Generalized r-Modes of the Maclaurin Spheroids

Lee Lindblom
Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, CA 91125

James R. Ipser
Department of Physics, University of Florida, Gainesville, FL 32611

(February 7, 2008)

Analytical solutions are presented for a class of generalized $r-$modes of rigidly rotating uniform density stars—the Maclaurin spheroids—with arbitrary values of the angular velocity. Our analysis is based on the work of Bryan [5]; however, we derive the solutions using slightly different coordinates that give purely real representations of the $r-$modes. The class of generalized $r-$modes is much larger than the previously studied 'classical' $r-$modes. In particular, for each l and m we find $l-m$ (or $l-1$ for the $m=0$ case) distinct $r-$modes. Many of these previously unstudied $r-$modes (about 30% of those examined) are subject to a secular instability driven by gravitational radiation. The eigenfunctions of the 'classical' $r-$modes, the $l=m+1$ case here, are found to have particularly simple analytical representations. These $r-$modes provide an interesting mathematical example of solutions to a hyperbolic eigenvalue problem.

PACS Numbers: 97.10.Sj, 97.10.Kc, 04.30.Db

I. INTRODUCTION

During the past year the $r-$modes of rotating neutron stars have been found to play in interesting and important role in relativistic astrophysics. Andersson [1] and Friedman and Morsink [2] showed that these modes would be driven unstable by gravitational radiation reaction in the absence of internal fluid dissipation. Lindblom, Owen, and Morsink [3] have subsequently shown that this instability will in fact play an important role in the evolution of hot young neutron stars. The gravitational radiation reaction force in these modes was shown to be sufficiently strong to overcome the internal fluid dissipation present in neutron stars hotter than about 10^9 K. Hot young rapidly rotating neutron stars are expected therefore to radiate away most (i.e. up to about 90%) of their angular momentum via gravitational radiation in a period of about one year. Owen, et al. [4] have shown that the gravitational radiation emitted during this spindown process is expected to be one of the more promising potential sources for the ground based laser interferometer gravitational wave detectors (e.g. LIGO, VIRGO, etc.) now under construction.

To date the various analyses of the $r-$modes and their instability to gravitational radiation reaction have all been based on small angular velocity approximations. This instability is of primary importance in astrophysics for rapidly rotating stars. The purpose of this paper is to provide the first look at the properties of these important modes in stars of large angular velocity. We do this by solving the stellar pulsation equations for the $r-$modes of the rapidly rotating uniform density stellar models which are known as the Maclaurin spheroids. The pulsations of these models were studied over a century ago by Bryan [5], who showed how analytical expressions for all of the modes of these stars could be found. We follow the general strategy developed by Bryan to derive analytical expressions for the $r-$modes of these stars. We use somewhat different coordinates than Bryan, however, in order to obtain real representations of the $r-$modes (of primary interest to us here) using purely real coordinates.

We generalize the traditional definition of $r-$mode to include any mode whose frequency vanishes linearly with the angular velocity of the star. Such modes have as their principal restoring force the Coriolos force, and hence it is appropriate to call them rotation modes or generalized $r-$modes [6]. We find a very large number of generalized $r-$modes in the Maclaurin spheroids [7]. In particular for each pair of integers l and m (with $l \geq m \geq 0$) we find $l-m$ (or $l-1$ for the $m=0$ case) distinct $r-$modes. The 'classical' $r-$modes as studied for example by Papaloizou and Pringle [8] correspond here to the case $l=m+1$ [9]. We find that many of the previously unstudied $r-$modes (about 30% of those examined) are also subject to the gravitational radiation driven instability in stars without internal fluid dissipation. These new $r-$modes couple to higher order gravitational multipoles and consequently are expected to be of less astrophysical importance than the $l=m+1=3$ mode that is of primary importance in the instabilty discussed by Lindblom, Owen, and Morsink [3].

In Section II we review a few important facts about the equilibrium structures of the Maclaurin spheroids. In Sec. III we present the equations for the modes of rapidly rotating stars using the two-potential formalism [10]. This formalism determines all of the properties of the modes of rotating stars from a pair of scalar potentials: a hydrodynamic potential δU and the gravitational potential $\delta\Phi$. The equations satisfied by these potentials are (coupled) second-order partial differential equations with suitable boundary conditions at the surface of the star and at infinity. These equations become extremely simple in the case of uniformly rotating uniform-density stars. In Sec. IV we introduce coordinates which allow the equations for the two potentials to be solved analytically. And following in the footsteps of Bryan [5], we present the general solutions to these equations for the generalized $r-$modes of the Maclaurin spheroids. In Sec. V we give expressions for the various boundary conditions in the coordinates adapted to this problem. In Sec. VI we deduce the eigenvalue equation that determines when the boundary conditions may be satisfied. In Sec. VI we also present explicit solutions to this eigenvalue equation for a large number of generalized $r-$modes. In the limit of small angular velocity we tabulate a complete set of solutions for all generalized $r-$modes with $l \leq 6$. We also present graphically the angular velocity dependence of each of these $r-$modes which is unstable to the gravitational radiation instability. In Sec. VII we analyze the analytical expressions for the eigenfunctions of these modes. We show that in the case of the 'classical' $r-$modes, the $l = m + 1$ case here, the eigenfunctions δU and $\delta\Phi$ have particularly simple forms. In particular, each of these eigenfunctions is simply $r^{m+1}Y_{m+1\,m}(\theta,\varphi)$ (where r, θ and φ are the standard spherical coordinates) multiplied by some angular velocity dependent normalization. We also show that these 'classical' $r-$mode eigenfunctions have the unexpected property that Δp, the Lagrangian pressure perturbation, vanishes identically throughout the star. In Sec. VIII we discuss some of the interesting implications of this analysis. And finally, in the Appendix we explore in some detail the properties of the rather unusual bi-spheroidal coordinate system needed in Sec. IV to solve the pulsation equations for the hydrodynamic potential δU.

II. THE MACLAURIN SPHEROIDS

The uniformly rotating uniform-density equilibrium stellar models are called Maclaurin spheroids. The structures of these stars are determined by solving the time independent Euler equation

$$0 = \nabla_a \left[\tfrac{1}{2}(x^2 + y^2)\Omega^2 + \frac{p}{\rho} - \Phi \right]. \tag{2.1}$$

In this equation p is the pressure, ρ is the density, Ω is the angular velocity, and Φ is the gravitational potential of the equilibrium star. Using the expression for the gravitational potential of a uniform-density spheroid [11], it is straightforward to show that the solution to Eq. (2.1) for the pressure is

$$p = 2\pi G\rho^2\zeta_o^2(1 + \zeta_o^2)(1 - \zeta_o \cot^{-1}\zeta_o) \times \left(a^2 - \frac{x^2 + y^2}{1 + \zeta_o^2} - \frac{z^2}{\zeta_o^2} \right), \tag{2.2}$$

where a is the focal radius of the spheroid, G is Newton's constant, and ζ_o is related to the eccentricity e of the spheroid by $e^2 = 1/(1+\zeta_o^2)$. Similarly, it follows that the angular velocity of the star is related to the shape of the spheroid by

$$\Omega^2 = 2\pi G\rho\,\zeta_o \left[(1 + 3\zeta_o^2)\cot^{-1}\zeta_o - 3\zeta_o \right]. \tag{2.3}$$

We note that small angular velocities, Ω, correspond to small eccentricities e and large ζ_o.

