 13\}$ can be used to generate a HOP transversal in the frame of type 2^8 constructed by the following 2-IFSA.

S	1,3	4,10	5,9	8,13	∞_1,2	∞_2,6
A	5	8	4	2	9	11
C	11,12					
R	2,5					

Lemma 2.11 *There exists a frame of type 2^n with a set T of ℓ disjoint HOP transversals for the following parameter sets (n, ℓ): (8,1), (9,1), (10,2), (11,3), (12,3), (13,4), (14,4), (15,5), (16,5), (17,6), (18,6), and (19,7).*

Proof: Each of these is constructed from an intransitive frame starter and adder pair. Table 2.2 contains a list of the references for the IFSAs.□

Table 2.2
2-IFSAs for frames of type 2^n, $8 \le n \le 19$

frame	HOPs	reference
2^8	1	Example above
2^9	1	Theorem 2.9
2^{10}	2	[17]
2^{11}	3	[15]
2^{12}	3	Appendix 1
2^{13}	4	Appendix 1
2^{14}	4	[17]
2^{15}	5	[15]
2^{16}	5	Appendix 1
2^{17}	6	Appendix 1
2^{18}	6	[17]
2^{19}	7	[15]

3 Small Orders

In this section we discuss the computer methods used to construct some of the frames of small orders. In order to describe these algorithms we must first give some definitions and background.

Let V be a set of v vertices, and let $\{V_1, \ldots, V_n\}$ be a partition of V, where $|V_i| = v_i$. Let $K_{v_1,\ldots,v_n}$ denote the complete multipartite graph with vertices V and with parts $\{V_1, \ldots, V_n\}$. These parts are the *holes* since $K_{v_1,\ldots,v_n}$ does not contain any edges with both vertices in V_i, $1 \leq i \leq n$. A *holey factor* of $K_{v_1,\ldots,v_n}$, missing hole V_i, is a one-factor of the graph $K_{v_1,\ldots,v_n} \setminus V_i$ (i.e. a set of edges such that each vertex of $K_{v_1,\ldots,v_n} \setminus V_i$ is on exactly one of these edges and there is no edge between any two vertices in the same hole). A *holey factorization* of $K_{v_1,\ldots,v_n}$ is a partition of the edges of the graph into holey factors such that for each i, $1 \leq i \leq n$, there are exactly v_i holey factors missing hole V_i.

The *type* of a holey factorization with holes $\{V_1, V_2, \ldots, V_n\}$ is the multiset $\{|V_1|, |V_2|, \ldots, |V_n|\}$. A holey factorization has *type* $T = t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$ if there are u_i V_j's of cardinality t_i, $1 \leq i \leq k$. Note that the rows (and the columns) of a Room frame of type T constitute a holey factorization of type T.

Two holey factorizations F and G, both of type T, are *orthogonal* if the following properties are satisfied:

1. if e_1 and e_2 are any two edges of the underlying graph which are in the same holey factor in F, then they are different holey factors of G; and

2. any holey factor in F and any holey factor in G which are missing the same hole have no edges in common.

A *complete transversal* in a holey factorization is a set of edges with the property that every vertex in the underlying graph occurs twice and every holey factor contains exactly one edge of the transversal. *Holey transversals with respect to a hole* V_i are defined similarly. *Ordered* and *partitionable transversals* of holey factorizations are defined as they are for Room frames. The following theorem describes the relationship between holey factorizations with transversals and frames with transversals.

Theorem 3.1 *[3] The existence of a pair of orthogonal holey factorizations of type T which both contain the same COP (HOP, respectively) transversal* *is equivalent to the existence of a Room frame of type T with a COP (HOP, respectively) transversal.*

In [8] a hill-climbing algorithm is presented for the construction of one-factorizations and orthogonal one-factorizations of K_n. This program is easily modified to find orthogonal holey factorizations of type T. This modified algorithm has been used successfully in the past to find many special frames of small orders; see for example [6, 9]. For this project it was necessary to modify the program to find frames with transversals.

Three different algorithms were implemented to deal with the small cases of frames with transversals.

Algorithm 3.2 *Frame.transversal.*

1. Generate a random Room frame R by hill-climbing;
2. Find all COP (HOP) transversals for R by an exhaustive backtracking search;
3. Use a clique program to find the maximal set of disjoint transversals in R.

Algorithm 3.3 *Fixed.*

1. Assume the underlying graph has v vertices. Fix some set of prescribed edges $E = e_1, e_2, \ldots e_r$ and give two assignments $h_i : \{1, 2, \ldots r\} \rightarrow \{1, 2, \ldots v\}$, $i = 1, 2$.
2. Using hill-climbing, generate a holey factorization $F_1 = f_1, f_2, \ldots f_v$ with the property that for each i with $1 \leq i \leq r$, $e_i \in f_{h_1(i)}$. (This is easily implemented by putting the edges of E in their prescribed factors and *never* allowing them to be moved during the hill-climb.)
3. Using hill-climbing, generate a holey factorization $F_2 = g_1, g_2, \ldots g_v$ with the property that for each i, $1 \leq i \leq r$, $e_i \in g_{h_2(i)}$ and with the additional property that F_2 is orthogonal to F_1.