The surfaces of these stellar models are the surfaces on which the pressure vanishes:

$$\frac{x^2 + y^2}{\zeta_o^2 + 1} + \frac{z^2}{\zeta_o^2} = a^2. \tag{2.4}$$

This surface is an oblate spheroid. Let us denote the equatorial and polar radii of this spheroid as R_e and R_p respectively. We see from Eq. (2.4) that

$$R_e^2 = a^2(\zeta_o^2 + 1), \tag{2.5}$$

$$R_p^2 = a^2\zeta_o^2. \tag{2.6}$$

If we consider a sequence of uniformly rotating spheroids having the same total mass, then the volume of each of the spheroids in the sequence is the same (since the density is constant). Let R denote the average radius of the spheroid: $R^3 = R_e^2 R_p$. It follows that R is constant along this sequence since the volume of a spheroid is $V = 4\pi R^3/3$. Thus the angular velocity dependence of the focal radius a is determined by

$$a^3 = \frac{R^3}{\zeta_o(\zeta_o^2 + 1)}, \tag{2.7}$$

since ζ_o is related to the angular velocity of the spheroid Ω by Eq. (2.3). This expression determines then the angular velocity dependencies of the equatorial and polar radii of the spheroid:

$$R_e = R\left(\frac{\zeta_o^2 + 1}{\zeta_o^2}\right)^{1/6}, \tag{2.8}$$

$$R_p = R\left(\frac{\zeta_o^2}{\zeta_o^2 + 1}\right)^{1/3}. \tag{2.9}$$

In the work that follows we will need the quantity $n^a\nabla_a p$, where n^a is the outward directed unit normal to the surface of the star, in order to evaluate certain boundary conditions associated with the stellar pulsations. Since $\nabla_a p$ is also normal to the surface of the star, we may use the expression $(n^a\nabla_a p)^2 = \nabla^a p \nabla_a p$ and Eq. (2.2) to obtain

$$n^a\nabla_a p = -4\pi G\rho^2\zeta_o^2(1+\zeta_o^2)(1 - \zeta_o\cot^{-1}\zeta_o) \times \left[\frac{x^2 + y^2}{(1 + \zeta_o^2)^2} + \frac{z^2}{\zeta_o^4} \right]^{1/2}, \tag{2.10}$$

where (x, y, z) are to be confined to the surface defined by Eq. (2.4).

III. THE PULSATION EQUATIONS

The modes of any uniformly rotating barotropic stellar model are determined completely in terms of two scalar potentials δU and $\delta\Phi$ [10]. The potential $\delta\Phi$ is the Newtonian gravitational potential, while δU is a potential that primarily describes the hydrodynamic perturbations of the star:

$$\delta U = \frac{\delta p}{\rho} - \delta\Phi, \tag{3.1}$$

where δp is the Eulerian pressure perturbation, and ρ is the unperturbed density of the equilibrium stellar model. We assume here that the time dependence of the mode is $e^{i\omega t}$ and that its azimuthal angular dependence is $e^{im\varphi}$, where ω is the frequency of the mode and m is an integer. The velocity perturbation δv^a is determined by solving Euler's equation. This reduces in this case to an algebraic relationship between δv^a and the potential δU:

$$\delta v^a = iQ^{ab}\nabla_b \delta U. \tag{3.2}$$

The tensor Q^{ab} depends on the frequency of the mode, and the angular velocity of the equilibrium star Ω:

$$Q^{ab} = \frac{1}{(\omega+m\Omega)^2 - 4\Omega^2} \times \left[(\omega+m\Omega)\delta^{ab} - \frac{4\Omega^2}{\omega+m\Omega}z^a z^b - 2i\nabla^a v^b\right]. \tag{3.3}$$

In Equation (3.3) the unit vector z^a points along the rotation axis of the equilibrium star, δ^{ab} is the Euclidean metric tensor (the identity matrix in Cartesian coordinates), and v^a is the velocity of the equilibrium stellar model. The potentials δU and $\delta\Phi$ are determined then by solving the perturbed mass conservation and gravitational potential equations. In this case, these reduce to the following system of partial differential equations [10]

$$\nabla_a(\rho Q^{ab}\nabla_b\delta U) = -(\omega+m\Omega)\rho\frac{d\rho}{dp}(\delta U + \delta\Phi), \tag{3.4}$$

$$\nabla^a\nabla_a\delta\Phi = -4\pi G\rho\frac{d\rho}{dp}(\delta U+\delta\Phi). \tag{3.5}$$

In addition these potentials are subject to the appropriate boundary conditions at the surface of the star for δU and at infinity for $\delta\Phi$.

The stellar pulsation Eqs. (3.4) and (3.5) simplify considerably for the case of uniformly rotating uniform-density stellar models. In this case $d\rho/dp = \rho\,\delta(p)$ [where $\delta(p)$ is the Dirac delta function]. Thus the right sides of Eqs. (3.4) and (3.5) vanish except on the surface of the star. Further, the density ρ that appears on the left side of Eq. (3.4) may be factored out. The resulting equations then in the uniform-density case are simply

$$\kappa^2\nabla^a\nabla_a\delta U - 4z^a z^b\nabla_a\nabla_b\delta U = 0, \tag{3.6}$$

$$\nabla^a\nabla_a\delta\Phi = -4\pi G\rho^2\delta(p)\,(\delta U+\delta\Phi)\,, \tag{3.7}$$

where κ is related to the frequency of the mode by

$$\kappa\Omega = \omega + m\Omega. \tag{3.8}$$

These equations are equivalent to those used by Bryan [5] in his analysis of the oscillations of the Maclaurin spheroids.

Next we wish to consider the boundary conditions to which the functions δU and $\delta\Phi$ are subject. Let Σ denote a function which vanishes on the surface of the star, and which has been normalized so that its gradient, $n_a = \nabla_a\Sigma$, is the outward directed unit normal vector there, $n^a n_a = 1$. First, the function δU must be constrained at the surface of the star, $\Sigma = 0$, in such a way that the Lagrangian perturbation in the pressure vanishes there, $\Delta p = 0$. This condition can be written in terms of the variables used here by noting that

$$\Delta p = \delta p + \left(\frac{\delta v^a}{i\kappa\Omega}\right)\nabla_a p. \tag{3.9}$$

Then using Eqs. (3.1) and (3.2) the boundary condition can be written in terms of δU and $\delta\Phi$ as

$$0 = \Big[\rho\,\kappa\Omega(\delta U+\delta\Phi) + Q^{ab}\nabla_a p\nabla_b\delta U\Big]_{\Sigma\uparrow 0}. \tag{3.10}$$

The perturbed gravitational potential $\delta\Phi$ must vanish at infinity, $\lim_{r\to\infty}\delta\Phi = 0$. In addition $\delta\Phi$ must as a consequence of Eq. (3.7) have a finite discontinuity in its first derivative at the surface of the star. In particular the derivatives must satisfy

$$\Big[n^a\nabla_a\delta\Phi\Big]_{\Sigma\downarrow 0} = \left[n^a\nabla_a\delta\Phi + \frac{4\pi G\rho^2(\delta U+\delta\Phi)}{n^a\nabla_a p}\right]_{\Sigma\uparrow 0}. \tag{3.11}$$

The problem of finding the modes of uniform-density stars is reduced therefore to finding the solutions to Eqs. (3.6) and (3.7) subject to the boundary conditions in Eqs. (3.10) and (3.11).

IV. SOLVING FOR THE POTENTIALS

In this section we find the general solutions for the potentials δU and $\delta\Phi$ that satisfy Eqs. (3.6) and (3.7). This analysis basically follows that of Bryan [5] except for some changes to modernize notation, and a change of coordinates to express in a purely real manner the solutions of interest to us here. Our primary concern here is to find expressions for the generalized $r-$modes of the Maclaurin spheroids.

We first introduce a system of spheroidal coordinates that are useful in solving for the gravitational potential $\delta\Phi$. Thus we introduce the coordinates (μ, ζ, φ) that are related to the usual Cartesian coordinates (x, y, z) by the transformation:

$$x = a\sqrt{(\zeta^2+1)(1-\mu^2)}\cos\varphi, \tag{4.1}$$

$$y = a\sqrt{(\zeta^2+1)(1-\mu^2)}\sin\varphi, \tag{4.2}$$

$$z = a\zeta\mu, \tag{4.3}$$

where a is defined in Eq. (2.4) above. These are the standard oblate spheroidal coordinates. It is straightforward to show that

$$\frac{x^2+y^2}{1+\zeta^2} + \frac{z^2}{\zeta^2} = a^2. \tag{4.4}$$

Thus the surfaces of constant ζ are oblate spheroids, with $\zeta = \zeta_o$ corresponding to the surface of the star. The coordinate ζ has the range $0 \le \zeta \le \zeta_o$ within the star, and $\zeta \ge \zeta_o$ outside. The surface $\zeta = 0$ corresponds to a disk of radius a within the equatorial plane of the star. The nature of the surfaces of constant μ can similarly be explored by noting that

$$\frac{x^2+y^2}{1-\mu^2} - \frac{z^2}{\mu^2} = a^2. \tag{4.5}$$

Thus the constant μ surfaces are hyperbolas. The coordinate μ is confined to the range $-1 \le \mu \le 1$, with $\mu^2 = 1$ corresponding to the rotation axis of the star, and $\mu = 0$ to the portion of the equatorial plane outside the disk of radius a. The coordinate φ measures angles about the rotation axis.

The Equation (3.7) for the gravitational potential in these spheroidal coordinates becomes

$$\frac{\partial}{\partial\zeta}\left[(\zeta^2+1)\frac{\partial\delta\Phi}{\partial\zeta}\right] + \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\delta\Phi}{\partial\mu}\right] + \frac{\zeta^2+\mu^2}{(\zeta^2+1)(1-\mu^2)}\frac{\partial^2\delta\Phi}{\partial\varphi^2} = 0, \tag{4.6}$$

except on the surface of the star $\zeta = \zeta_o$. The separable solutions to these equations are functions of the form $P_l^m(i\zeta)P_l^m(\mu)e^{im\varphi}$ and $Q_l^m(i\zeta)P_l^m(\mu)e^{im\varphi}$. The associated Legendre functions $Q_l^m(\mu)$ diverge at $\mu^2 = 1$, consequently only the functions $P_l^m(\mu)$ appear in these solutions. The associated Legendre functions $P_l^m(i\zeta)$ diverge as ζ^l while the $Q_l^m(i\zeta)$ vanish as $\zeta^{-(l+1)}$ in the limit $\zeta \to \infty$. Thus the gravitational potential in the exterior of the star, $\zeta \ge \zeta_o$, must have the form:

$$\delta\Phi = \alpha\frac{Q_l^m(i\zeta)}{Q_l^m(i\zeta_o)}P_l^m(\mu)e^{im\varphi}. \tag{4.7}$$

for some constant α. In the interior of the star $\zeta \le \zeta_o$ the situation is more complicated. Both $P_l^m(i\zeta)$ and $Q_l^m(i\zeta)$ are bounded in the limit $\zeta \to 0$. However, we must insure that the solution is smooth across the disk $\zeta = 0$. The functions $Q_l^m(i\zeta)$ are non-zero at $i\zeta = 0$ (see Bateman [12] Eqs. 3.4.9, 3.4.20 and 3.4.21). For the case of $l+m$ odd the function $Q_l^m(i\zeta)P_l^m(\mu)$ is therefore discontinuous at the disk $\zeta = 0$, and consequently it does not satisfy Laplace's equation there. Similarly for $l+m$ even the function $Q_l^m(i\zeta)P_l^m(\mu)$ is continuous but not differentiable at $\zeta = 0$, and so it does not satisfy Laplace's equation there either. Thus, in the interior of the star, $\zeta \le \zeta_o$, the solution to Eq. (3.7) for given l and m is

$$\delta\Phi = \alpha\frac{P_l^m(i\zeta)}{P_l^m(i\zeta_o)}P_l^m(\mu)e^{im\varphi}. \tag{4.8}$$

The potentials in Eqs. (4.7) and (4.8) have been normalized so that $\delta\Phi$ is continuous at the surface of the star $\zeta = \zeta_o$.

Following the analysis of the gravitational potential equation, we introduce a second system of coordinates $(\xi, \tilde{\mu}, \varphi)$ which allow the equation for the hydrodynamic potential, Eq. (3.6), to be written in a convenient form. These coordinates are related but not identical to those used by Bryan [5]:

$$x = b\sqrt{(1-\xi^2)(1-\tilde{\mu}^2)}\cos\varphi, \tag{4.9}$$

$$y = b\sqrt{(1-\xi^2)(1-\tilde{\mu}^2)}\sin\varphi, \tag{4.10}$$

$$z = b\xi\tilde{\mu}\frac{\sqrt{4-\kappa^2}}{\kappa}. \tag{4.11}$$

Here the focal radius b is related to a by

$$b^2 = \frac{a^2}{4-\kappa^2}\left[4(1+\zeta_o^2)-\kappa^2\right]. \tag{4.12}$$

The parameter b is real for frequencies, such as those of the $r-$modes, which satisfy $\kappa^2 < 4$. As we shall see, the Eq. (3.6) for the potential δU separates nicely in terms of these coordinates. However these new coordinates are rather unusual, so we present an in depth discussion of them in the Appendix. In summary, the coordinates ξ and $\tilde{\mu}$ cover the interior of the star when their values are confined to the domains $\xi_o \le \xi \le 1$ and $-\xi_o \le \tilde{\mu} \le \xi_o$, where ξ_o is defined as

$$\xi_o^2 = \frac{a^2\zeta_o^2}{b^2}\frac{\kappa^2}{4-\kappa^2} = \frac{\zeta_o^2\kappa^2}{4(1+\zeta_o^2)-\kappa^2} < \frac{\kappa^2}{4} < 1. \tag{4.13}$$

The surface $\xi = 1$ corresponds to the rotation axis of the star, and the surface $\tilde{\mu} = 0$ corresponds to the equatorial plane of the star. The surface of the star, $\zeta = \zeta_o$, is divided into three regions in this coordinate system. The portions of the stellar surface nearest the two branches of the rotation axis correspond to the surfaces $\tilde{\mu} = \pm\xi_o$, while the portion of the stellar surface that includes the equator corresponds to the surface $\xi = \xi_o$. The coordinate $\tilde{\mu}$ coincides with the value of the coordinate μ in that portion of the surface of the star where $\xi = \xi_o$. In the other portions of the surface of the star the value of μ coincides with $\pm\xi$. These facts will be essential in imposing the boundary conditions in the next section.