Algorithm 3.4 *Transv.*

1. Input a Room frame R.
2. Perform an exhaustive backtracking search of R to find one transversal.

We are now in a position to discuss the construction of several Room frames that were found using these algorithms. We do this by order of the frame. All of the specific frames and the transversals for this section can be found at the URL; `http://www.emba.uvm.edu/~dinitz/transv.html`.

Lemma 3.5 *There exists a frame of type* 2^5 *with four disjoint COP transversals.*

Proof: There are exactly 64 nonisomorphic Room frames of type 2^5, see [3]. For each of these frames steps 2 and 3 of *Frame.transversal* were run. Frame number 63 is the only frame of type 2^5 with a set of four disjoint COP transversals. The frame and the four transversals T^i, $1 \leq i \leq 4$ are displayed in Example 1.3.

Lemma 3.6 *There exists a frame of type* 2^6 *with a COP transversal.*

Discussion: The program *Frame.transversal* was used. 163 different 2^6 frames were found before one was found with a COP transversal. Several were found with *nonpartionable* transversals before finding the solution. This frame is displayed in Example 1.1.

Lemma 3.7 *There exists a frame of type* 2^7 *with a COP transversal.*

Discussion: The program *Fixed* was used with one fixed transversal. At least 245,225 runs were made before a frame with a COP transversal was found. Step 2 of the algorithm always completed, but the algorithm failed in step 3 with an average deficit of 4.79. This number represents the number of vertices not yet placed in the square. See [6, 8] for a discussion of the significance of this deficit number. In addition, several attempts were made using the program *Fixed* with one fixed transversal plus several fixed cells of a hoped-for second transversal; none of these attempts were successful.

Lemma 3.8 *There exists a frame of type* 2^9 *with a set of five disjoint COP transversals.*

Discussion: For this result we input the very nice frame of type 2^9 constructed in [16], and then we ran steps 2 and 3 of *Frame.transversal*. 62 COP transversals were found for this frame, and there are several maximal sets of five disjoint COP transversals. In addition, there exists a set of six disjoint CO (but not partitionable) transversals in this frame.

Lemma 3.9 *There exists a frame of type* 2^{10} *with a set of three disjoint COP transversals.*

Discussion: The program *Frame.transversal* was used. It was run four times, each run taking about 3 hours (on a 15 MIP workstation). The table below gives the values of the total number of COP transversals found and the maximum number of disjoint COP transversals for each of the four runs.

Frame number	1	2	3	4
Total number of COP transversals	15	7	19	12
Maximum number of disjoint COP transversals	3	2	2	2

Lemma 3.10 *There exists a frame of type* 2^{11} *with a set of four disjoint COP transversals.*

Discussion: We again used the program *Frame.transversal*. Now, however, the exhaustive search for all COP transversals (step 2) took 3 days of CPU time. The program was run three times. The table below summarizes the results of these three runs.

Frame number	1	2	3
Total number of COP transversals	95	107	111
Maximum number of disjoint COP transversals	3	3	4

Lemma 3.11 *There exists a frame of type* 2^{14} *with a set of five disjoint COP transversals.*

Discussion: A Room frame R of type 2^{14} was generated by the hill-climbing algorithm. The algorithm *Transv* was then run and a transversal T_1 was found. The algorithm *Transv* was again run, now on $R - T_1$. A transversal T_2 was found. The algorithm *Transv* was run again on $R - T_1 - T_2$. This process was repeated until five disjoint COP transversals were found in R.

Lemma 3.12 *There exists a frame of type* 2^{15} *with a set of four disjoint COP transversals.*

Discussion: The algorithm *Fixed* was used to find a frame F of type 2^{15} with three disjoint COP transversals. This took 96,765 attempts where the average deficit (in step 3) of those attempts which failed was 4.85. The algorithm *Transv* was then run on R (minus the three existing transversals) and in about a week of CPU time a fourth transversal was found.

Lemma 3.13 *There exists a frame of type* 2^{18} *with a set of four disjoint COP transversals.*

Discussion: The algorithm *Fixed* was used to find this frame.

Lemma 3.14 *There exists a frame of type* 2^{19} *with a set of four disjoint COP transversals.*

Discussion: The algorithm *Fixed* was used to find a frame F of type 2^{19} with four disjoint COP transversals. The algorithm was run 184 times with the average deficit (in step 3) of those attempts which failed was 5.29.

We now turn our attention to two small cases for Room frames with HOP transversals.

Lemma 3.15 *There exists a frame of type* 2^6 *with a HOP transversal.*

Discussion: The algorithm *Fixed* was used to find a frame F of type 2^6 with a HOP transversal. This took about 28,000 attempts where the average deficit (in step 3) of those attempts which failed was 4.9.

Lemma 3.16 *There exists a frame of type* 2^7 *with a HOP transversal.*

Discussion: The algorithm *Fixed* was used to find a frame F of type 2^6 with a HOP transversal. This took only 100 attempts (about 20 minutes on a slow PC). The average deficit (in step 3) of those attempts which failed was 4.32.

4 Recursive Constructions

The "filling in the holes" construction for frames can be used to find sets of disjoint COP and HOP transversals.

Theorem 4.1 *Let F be an $\{S_1, S_2, \ldots, S_n\}$-frame where $|S_i| = 2t_i$ for $i = 1, 2, \ldots, n$ and let $t = \sum_{i=1}^{n} t_i$.*

(a) If there exists a frame of type 2^{t_i} for each i, $1 \leq i \leq n$, then there is a frame of type 2^t with a set of $min\{C(2^{t_i}) : i = 1, 2, \ldots, n\}$ disjoint COP transversals.

(b) If there exists a frame of type 2^{t_i} for each i, $1 \leq i \leq n$, then there is a frame of type 2^t with a set of $max\{h(2^{t_i}) : i = 1, 2, \ldots, n\}$ disjoint HOP transversals with respect to the same hole, where $h(2^{t_i}) = min\{C(2^{t_1}), \ldots, C(2^{t_{i-1}}), H(2^{t_i}), C(2^{t_{i+1}}), \ldots, C(2^{t_n})\}$

(c) If there exists a frame of type 2^{t_i+1} for each i, $1 \leq i \leq n$, then there is a frame of type 2^{t+1} with a set of $max\{c(2^{t_i+1}) : i = 1, 2, \ldots, n\}$ disjoint COP transversals, where $c(2^{t_i+1}) = min\{H(2^{t_1+1}), \ldots, H(2^{t_{i-1}+1}), C(2^{t_i+1}), H(2^{t_{i+1}+1}), \ldots, H(2^{t_n+1})\}$.

(d) If there exists a frame of type 2^{t_i+1} for each i, $1 \leq i \leq n$, then there is a frame of type 2^{t+1} with a set of $min\{H(2^{t_i+1})\}$ disjoint HOP transversals with respect to the same hole.

The frame product can also be used to find sets of COP and HOP transversals.

Theorem 4.2 *If there is a frame of type 2^n and a pair of mutually orthogonal Latin squares of order m ($m \geq 5$), then there is a frame of type $(2^m)^n = 2^{mn}$ with $C(2^m)$ disjoint COP transversals and $min\{C(2^m), H(2^m)\}$ disjoint HOP transversals with respect to the same hole.*

Proof: First construct a frame of type $(2m)^n$ by expanding by m, then fill in the holes with frames of type 2^m (Theorem 4.1). □

The product can be modified slightly to construct larger sets of disjoint COP transversals.

Theorem 4.3 *If there exists a frame of type* 2^n *with a set of* t *disjoint COP transversals and a set of three mutually orthogonal Latin squares of order* m *where* $m \geq 5$*, then there is a frame of type* 2^{mn} *with a set of* $tm + C(2^m)$ *disjoint COP transversals.*

Proof: Expand by a factor of m to construct a frame F of type $(2m)^n$. The third orthogonal Latin square is used to expand each of the t COP transversals into a set of m disjoint COP transversals for F. We fill in the holes of F with frames of type 2^m; this gives us $tm + C(2^m)$ disjoint COP transversals for the resulting frame of type 2^{mn}. □

Similarly, we can use a frame of type 2^n with HOP transversals.

Theorem 4.4 *If there exists a frame of type* 2^n *with a set of* t *disjoint HOP transversals with respect to the same hole and a set of three mutually orthogonal Latin squares of order* m *where* $m \geq 5$*, then there is a frame of order* 2^{mn} *with a set of min*$\{tm + C(2^m), H(2^m)\}$ *disjoint HOP transversals with respect to the same hole.*

Proof: Expand by a factor of m to construct a frame F of type $(2m)^n$. Let H_1 denote the first hole in F, $|H_1| = 2m$. The third orthogonal Latin square is used to expand each of the t HOP transversals into a set of m disjoint HOP transversals in F with respect to H_1. This gives us a set of tm disjoint HOP transversals with respect to H_1. Fill in the holes of F with frames of type 2^m. The resulting frame of type 2^{mn} has at least $\min\{tm + C(2^m), H(2^m)\}$ disjoint HOP transversals with respect to a hole of size 2. □

Note that Theorem 4.3 can be used to find other examples where the upper bound for $C(2^m)$, namely $C(2^m) = m - 1$, can be met. For example, let $n = 5$, $t = 4$, and $m = 5$. Then using Theorem 4.3, we can construct a set of 24 disjoint COP transversals for a frame of type 2^{25}.

Our remaining constructions are corollaries of two very powerful constructions for frames, see [13, 4]. These constructions use incomplete orthogonal arrays, $IA(n, k, s)$s. Let V be a finite set of size n. Let K be a subset of V of size k. An incomplete orthogonal array $IA(n, k, s)$ is an $(n^2 - k^2) \times s$ array written on the symbol set V such that every ordered pair of $(V \times V) - (K \times K)$ occurs in any ordered pair of columns from the array. An $IA(n, k, s)$ is equivalent to a set of $s - 2$ mutually orthogonal Latin squares which are missing a subsquare of order k. (We need not be able to fill in the $k \times k$ missing subsquares with Latin squares of order k.) An $IA(n, k, s)$ is also known as a $TD(s, n) - TD(s, k)$.

The first construction expands and adds a new hole to a Room frame.