The Equation (3.6) for the potential δU reduces to the following in terms of the coordinates $(\xi, \tilde{\mu}, \varphi)$

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)\frac{\partial \delta U}{\partial \xi}\right]+\frac{\partial}{\partial \tilde{\mu}}\left[(1-\tilde{\mu}^2)\frac{\partial \delta U}{\partial \tilde{\mu}}\right] +\frac{\xi^2-\tilde{\mu}^2}{(\xi^2-1)(1-\mu^2)}\frac{\partial^2 \delta U}{\partial \varphi^2}=0. \quad (4.14)$$

The separated solutions of this equation are associated Legendre functions of ξ and $\tilde{\mu}$. The coordinate ξ includes $\xi = 1$ in its range, so the non-singular separated solutions to Eq. (4.14) are $P_l^m(\xi)P_l^m(\tilde{\mu})e^{im\varphi}$ and $P_l^m(\xi)Q_l^m(\tilde{\mu})e^{im\varphi}$. The "angular" coordinate $\tilde{\mu}$ does not include ± 1 in its range. Thus at this stage, it is not possible to eliminate the $Q_l^m(\tilde{\mu})$ solution without imposing the boundary conditions.

V. IMPOSING THE BOUNDARY CONDITIONS

In order to obtain the physical solutions to the stellar pulsation equations, we must now impose the boundary conditions, Eqs. (3.10) and (3.11). The simplest boundary condition is the one that involves the derivatives of $\delta\Phi$, Eq. (3.11). In the coordinates (μ, ζ, φ) used to find the solution for $\delta\Phi$ in Eqs. (4.7) and (4.8), the unit normal vector to the surface of the spheroid n^a has only one nonvanishing component, n^ζ:

$$n^\zeta = \frac{1}{a}\sqrt{\frac{\zeta_o^2+1}{\zeta_o^2+\mu^2}}. \quad (5.1)$$

Thus, the normal derivatives that appear in the boundary condition, $n^a\nabla_a\delta\Phi$, can be expressed simply as $n^\zeta\partial_\zeta\delta\Phi$. The gradient of the pressure, $n^a\nabla_a p$ that appears in Eq. (3.11) is given by Eq. (2.10). When evaluated at the surface of the spheroid this reduces to

$$n^a\nabla_a p = -4\pi G\rho^2 a\,\zeta_o\sqrt{(\zeta_o^2+1)(\zeta_o^2+\mu^2)} \times(1-\zeta_o\cot^{-1}\zeta_o). \quad (5.2)$$

Thus, using Eqs. (4.7), (4.8), (5.1) and (5.2) the boundary condition Eq. (3.11) on $\delta\Phi$ is equivalent to

$$\alpha\frac{\zeta_o^2+1}{Q_l^m(i\zeta_o)}\frac{dQ_l^m(i\zeta_o)}{d\zeta}P_l^m(\mu)e^{im\varphi} = \alpha\frac{\zeta_o^2+1}{P_l^m(i\zeta_o)}\frac{dP_l^m(i\zeta_o)}{d\zeta}P_l^m(\mu)e^{im\varphi} -\frac{\alpha P_l^m(\mu)e^{im\varphi}+\delta U}{\zeta_o(1-\zeta_o\cot^{-1}\zeta_o)}. \quad (5.3)$$

The first immediate consequence of this boundary condition is that the potential δU must be proportional to $P_l^m(\mu)$ on the surface of the star. In the last section we found that the potential δU was some linear combination of $P_l^m(\xi)P_l^m(\tilde{\mu})e^{im\varphi}$ and $P_l^m(\xi)Q_l^m(\tilde{\mu})e^{im\varphi}$. As we show in the Appendix, the surface of the star is somewhat complicated in the $(\xi, \tilde{\mu}, \varphi)$ coordinate system. For the portion of the surface of the star that includes the equator, we found that $\xi = \xi_o$ and $\tilde{\mu} = \mu$. This fixes the angular dependence of δU. Therefore, throughout the star δU must have the form

$$\delta U = \beta\frac{P_l^m(\xi)}{P_l^m(\xi_o)}P_l^m(\tilde{\mu})e^{im\varphi}, \quad (5.4)$$

where β is an arbitrary constant. On the portion of the surface of the star that includes the equator, this expression reduces to $\delta U = \beta P_l^m(\tilde{\mu})e^{im\varphi} = \beta P_l^m(\mu)e^{im\varphi}$. On the portion of the surface that includes the positive rotation axis, $\tilde{\mu} = \xi_o$, the expression for δU reduces to $\delta U = \beta P_l^m(\xi)e^{im\varphi} = \beta P_l^m(\mu)e^{im\varphi}$ since $\xi = \mu$ here. Finally, on the portion of the surface that includes the negative rotation axis, $\tilde{\mu} = -\xi_o$, the expression for δU also reduces to $\delta U = \beta P_l^m(-\xi)e^{im\varphi} = \beta P_l^m(\mu)e^{im\varphi}$ since $\xi = -\mu$ here. Consequently, the potential δU reduces to the expression $\delta U = \beta P_l^m(\mu)e^{im\varphi}$ everywhere on the surface of the star. Thus, the boundary condition on $\delta\Phi$ reduces to

$$\alpha\frac{\zeta_o^2+1}{Q_l^m(i\zeta_o)}\frac{dQ_l^m(i\zeta_o)}{d\zeta} = \alpha\frac{\zeta_o^2+1}{P_l^m(i\zeta_o)}\frac{dP_l^m(i\zeta_o)}{d\zeta} -\frac{\alpha+\beta}{\zeta_o(1-\zeta_o\cot^{-1}\zeta_o)}. \quad (5.5)$$

We now see why it was necessary to obtain the solutions for δU in the strange and complicated $(\xi, \tilde{\mu}, \varphi)$ coordinate system. These unusual coordinates have two essential properties: first they allow the solutions to Eq. (3.6) to be found in separated form, and second they have the property that one of the coordinates reduces on the surface of the star to the angular coordinate μ. This last property was needed to allow us to satisfy the boundary conditions using simple separated solutions for both δU and $\delta\Phi$.