Theorem 4.5 *[13] Suppose there exists*

(1) a Room frame of type $(2t)^n$ with a set of ℓ disjoint COP transversals,

(2) a pair of orthogonal Latin squares of side m, and

(3) $IA(m + k_j, k_j, 4)$ where $\sum_{j=1}^{\ell} k_j = k$.

Then there exists a Room frame of type $(2tm)^n(2k)$.

We note that in the event that every filled cell of the Room frame in condition (1) of Theorem 4.5 is in a COP transversal and if for all $1 \leq j \leq \ell$, $k_j \neq 0$, then condition (2) of the theorem is not necessary. This is true as well in the corollaries to this theorem.

The first corollary uses Theorem 4.5 to construct the frame and then fills in the holes of this frame (see Theorem 4.1).

Corollary 4.6 *Let m and k be positive integers with $m, k > 4$. Suppose there exists a Room frame of type 2^n with a set of ℓ COP transversals, a pair of mutually orthogonal Latin squares of side m, and $IA(m+k_i, k_i, 4)$, $1 \leq i \leq \ell$. Let $\sum k_i = k$. Then there exists a frame of type 2^{mn+k} with $min\{C(2^m), C(2^k)\}$ disjoint COP transversals and $max\{C_1, C_2\}$ disjoint HOP transversals where $C_1 = min\{C(2^m), H(2^k)\}$ and $C_2 = min\{C(2^m),\ H(2^m), C(2^k)\}$.*

Corollary 4.7 *Let m and k be positive integers with $m, k > 4$. Suppose there exists a Room frame of type 2^n with a set of ℓ COP transversals, three mutually orthogonal Latin squares of side m, and $IA(m + k_i, k_i, 4)$ where $k_i = 0$ for at least one i, $1 \leq i \leq \ell$. Let $\sum k_i = k$. Then there exists a frame of type 2^{mn+k} with $min\{m + C(2^m), C(2^k)\}$ disjoint COP transversals and $min\{m + C(2^m), H(2^k)\}$ disjoint HOP transversals with respect to the same hole.*

Proof: We use Theorem 4.5 to construct a frame of type $(2m)^n(2k)$ defined on a set $V \cup W$ where $|V| = 2mn$ and $|W| = 2k$. Since $k_i = 0$ for some i, we expand along the ith COP transversal using 3 mutually orthogonal Latin squares of order m instead of an incomplete orthogonal array. This provides a set of m COP transversals of the $2mn \times 2mn$ array indexed by V. Now fill in the holes with frames of type 2^m and 2^k. We have $C(2^m) + m$ COP transversals of the $2mn \times 2mn$ array indexed by V, and therefore $min\{C(2^m)+m, C(2^k)\}$ disjoint COP transversals in the frame of type 2^{mn+k}. □

Note that if $k_i = 0$ for $i = 1, 2, \ldots, u$ $(u \leq \ell)$, then there is a frame of type 2^{mn+k} with $min\{C(2^m) + um, C(2^k)\}$ disjoint COP transversals.

Corollary 4.8 *Let m and k be positive integers with $m, k > 4$. Suppose there exists a Room frame of type 2^n with a set of ℓ COP transversals $\{T_1, T_2, \ldots, T_\ell\}$, a pair of mutually orthogonal Latin squares of side m, and $IA(m + k_i, k_i, 4 + w_i)$ where $w_i > 0$ for some i, $1 \leq i \leq \ell$. Let $\sum k_i = k$. Then there exists a frame of type 2^{mn+k} with $min\{\sum_{i=1}^{\ell} w_i k_i, C(2^k)\}$ disjoint COP transversals and $min\{\sum_{i=1}^{\ell} w_i k_i, H(2^k)\}$ disjoint HOP transversals with respect to the same hole.*

Proof: We use Theorem 4.5 to construct a frame of type $(2m)^n(2k)$ defined on a set $V \cup W$ where $|V| = 2mn$ and $|W| = 2k$. Since there is an $IA(m + k_i, k_i, 4 + w_i)$ (where $w_i > 0$ for some i), we can construct a set of w_i disjoint COP transversals for each of the k_i new elements added when we expand along the ith transversal T_i. This gives us a set of $min\{\sum_{i=1}^{\ell} w_i k_i, C(2^k)\}$ disjoint COP transversals and a set of $min\{\sum_{i=1}^{\ell} w_i k_i, H(2^k)\}$ disjoint HOP transversals with respect to a hole of size 2 for the frame of type 2^{mn+k}. □

The second basic construction for frames expands and enlarges one hole of a Room frame.

Theorem 4.9 *[13] Suppose there exists*

(1) a Room frame of type $(2t)^n$ with a set of ℓ disjoint HOP transversals with respect to a hole H,

(2) a pair of orthogonal Latin squares of side m, and

(3) $IA(m + k_j, k_j, 4)$ where $\sum_{j=1}^{\ell} k_j = k$.

Then there exists a Room frame of type $(2tm)^{n-1}(2tm+2k)$.

The next three corollaries are the analogues of Corollaries 4.6, 4.7 and 4.8, respectively.

Corollary 4.10 *Let* m *and* k *be positive integers with* $m, k > 4$. *Suppose there exists a Room frame of type* 2^n *with a set of* ℓ *HOP transversals with respect to a hole* H, *a pair of mutually orthogonal Latin squares of side* m, *and* $IA(m+k_i, k_i, 4)$, $1 \leq i \leq \ell$. *Let* $\sum k_i = k$. *Then there exists a frame of type* 2^{mn+k} *with* $min\{C(2^m), C(2^{m+k})\}$ *disjoint COP transversals and* $max\{C_1, C_2\}$ *HOP transversals where* $C_1 = min\{C(2^m), H(2^{m+k})\}$ *and* $C_2 = min\{C(2^m), C(2^{m+k}), H(2^m)\}$.

Corollary 4.11 *Let* m *and* k *be positive integers with* $m, k > 4$. *Suppose there exists a Room frame of type* 2^n *with a set of* ℓ *HOP transversals with respect to a hole* H, *three mutually orthogonal Latin squares of side* m, *and and* $IA(m+k_i, k_i, 4)$ *where* $k_i = 0$ *for at least one* i, $1 \leq i \leq \ell$. *Let* $\sum k_i = k$. *Then there exists a frame of type* 2^{mn+k} *with* $min\{m + C(2^m), C(2^{m+k})\}$ *disjoint COP transversals and* $min\{m + C(2^m), H(2^{m+k})\}$ *disjoint HOP transversals.*

Corollary 4.12 *Let* m *and* k *be positive integers with* $m, k > 4$. *Suppose there exists a Room frame of type* 2^n *with a set of* ℓ *HOP transversals* $\{T_1, T_2, \ldots, T_\ell\}$ *with respect to a hole* H, *a pair of mutually orthogonal Latin squares of side* m, *and* $IA(m+k_i, k_i, 4+w_i)$ *where* $w_i > 0$ *for some* i, $1 \leq i \leq \ell$. *Let* $\sum k_j = k$. *Then there exists a frame of type* 2^{mn+k} *with* $min\{\sum_{i=1}^{\ell} w_i k_i, C(2^{k+m})\}$ *disjoint COP transversals and* $min\{\sum_{i=1}^{\ell} w_i k_i, H(2^{k+m})\}$ *disjoint HOP transversals.*

5 General Results

In this section, we find lower bounds for $C(2^n)$ and $H(2^n)$ for all n. Applications of the recursive constructions require existence results for sets of mutually orthogonal Latin squares, incomplete orthogonal arrays and uniform frames. For the most recent results on sets of mutually orthogonal Latin squares and incomplete orthogonal arrays, we refer to the tables in [1] and [2], respectively. The existence of uniform Room frames was recently settled with one possible exception.

Theorem 5.1 *[9, 10, 5] Suppose t and u are positive integers, $u \geq 4$, and $(t,u) \neq (1,5),(2,4)$. Then there exists a frame of type t^u if and only if $t(u-1)$ is even, except possibly when $u = 4$ and $t = 14$.*

We first collect our results for frames of type 2^n for $5 \leq n \leq 20$ in Table 5.1.

Table 5.1
COPs and HOPs in frames of type 2^n for $5 \leq n \leq 20$

n	construction	COPs	construction	HOPs
5	3.5, computer	4	2.9	1
6	3.6, computer	1	3.15, computer	1
7	3.7, computer	1	3.16, computer	1
8	2.7, starter-adder	4	2.11	1
9	3.8, computer	5	2.9	1
10	3.9, computer	3	2.11	2
11	3.10, computer	4	2.11	3
12	2.7, starter-adder	6	2.11	3
13	2.5, starter-adder	6	2.11	4
14	3.11, computer	5	2.11	4
15	3.12, computer	4	2.11	5
16	2.7, starter-adder	8	2.11	5
17	2.5, starter-adder	8	2.11	6
18	3.13, computer	4	2.11	6
19	3.14, computer	4	2.11	7
20	2.7, starter-adder	10	4.1b, $(10)^4$ frame	1

We now construct sets of COP transversals for frames of type 2^n. When $n \equiv 0 \pmod 5$, we can use Theorem 4.3 to construct frames with a large number of disjoint COP transversals.

Lemma 5.2 *There exists a frame of type 2^{5m} with $4m + C(2^m)$ disjoint COP transversals for $m \geq 4$, $m \neq 6, 10$.*

Proof: We apply Theorem 4.3 with $n = 5$, $t = 4$, and three mutually orthogonal Latin squares of side m. □

Table 5.2 describes constructions for sets of disjoint COP transversals for frames of type 2^n for $21 \leq n \leq 71$.