The boundary condition, Eq. (3.10), on the potential δU is unfortunately somewhat more complicated. The tensor Q^{ab} that is used in Eq. (3.10) is most simply expressed in cylindrical coordinates (see Eq. 3.3). Therefore it is simplest to consider the boundary conditions on δU in these coordinates. Let $\varpi^2 = x^2 + y^2$ denote the cylindrical radial coordinate. Then, the boundary condition Eq. (3.10) can be expressed as

$$(\kappa^2-4)n^z\partial_z\delta U+\kappa^2 n^\varpi\partial_\varpi\delta U+\frac{2m\kappa}{\varpi}n^\varpi\delta U = -\kappa^2(\kappa^2-4)\Omega^2\rho\frac{\delta U+\delta\Phi}{n^a\nabla_a p}. \quad (5.6)$$

The components of the unit normal vector to the surface of the spheroid, n^a, that appear in Eq. (5.6) can be obtained by taking the gradient of the function that appears on the left side of Eq. (2.4):

$$n^{\varpi} = \sqrt{\frac{\zeta_o^2(1-\mu^2)}{\zeta_o^2+\mu^2}}, \tag{5.7}$$

$$n^z = \mu\sqrt{\frac{\zeta_o^2+1}{\zeta_o^2+\mu^2}}. \tag{5.8}$$

The partial derivatives $\partial_z \delta U$ and $\partial_\varpi \delta U$ that appear in Eq. (5.6) are more difficult to evaluate. To do this we must evaluate the Jacobian matrix that determines the coordinate transformation defined in Eqs. (4.9) through (4.11). The needed partial derivatives are given in the Appendix as Eqs. (A13) through (A16). These expressions can now be used to transform the derivatives ∂_z and ∂_ϖ needed in the boundary condition into expressions in terms of ∂_ξ and $\partial_{\tilde\mu}$. When evaluated on the portion of the surface of the star with $\xi = \xi_o$ and $\tilde\mu = \mu$ (using the relationships in Eqs. 4.12 and 4.13), we obtain the following

$$\begin{aligned}(\kappa^2-4)n^z\partial_z\delta U + \kappa^2 n^\varpi \partial_\varpi \delta U \\ = -\frac{\kappa(4-\kappa^2)}{a\sqrt{4(\zeta_o^2+1)-\kappa^2}}\sqrt{\frac{\zeta_o^2+1}{\zeta_o^2+\mu^2}}\partial_\xi \delta U.\end{aligned} \tag{5.9}$$

Similarly, when evaluated on the portions of the surface with $\tilde\mu = \pm\xi_o$ and $\xi = \pm\mu$ we find

$$\begin{aligned}(\kappa^2-4)n^z\partial_z\delta U + \kappa^2 n^\varpi \partial_\varpi \delta U \\ = \mp\frac{\kappa(4-\kappa^2)}{a\sqrt{4(\zeta_o^2+1)-\kappa^2}}\sqrt{\frac{\zeta_o^2+1}{\zeta_o^2+\mu^2}}\partial_{\tilde\mu} \delta U.\end{aligned} \tag{5.10}$$

Finally then, we may combine the results of Eqs. (4.7), (5.4), (5.9), and (5.10) to obtain the following representation of the boundary condition Eq. (5.6):

$$\begin{aligned}\frac{\zeta_o^2+1}{P_l^m(\xi_o)\sqrt{4(\zeta_o^2+1)-\kappa^2}}\frac{dP_l^m(\xi_o)}{d\xi}\beta - \frac{2m\zeta_o}{4-\kappa^2}\beta \\ = \frac{(1+3\zeta_o^2)\cot^{-1}\zeta_o - 3\zeta_o}{2(1-\zeta_o\cot^{-1}\zeta_o)}\kappa(\alpha+\beta).\end{aligned} \tag{5.11}$$

We point out that Eq. (5.11) is valid on the entire surface of the star. For the portion of the surface where $\tilde\mu = -\xi_o$ and $\xi = -\mu$ on the surface, it is helpful to remember that $P_l^m(-\xi_o)P_l^m(-\mu) = P_l^m(\xi_o)P_l^m(\mu)$, while $P_l^m(-\mu)\, dP_l^m(-\xi_o)/d\xi = -P_l^m(\mu)\, dP_l^m(\xi_o)/d\xi$.

In summary then the boundary conditions, Eqs. (5.5) and (5.11) are given by

$$B(\zeta_o)\alpha = -\frac{\alpha+\beta}{\zeta_o(1-\zeta_o\cot^{-1}\zeta_o)}, \tag{5.12}$$

$$A(\kappa,\zeta_o)\beta = \frac{(1+3\zeta_o^2)\cot^{-1}\zeta_o - 3\zeta_o}{2(1-\zeta_o\cot^{-1}\zeta_o)}\kappa(\alpha+\beta), \tag{5.13}$$

where $A(\kappa,\zeta_o)$ and $B(\zeta_o)$ are defined by

$$A(\kappa,\zeta_o) = \frac{\zeta_o^2+1}{P_l^m(\xi_o)\sqrt{4(\zeta_o^2+1)-\kappa^2}}\frac{dP_l^m(\xi_o)}{d\xi} - \frac{2m\zeta_o}{4-\kappa^2}, \tag{5.14}$$

$$B(\zeta_o) = \frac{\zeta_o^2+1}{Q_l^m(i\zeta_o)}\frac{dQ_l^m(i\zeta_o)}{d\zeta} - \frac{\zeta_o^2+1}{P_l^m(i\zeta_o)}\frac{dP_l^m(i\zeta_o)}{d\zeta}. \tag{5.15}$$

Note that ξ_o that appears on the right side of Eq. (5.14) is an implicit function of κ and ζ_o as given in Eq. (4.13).

VI. THE EIGENVALUES

The boundary conditions Eqs. (5.12) and (5.13) can be satisfied for only certain values of the eigenvalue κ. It is easy to see that the necessary and sufficient condition that there exists a solution to the boundary conditions is

$$\begin{aligned}0 = F(\kappa,\zeta_o) \equiv A(\kappa,\zeta_o)B(\zeta_o) + \\ \frac{2A(\kappa,\zeta_o) - \kappa\zeta_o B(\zeta_o)\left[(1+3\zeta_o^2)\cot^{-1}\zeta_o - 3\zeta_o\right]}{2\zeta_o(1-\zeta_o\cot^{-1}\zeta_o)}.\end{aligned} \tag{6.1}$$

We have verified that Eq. (6.1) is exactly equivalent (i.e. up to changes in notation) to Eq. (60) of Bryan [5].

We now wish to evaluate this eigenvalue equation analytically for the generalized $r-$modes of slowly rotating stars. The parameter ζ_o determines the angular velocity of the star through Eq. (2.3). Large values of ζ_o correspond to small angular velocities, and so we may expand our equations in inverse powers of ζ_o. The leading order terms in such an expression for the angular velocity are

$$\frac{\Omega^2}{\pi G\rho} = \frac{8}{15\zeta_o^2}\left[1 - \frac{6}{7\zeta_o^2} + \mathcal{O}(\zeta_o^{-4})\right]. \tag{6.2}$$

We now wish to obtain solutions to the eigenvalue equation $F(\kappa,\zeta_o) = 0$ for large values of ζ_o. We do this by expanding the expressions on the right side of Eq. (6.1) in inverse powers of ζ_o:

$$\begin{aligned}F(\kappa,\zeta_o) = -(l-1)\zeta_o^2\left[\frac{1}{P_l^m(\kappa/2)}\frac{dP_l^m(\kappa/2)}{d\xi} - \frac{4m}{4-\kappa^2}\right] \\ \times[1+\mathcal{O}(\zeta_o^{-2})].\end{aligned} \tag{6.3}$$

Setting the coefficient of this lowest-order term to zero, we obtain an equation for the lowest-order expression for the eigenvalue, κ_o:

$$\frac{dP_l^m(\kappa_o/2)}{d\xi} = \frac{4m}{4-\kappa_o^2}P_l^m(\kappa_o/2). \tag{6.4}$$

TABLE I. The eigenvalues κ_o of the r-modes of the Maclaurin spheroids in the limit of low angular velocities. The frequencies of these modes are related to κ_o by $\omega = (\kappa_o - m)\Omega$ in the low angular velocity limit. Those frequencies denoted with $*$ have the property that $\omega(\omega + m\Omega) < 0$. These $*$ modes are subject to a gravitational radiation driven secular instability.

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$l=2$	0.0000	1.0000			
$l=3$	0.8944	1.5099	0.6667$*$		
	-0.8944	-0.1766			
$l=4$	1.3093	1.7080	1.2319$*$	0.5000$*$	
	0.0000	0.6120$*$	-0.2319		
	-1.3093	-0.8200			
$l=5$	1.5301	1.8060	1.4964$*$	1.0532$*$	0.4000$*$
	0.5705	1.0456	0.4669$*$	-0.2532	
	-0.5795	-0.0682	-0.7633		
	-1.5301	-1.1834			
$l=6$	1.6604	1.8617	1.6434$*$	1.3402$*$	0.9279$*$
	0.9377	1.3061	0.8842$*$	0.3779$*$	-0.2613
	0.0000	0.4405$*$	-0.1018	-0.7181	
	-0.9377	-0.5373	-1.0926		
	-1.6604	-1.4042			

Using the Rodrigues formula for the associated Legendre functions, this equation can be transformed (for the $m \geq 0$ modes) into the form

$$0 = m\frac{d^m P_l(\kappa_o/2)}{d\xi^m} + \left(\frac{\kappa_o}{2} - 1\right)\frac{d^{m+1}P_l(\kappa_o/2)}{d\xi^{m+1}}, \quad (6.5)$$

where P_l is the Legendre polynomial of degree l. Equation (6.5) is equivalent to Bryan's Eq. (83) [5]. This equation admits $l-m$ (or $l-1$ for the $m=0$ case) distinct roots all of which lie in the interval $-2 < \kappa_o < 2$ [13]. For the case $l = m+1$ the single root of Eq. (6.5) is $\kappa_o = 2/(m+1)$. This agrees with the frequency of the classical r-mode of order m as found for example by Papaloizou and Pringle [8]. The modes with $m < 0$ are equivalent to those with $m > 0$: if κ_o is a solution to Eq. (6.4) for some m, then $-\kappa_o$ is a solution for $-m$. In Table I we present numerical solutions to Eq. (6.4) for a range of different values of l and m. We see that for each value of m there exist solutions of this equation for each value of $l \geq m+1$. For each value of l and m there are $l-m$ different solutions. Thus, there exist a vast number of modes whose frequencies vanish linearly with the angular velocity of the star. We indicate with a $*$ those frequencies in Table I that satisfy the condition $\omega(\omega + m\Omega) < 0$. The modes satisfying this condition would be driven unstable by gravitational radiation reaction in the absence of internal fluid dissipation (i.e. viscosity) [3].

Next we wish to extend the formula for the eigenvalue to higher angular velocity. We define the next term in the expansion of the eigenvalue as

$$\kappa = \kappa_o + \kappa_2\zeta_o^{-2} + \mathcal{O}(\zeta_o^{-4}). \quad (6.6)$$

Using this definition and Eq. (6.4) for κ_o we find the next order term in $F(\kappa, \zeta_o)$ to be

$$F(\kappa,\zeta_o) = -(l-1)\left[\frac{m}{2} + \frac{\kappa_o l(l+1)}{4} - \frac{2\kappa_2 l(l+1)}{4-\kappa_o^2}\right] + \tfrac{2}{5}(2l+1)\kappa_o + \mathcal{O}(\zeta_o^{-2}). \quad (6.7)$$

We can now determine the second order correction to the eigenvalue of the mode by solving $F(\kappa,\zeta_o) = 0$ for κ_2:

$$\kappa_2 = \frac{\kappa_o(4-\kappa_o^2)}{2l(l+1)}\left[\frac{m}{2\kappa_o} + \frac{l(l+1)}{4} - \frac{2}{5}\frac{2l+1}{l-1}\right]. \quad (6.8)$$

In the case of the classical r-modes with $l = m+1$, this expression reduces to:

$$\kappa_2 = \frac{4m}{(m+1)^4}\left[\frac{(m+1)^2}{2} - \frac{2}{5}\frac{2m+3}{m}\right]. \quad (6.9)$$

Thus, for the classical $r-$modes the frequency of the mode is given by [14]

$$\omega + m\Omega = \Omega\left\{\frac{2}{m+1} + \frac{3m}{(m+1)^2}\left[\frac{5}{4} - \frac{2m+3}{m(m+1)^2}\right]\frac{\Omega^2}{\pi G\rho}\right\} + \mathcal{O}(\Omega^5). \quad (6.10)$$

For rapidly rotating stars we must solve the eigenvalue Eq. (6.1) numerically. This presents certain unusual numerical difficulties. In particular the associated Legendre functions $P_l^m(i\zeta)$ and $Q_l^m(i\zeta)$ that appear in Eq. (5.15) are not commonly encountered. Thus, we digress briefly here to describe how these functions may be evaluated numerically. First we note that these functions are essentially real. In particular the functions $\tilde{P}_l^m(\zeta)$ and $\tilde{Q}_l^m(\zeta)$ defined by

$$P_l^m(i\zeta) = i^l\tilde{P}_l^m(\zeta), \quad (6.11)$$

$$Q_l^m(i\zeta) = i^{l+1}\tilde{Q}_l^m(\zeta), \quad (6.12)$$

are real for real values of ζ. The functions $\tilde{P}_l^m(\zeta)$ can be evaluated numerically using essentially the same algorithms used to evaluate their counterparts on the real axis. In particular

$$\tilde{P}_m^m(\zeta) = (2m-1)!!(\zeta^2+1)^{m/2}, \quad (6.13)$$

$$\tilde{P}_{m+1}^m(\zeta) = (2m+1)\zeta\tilde{P}_m^m(\zeta). \quad (6.14)$$

These expressions can be used as initial values for the recursion formula,

$$(l-m)\tilde{P}_l^m(\zeta) = (2l-1)\zeta\tilde{P}_{l-1}^m(\zeta) + (l+m-1)\tilde{P}_{l-2}^m(\zeta), \quad (6.15)$$

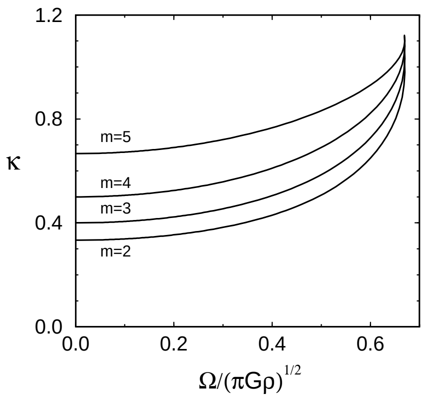

FIG. 1. Angular velocity dependence of the eigenvalues κ of the classical $r-$modes, i.e. those modes with $l = m + 1$ for $l \le 6$.. The frequencies of these modes are related to κ by $\omega = (\kappa - m)\Omega$.