Table 5.2
Constructions for COPs in frames of type 2^n for $21 \le n \le 71$

n	Authority	comments	COPs
21	4.1c	$n = 5, t_i = 4$	1
22	4.9	$t = 1, n = 5, \ell = 1, m = 4, k_1 = 1$	
	4.1c	$n = 5, t_i = 4, 1 \le i \le 4, t_5 = 5$	1
23	4.9	$t = 1, n = 5, \ell = 1, m = 4, k_1 = 2$	
	4.1c	$n = 5, t_i = 4, 1 \le i \le 4, t_5 = 6$	1
24	4.1a	$n = 4, t_i = 6$	1
25	5.2	$m = 5$	24
26	4.1c	$n = 5, t_i = 5$	1
27	4.9	$t = 1, n = 5, \ell = 1, m = 5, k_1 = 2$	
	4.1a	$n = 5, t_i = 5, 1 \le i \le 4, t_5 = 7$	1
28	4.9	$t = 1, n = 5, \ell = 1, m = 5, k_1 = 2$	
	4.1c	$n = 5, t_1 = t_2 = t_3 = t_4 = 5, t_5 = 7$	1
29	2.5	prime	14
30	4.3	$n = 6, t = 1, m = 5$	9
31	4.9	$t = 1, n = 7, \ell = 1, m = 4, k_1 = 2$	
	4.1c	$n = 7, t_i = 4, 1 \le i \le 6, t_7 = 6$	1
32	4.1a	$n = 4, t_i = 8$	4
33	4.1c	$n = 8, t_i = 4$	1
34	4.9	$t = 1, n = 8, \ell = 1, m = 4, k_1 = 1$	
	4.1c	$n = 8, t_i = 4, 1 \le i \le 7, t_8 = 6$	1
35	5.2	$m = 7$	29
36	4.1a	$n = 4, t_i = 9$	5
37	2.5	prime	18
38	4.8	$m = 6, n = 5, \ell = 4,$	
		$k_1 = 1, k_2 = k_3 = 2, k_4 = 3, (w_1 = 6)$	4
39	4.8	$m = 6, n = 5, \ell = 4,$	
		$k_1 = 1, k_2 = 2, k_3 = k_4 = 3, (w_1 = 6)$	4
40	5.2	$m = 8$	36
41	2.5	prime	20
42	4.3	$n = 6, t = 1, m = 7$	8
43	4.7	$n = 5, \ell = 4, m = 7, k_1 = 0, k_2 = k_3 = 3, k_4 = 2$	4
44	4.7	$n = 5, \ell = 4, m = 7, k_1 = 0, k_2 = k_3 = k_4 = 3$	5
45	5.2	$m = 9$	41

n	Authority	comments	COPs
46	4.10	$n=5, \ell=1, m=9, k_1=1$	3
47	4.10	$n=5, \ell=1, m=9, k_1=2$	4
48	4.3	$n=6, t=1, m=8$	12
49	2.5	prime power	24
50	4.3	$n=10, t=3, m=5$	19
51	4.7	$n=5, \ell=4, m=8, k_1=3, k_2=k_3=4, k_4=0$	4
52	4.6	$n=5, \ell=4, m=8, k_1=1, k_2=k_3=4, k_4=3$	4
53	2.5	prime	26
54	4.3	$n=6, t=1, m=9$	14
55	5.2	$m=11$	48
56	4.3	$n=8, t=4, m=7$	29
57	4.6	$n=5, \ell=4, m=9, k_1=k_2=k_3=4, k_4=0$	5
58	4.6	$n=5, \ell=4, m=9, k_1=k_2=k_3=4, k_4=1$	5
59	4.6	$n=5, \ell=4, m=9, k_1=k_2=k_3=4, k_4=2$	5
60	5.2	$m=12$	54
61	2.5	prime	30
62	4.8	$m=10, n=5, \ell=4,$ $k_1=1, k_2=k_3=4, k_4=3, w_1=6$	6
63	4.3	$n=9, t=5, m=7$	36
64	4.3	$n=8, t=4, m=8$	36
65	5.2	$m=13$	58
66	4.10	$n=5, \ell=1, m=13, k_1=1$	5
67	2.5	prime	33
68	4.7	$n=5, \ell=4, m=11, k_1=0, k_2=k_3=5, k_4=3$	6
69	4.7	$n=5, \ell=4, m=11, k_1=0, k_2=k_3=5, k_4=4$	5
70	4.3	$n=14, t=1, m=5$	9
71	2.5	prime	35

Theorem 5.3 *Let n be a positive integer, $n \geq 5$. There exists a frame of type 2^n with 4 disjoint COP transversals except possibly for $n \in \{6, 7, 10, 21, 22, 23, 24, 26, 27, 28, 31, 33, 34, 46\}$.*

Proof: Lower bounds for $C(2^n)$ for $5 \leq n \leq 71$ are listed in Tables 5.1 and 5.2.

Let $n \geq 72$. If $n \equiv 0 \pmod 5$, then $C(2^n) \geq 52$ by Lemma 5.2. We write $n = 5m + \sum_{i=1}^{4} k_i$ where $m \geq 12$ and $\sum_{i=1}^{4} k_i \in \{11, 12, 13, 14\}$.

Now apply Corollary 4.7 using the 2^5 frame with a set of 4 disjoint COP transversals, $k_1 = 0$, and $k_i \in \{3,4,5\}$ for $i = 2,3,4$. This gives us $min\{m, C(2^{11}), C(2^{12}), C(2^{13}), C(2^{14})\} \geq 4$ COP transversals for frames of type 2^n and $n \geq 65$. $\square$

Theorem 5.4 *There exists a frame of type 2^n with 5 disjoint COP transversals for all $n \geq 53$.*

Proof: . The proof is identical to the one above except that now assume $m \geq 12$ and $\sum_{i=1}^{4} k_i \in \{12,13,14,16\}$ where $k_1 = 0$ and $k_i \in \{4,5,6\}$ for $i = 2,3,4$. $\square$

The next table, Table 5.3, describes constructions for sets of HOP transversals for frames of type 2^n for $21 \leq n \leq 60$.