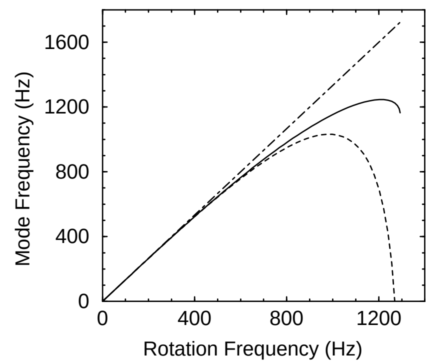

FIG. 2. Angular velocity dependence of the frequency $\omega = (\kappa - m)\Omega$ of the classical $m = 2$ $r-$mode. The solid curve gives ω corresponding to the exact solution of Eq. (6.1), while the dot-dashed and dashed curves correspond to the first and second order approximations respectively from Eq. (6.10).

from which the higher order functions can be determined. This approach does not work for the $\tilde{Q}_l^m$ however. The problem is that we need to evaluate these functions over a fairly wide range of their arguments, e.g. $0.25 \lesssim \zeta \lesssim 75$. For large values of ζ the recursion for the $\tilde{Q}_l^m$ involves a high degree of cancellation among the various terms. Standard computers simply do not have the numerical precision to perform these calculations to sufficient accuracy. Instead we rely on an integral representation of $\tilde{Q}_l^m$. Based on Bateman's Eq. (3.7.5) [12], we find

$$\tilde{Q}_l^m(\zeta) = \frac{(-1)^{l+m+1}(l+m)!}{2^{l+1}l!(\zeta^2+1)^{m/2}} \times \int_{-1}^{1} (\zeta^2+t^2)^{(m-l-1)/2}(1-t^2)^l \times \cos\big[(l+1-m)\tan^{-1}(t/\zeta)\big]dt. \quad (6.16)$$

The integrand in this expression is well behaved for all values of $\zeta > 0$, and the integrals may be determined numerically quite easily.

Using these numerical techniques, then, it is straightforward to solve for the eigenvalues of Eq. (6.1), $F(\kappa, \zeta_o) = 0$, over the relevant range of the parameter $0.25 \le \zeta_o \le 75$. Figures 1, 2, and 3 display the angular velocity dependence of the eigenvalue κ for a number of $r-$modes. Figure 1 depicts κ for the 'classical' $r-$modes, $l = m+1$, with $l \le 6$. This $l = m+1 = 3$ mode is the one found by Lindblom, Owen, and Morsink [3] to be sufficiently unstable due to gravitational radiation emission that it is expected to cause all hot young rapidly rotating neutron stars to spin down to low angular velocities within about one year. We have verified that our numerical solutions of Eq. (6.1) agree with those of Eq. (6.10), up to terms that scale as Ω^5.

Figure 2 presents the angular velocity dependence of the frequency of the classical $l = m + 1 = 3$ $r-$mode as measured in an inertial frame. The solid curve corresponds to the exact solution to Eq. (6.1), while the dot-dashed and dashed curves represent the first and second order approximations (respectively) given in Eq. (6.10). The units in Fig. 2 for both the vertical and horizontal axis scale as $\sqrt{\pi G \rho}$. The value of $\rho = 7 \times 10^{14}\mathrm{gm/cm}^3$ chosen here represents a typical average density for a neutron star. Figure 2 illustrates three interesting features about the frequencies of this mode in large angular velocity stars. First, the frequency is only about 2/3 that predicted by the first-order formula for stars rotating near their maximum angular velocity. This means that the gravitational radiation reaction force, which scales as $(\omega + m\Omega)\omega^5$, could be about 1/5 of that predicted by the first order formula. (Unless the mass and current multipoles of the rapidly rotating models are much larger than their non-rotating values.) Second, the frequencies of these modes first increase and then decrease as the angular velocity of the star is reduced. This means that the time evolution of the gravitational radiation signal from these sources will be more complicated than had been anticipated by Owen, et al. [4] on the basis of the first order expression for the frequency. In particular it appears that the evolution of the frequency will not be monotonic for the most rapidly rotating stars. Third, the accuracy of the second-order formula for the frequency is in fact considerably worse than that of the simple first-order formula for very rapidly rotating stars. This suggests that any application of low angular velocity expansions of the $r-$modes to the study of rapidly rotating stars is somewhat suspect.

Figure 3 depicts the angular velocity dependence of κ for several other previously unstudied $r-$modes. The modes depicted in Fig. 3 all have the property that $\omega(\omega + m\Omega) < 0$, and hence these modes would all be subject to the gravitational radiation secular instability in the absence of internal fluid dissipation (i.e. viscosity). These additional modes all couple to higher order gravitational moments than the classical $l = m + 1 = 3$ mode. Thus, these additional modes probably do not play a significant role in the astrophysical process which spins down hot young neutron stars.

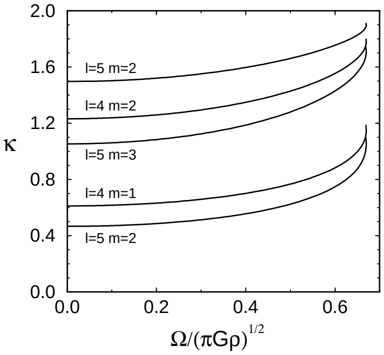

FIG. 3. Angular velocity dependence of the eigenvalues κ of $r-$modes with $5 \geq l > m+1$ which are unstable to the gravitational radiation instability. The frequencies of these modes are related to κ by $\omega = (\kappa - m)\Omega$.

VII. THE EIGENFUNCTIONS

The eigenfunctions associated with these modes are those given in Eqs. (4.7), (4.8), and (5.4). Inside the fluid we have

$$\delta\Phi = \alpha \frac{P_l^m(i\zeta)}{P_l^m(i\zeta_o)} P_l^m(\mu) e^{im\varphi} e^{i(\kappa-m)\Omega t}, \qquad (7.1)$$

$$\delta U = \beta \frac{P_l^m(\xi)}{P_l^m(\xi_o)} P_l^m(\tilde{\mu}) e^{im\varphi} e^{i(\kappa-m)\Omega t}, \qquad (7.2)$$

while outside we have

$$\delta\Phi = \alpha \frac{Q_l^m(i\zeta)}{Q_l^m(i\zeta_o)} P_l^m(\mu) e^{im\varphi} e^{i(\kappa-m)\Omega t}. \qquad (7.3)$$

The coefficients α and β are determined by solving the boundary conditions. The eigenvalue Eq. (6.1) is the consistency condition for the existence of these solutions. Given a solution of the eigenvalue equation then, the ratio of α and β can be determined from Eq. (5.12):

$$\beta = -\left[1 + \zeta_o(1 - \zeta_o \cot^{-1}\zeta_o) B(\zeta_o)\right]\alpha. \qquad (7.4)$$

It might appear at first glance that the eigenfunctions associated with the $l-m$ distinct $r-$modes are identical. However, the coordinates ξ and $\tilde{\mu}$ depend on the eigenvalue κ. Thus, the spatial dependence of δU will be different for each of these modes. Interestingly enough, however, the spatial dependence of the gravitational potential, $\delta\Phi$, depends only on l and m, and hence is the same for all $l-m$ distinct modes.