Table 5.3
Constructions for HOPs in frames of type 2^n for $21 \le n \le 60$

n	Authority	comments	HOPs
21	4.1d	$n = 5, t_i = 4$	1
22	4.9	$t = 1, n = 5, \ell = 1, m = 4, k_1 = 1$	
	4.1d	$n = 5, t_i = 4, 1 \le i \le 4, t_5 = 5$	1
23	4.9	$t = 1, n = 5, \ell = 1, m = 4, k_1 = 2$	
	4.1d	$n = 5, t_i = 4, 1 \le i \le 4, t_5 = 6$	1
24	4.1b	$n = 4, t_i = 6$	1
25	2.10	prime power	1
26	4.1d	$n = 5, t_i = 5$	1
27	4.9	$t = 1, n = 5, \ell = 1, m = 5, k_1 = 2$	
	4.1b	$n = 5, t_i = 5, 1 \le i \le 4, t_5 = 7$	1
28	4.9	$t = 1, n = 5, \ell = 1, m = 5, k_1 = 2$	
	4.1d	$n = 5, t_i = 5, 1 \le i \le 4, t_5 = 7$	1
29	2.10	prime	1
30	4.9	$t = 1, n = 7, \ell = 1, m = 4, k_1 = 1$	
	4.1d	$n = 7, t_i = 4, 1 \le i \le 6, t_7 = 5$	1
31	4.9	$t = 1, n = 7, \ell = 1, m = 4, k_1 = 2$	
	4.1d	$n = 7, t_i = 4, 1 \le i \le 6, t_7 = 6$	1
32	4.1b	$n = 4, t_i = 8$	1
33	4.1d	$n = 4, t_i = 8$	1
34	4.9	$t = 1, n = 8, \ell = 1, m = 4, k_1 = 1$	
	4.1d	$n = 8, t_i = 4, 1 \le i \le 7, t_8 = 5$	1
35	4.2	$n = 7, m = 5$	1
36	4.1b	$n = 4, t_i = 9$	1
37	2.10	prime	1
38	4.8	$m = 6, n = 5, \ell = 4,$	
		$k_1 = 1, k_2 = k_3 = 2, k_4 = 3, w_1 = 4$	1

n	Authority	comments	HOPs
39	4.8	$m = 6, n = 5, \ell = 4,$	
		$k_1 = 1, k_2 = k_3 = 3, k_4 = 2, w_1 = 4$	1
40	4.8	$m = 6, n = 5, \ell = 4,$	
		$k_1 = 1, k_2 = k_3 = k_4 = 3, w_1 = 4$	2
41	4.8	$m = 6, n = 5, \ell = 4,$	
		$k_1 = 2, k_2 = k_3 = k_4 = 3, w_1 = 1$	2
42	4.10	$n = 5, \ell = 1, m = 8, k_1 = 2$	2
43	4.10	$n = 5, \ell = 1, m = 8, k_1 = 3$	3
44	4.10	$n = 5, \ell = 1, m = 8, k_1 = 4$	3
45	4.8	$m = 7, n = 5, \ell = 4,$	
		$k_1 = 1, k_2 = k_3 = k_4 = 3, w_1 = 5$	2
46	4.10	$n = 5, \ell = 1, m = 9, k_1 = 1$	2
47	4.10	$n = 5, \ell = 1, m = 9, k_1 = 2$	3
48	4.10	$n = 5, \ell = 1, m = 9, k_1 = 3$	3
49	4.10	$n = 5, \ell = 1, m = 9, k_1 = 4$	4
50	4.1b	$n = 5, t_i = 10$	2
51	4.6	$n = 5, \ell = 4, m = 8, k_1 = 3, k_2 = k_3 = 4, k_4 = 0$	3
52	4.6	$n = 5, \ell = 4, m = 8, k_1 = 3, k_2 = k_3 = 4, k_4 = 1$	3
53	4.6	$n = 5, \ell = 4, m = 8, k_1 = 3, k_2 = k_3 = 4, k_4 = 2$	4
54	4.6	$n = 5, \ell = 4, m = 8, k_1 = 3, k_2 = k_3 = 4, k_4 = 3$	4
55	4.6	$n = 5, \ell = 4, m = 8, k_1 = 3, k_2 = k_3 = k_4 = 4$	4
56	4.6	$n = 5, \ell = 4, m = 8, k_1 = k_2 = k_3 = k_4 = 4$	4
57	4.10	$n = 5, \ell = 1, m = 11, k_1 = 2$	4
58	4.10	$n = 5, \ell = 1, m = 11, k_1 = 3$	4
59	4.10	$n = 5, \ell = 1, m = 11, k_1 = 4$	4
60	4.10	$n = 5, \ell = 1, m = 11, k_1 = 5$	4

Theorem 5.5 *For every positive integer $n \geq 53$, there is a frame of type 2^n with 4 disjoint HOP transversals with respect to the same hole.*

Proof: Lower bounds for $H(2^n)$ for $5 \leq n \leq 60$ are listed in Tables 5.1 and 5.3.

Let $61 \leq n \leq 66$. Since there exists a frame of type 4^4 with 6 disjoint COP transversals (Example 1.2), we apply Theorem 4.5 with $m = 6$ and $k_i \in \{1, 2, 3\}$ for $i = 1, 2, \ldots, 6$ to construct a frame of type $(24)^4(2k)$ where

$26 \leq 2k \leq 36$. Note that $m = 6$ is permitted since each filled cell is in a COP transversal and $t_i > 0$ for all i. The number of HOP transversals for these frames is at least $min\{C(2^{12}), H(2^k)\} \geq 4$.

Let $67 \leq n \leq 72$. We apply Theorem 4.5 with $t = 2$, $n = 4$, $\ell = 6$, $m = 7$, $k_i \in \{0, 1, 2, 3\}$, and $11 \leq \sum_{i=1}^{6} k_i \leq 16$. The resulting frame is of type $(28)^4(2k)$ where $k = \sum_{i=1}^{6} k_i$. The number of HOP transversals is at least $max\{min\{C(2^{14}), H(2^{14}), C(2^k)\}, min\{C(2^{14}), H(2^k)\}\} \geq 4$.

Let $n \geq 73$. We write $n = 5m + k$ where $m \geq 12$, $k = \sum_{i=1}^{4} k_i$, $k_1 = 0$, $k_i \in \{4, 5, 6\}$ for $i = 2, 3, 4$ and $k \in \{13, 14, 15, 16, 17\}$. We apply Corollary 4.7 using a frame of type 2^5 and $\ell = 4$ to construct frames of type 2^n with at least $min\{m + 1, H(2^k)\} \geq 4$ disjoint HOP transversals. □

Combining the results in Tables 5.1, 5.2, and 5.3 and Theorems 5.3 and 5.4, we have the following.

Theorem 5.6 *For every positive integer $n \geq 5$ there exists a frame of type 2^n with a COP transversal.*

Theorem 5.7 *For every positive integer $n \geq 5$ there exists a frame of type 2^n with a HOP transversal.*

6 Almost Uniform Frames

The purpose of this section is to give a quick application of the results from the previous section. A Room frame of type T is *almost uniform* if $T = a^n b^1$. In the case where $a = 1$ these frames are called incomplete Room squares. Incomplete Room squares have been studied extensively and the spectrum has been determined with only one possible exception, see [9]. Frames of type 2^n with sets of disjoint COP transversals and sets of disjoint HOP transversals can be used together with Theorems 4.5 and 4.9 (respectively) to prove that almost uniform frames exist for many orders.

Lemma 6.1 *If $n \geq 47$ and $m \geq 3, m \neq 6$, then there exists a Room frame of type $(2m)^n(2k)^1$ for $0 \leq k \leq 2m - 2$.*

Proof: Since $n \geq 47$, there exists a frame of type 2^n with 4 disjoint COP transversals (Theorem 5.3). Since $m \neq 6$, there exist $IA(m + k, k, 4)$ where

$0 \le k \le \lfloor \frac{m}{2} \rfloor$, [2]. We apply Theorem 4.5 with $\ell = 4$ to construct frames of type $(2m)^n(2k)^1$, $0 \le k \le 2m-2$. □

Lemma 6.2 *If $n \ge 53$ and $m \ge 3, m \ne 6$, then there exists a Room frame of type $(2m)^{n-1}(2m+2k)^1$ for $0 \le k \le 2m-2$.*

Proof: The proof is similar to that of the previous lemma. In this case, we use the frames constructed in Theorem 5.5 with sets of disjoint HOP transversals and apply Theorem 4.9 to construct frames of type $(2m)^{n-1}(2m+2k)^1$, $0 \le k \le 2m-2$. □

Combining Lemmas 6.1 and 6.2, we have the following result for almost uniform frames.

Theorem 6.3 *If $n \ge 57$ and $m \ge 3, m \ne 6$, then there exists a Room frame of type $(2m)^n(2s)^1$ for $0 \le 2s \le 6m-4$.*

A more complete investigation of the existence of almost uniform Room frames is the subject of a further paper.

7 Appendix 1

2-IFSAs are listed for frames of type 2^n for $n \in \{12, 13, 16, 17\}$.

$n = 12$

S	20,21	12,14	3,7	1,6	13,19	8,16	9,18	5,15	∞_1,2	∞_2,4
A	15	4	5	3	6	21	14	12	18	2
C	10,17									
R	2,21									

$n = 13$

S	1,2	3,5	15,18	4,8	7,13	14,21	9,17	10,19
A	1	2	4	7	21	19	8	13
S	6,16	∞_1,20	∞_2,23					
A	14	18	22					
C	11,22							
R	13,18							

$n = 16$

S	1,2	3,5	6,9	4,8	21,26	11,17	22,29	10,18
A	1	2	3	6	29	10	24	16
S	14,24	16,28	7,20	13,27	∞_1,19	∞_2,25		
A	5	8	21	4	19	23		
C	12,23							
R	13,22							

$n = 17$

S	3,5	4,7	24,28	14,20	6,13	18,26	22,31	9,19
A	18	4	13	15	20	28	3	8
S	10,21	11,23	17,30	15,29	12,27	∞_1,8	∞_2,25	
A	10	7	30	9	24	5	17	
C	1,2							
R	7,12							

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