While the expressions for the eigenfunctions are quite simple in terms of the special spheroidal coordinates used here, they are rather complicated when expressed in more traditional coordinates. One exception to this is the case of the classical $r-$modes, i.e. those with $l = m+1$. In this case the needed associated Legendre function, P_{m+1}^m, has the simple expression

$$P_{m+1}^m(\zeta) = (-1)^m (2m+1)!!\,\zeta(1-\zeta^2)^{m/2}. \qquad (7.5)$$

Using the expressions for the bi-spheroidal coordinates $(\xi, \tilde{\mu})$ in terms of the cylindrical coordinates (ϖ, z) given in Eqs. (A6) and (A7), it follows that

$$P_{m+1}^m(\xi) P_{m+1}^m(\tilde{\mu}) = (-1)^m (2m+1)!!\, P_{m+1}^m(\xi_o) \frac{z}{R_p} \left(\frac{\varpi}{R_e}\right)^m. \qquad (7.6)$$

Thus, the hydrodynamic potential δU is given by

$$\delta U = \beta \frac{z}{R_p} \left(\frac{\varpi}{R_e}\right)^m e^{im\varphi} e^{i(\kappa-m)\Omega t}. \qquad (7.7)$$

Aside from the overall normalization then, the spatial dependence of δU is completely independent of the angular velocity of the star! A similar expression can be obtained for the gravitational potential $\delta\Phi$ within the star. Using the fact that

$$P_{m+1}^m(i\zeta) P_{m+1}^m(\mu) = (-1)^m (2m+1)!!\, P_{m+1}^m(i\zeta_o) \frac{z}{R_p} \left(\frac{\varpi}{R_e}\right)^m, \qquad (7.8)$$

the gravitational potential is given by

$$\delta\Phi = \alpha \frac{z}{R_p} \left(\frac{\varpi}{R_e}\right)^m e^{im\varphi} e^{i(\kappa-m)\Omega t}. \qquad (7.9)$$

Thus the spatial dependencies of the potentials δU and $\delta\Phi$ are identical for the classical $r-$modes! This spatial dependence can also be expressed in spherical coordinates as $r^{m+1} Y_{m+1\,m}(\theta,\varphi)$, up to an overall normalization. It is interesting that this same function satisfies both the hydrodynamic Eq. (3.6) and the gravitational potential Eq. (3.7).

The Eulerian perturbation in the pressure for these modes is determined from these two potentials by Eq. (3.1):

$$\frac{\delta p}{\rho} = (\alpha + \beta) \frac{z}{R_p} \left(\frac{\varpi}{R_e}\right)^m e^{im\varphi} e^{i(\kappa-m)\Omega t}. \qquad (7.10)$$

We note that the constants α and β used here have been scaled by the factor $(-1)^m(2m+1)!!$ compared to their original definitions in Eqs. (4.8) and (5.4). It is also instructive to evaluate the Lagrangian perturbation of the pressure, Δp, as defined in Eq. (3.9). Using the expressions in Eqs. (7.7) and (7.9), we find that

$$\frac{\Delta p}{\rho} = \left\{ 1 + \frac{\alpha}{\beta} - \frac{2(1-\zeta_o \cot^{-1}\zeta_o)[(2-\kappa)(1+\zeta_o^2) - m\kappa\zeta_o^2]}{\kappa^2(2-\kappa)\zeta_o[(1+3\zeta_o^2)\cot^{-1}\zeta_o - 3\zeta_o]} \right\} \delta U. \qquad (7.11)$$

The term enclosed in {} brackets in Eq. (7.11) depends only on the frequency of the mode κ, the angular velocity of the star (through ζ_o), and the amplitudes of the perturbations α and β. When the boundary condition Eq. (3.10) (or equivalently 5.13) is satisfied, this term vanishes. Thus, we find that the Lagrangian perturbation in the pressure Δp vanishes identically for the classical $r-$modes of the Maclaurin spheroids. And this result (which is a consequence of the extremely simple eigenfunctions for these modes) holds for stars with *any* angular velocity!

VIII. DISCUSSION

Our analysis, which follows closely in the footsteps of the remarkable analysis of Bryan, provides several interesting insights into the properties of the $r-$modes of rapidly rotating stars. It demonstrates for example, that these modes actually do exist in rapidly rotating barotropic stars, and are not just figments of the first-order perturbation theory (as claimed by some authors [15]). This analysis shows that some properties of the $r-$modes are not well approximated by the low angular velocity expansions. Figure 2 illustrates, for example, that the first-order expression for the angular velocity dependence of the frequency of the classical $r-$modes is in fact superior to the second-order expression in the most rapidly rotating stars. This analysis shows that the frequency evolution of the gravitational radiation emitted by the $r-$mode instability is likely to be more interesting than had previously been thought [4]. Figure 2 shows that the frequency of these modes will first increase and then decrease as the angular velocity of the star is reduced by the emission of gravitational radiation. This analysis shows that there is a much larger family of $r-$modes than the 'classical' $r-$modes studied for example by Papaloizou and Pringle [8]. For each pair of integers l and m (which satisfy $l \geq m \geq 0$) there exist $l-m$ (or $l-1$ in the $m=0$ case) distinct $r-$modes. This analysis shows that a significant fraction of these previously unstudied $r-$modes are subject to the gravitational radiation driven secular instability. This analysis has derived simple analytical expressions for the eigenfunctions of the classical $r-$modes. Both of the potentials δU and $\delta\Phi$ are proportional to $r^{m+1}Y_{m+1\,m}(\theta,\varphi)$ for the classical $r-$modes in Maclaurin spheroids of any angular velocity. This analysis shows that the Lagrangian variation in the pressure, Δp, associated with the classical $r-$modes vanishes identically in Maclaurin spheroids of arbitrary angular velocity. Thus, the $r-$modes of the Maclaurin spheroids provide a completely unsuitable model for the study of the effects of bulk viscosity on the $r-$modes. This analysis also provides an interesting mathematical example of a hyperbolic eigenvalue problem. The $r-$modes studied here have the property that $\kappa^2 < 4$. Thus, the equation satisfied by the potential δU, Eq. (3.6), is in fact hyperbolic. Nevertheless, the boundary condition imposed on the potential δU, Eq. (3.10), is of the mixed Dirchlet-Neumann type that is generally associated with elliptic problems. The analytical solutions given here illustrate that this unusual hyperbolic eigenvalue problem does nevertheless admit well behaved solutions.

ACKNOWLEDGMENTS

We particularly thank J. Friedman, K. Lockitch and N. Stergioulas for reading and criticizing the first draft of this manuscript. We also thank P. Brady, C. Cutler, B. J. Owen, and K. S. Thorne for helpful discussions concerning this work; and we thank S. Morsink for providing us with her notes on the non-existence of pure axial modes in the Maclaurin spheroids. This research was supported by NSF grants PHY-9408910 and PHY-9796079, and by NASA grants NAG5-3936 and NAG5-4093.

APPENDIX: BI-SPHEROIDAL COORDINATES

The coordinates ξ and $\tilde{\mu}$ defined in Eqs. (4.9) through (4.11) are rather unusual. The purpose of this appendix is to explore the geometrical properties of these coordinates. The coordinates ξ and $\tilde{\mu}$ cover the planes usually described with the spherical coordinates r and θ or equivalently the cylindrical coordinates $\varpi = \sqrt{x^2+y^2}$ and z. First, we note that it follows from Eqs. (4.9) through (4.11) that

$$\frac{\varpi^2}{1-\xi^2} + \frac{\kappa^2}{4-\kappa^2}\frac{z^2}{\xi^2} = b^2. \tag{A1}$$

Thus, the surfaces of constant ξ (with $\xi^2 < 1$) are spheroids for the $r-$modes which have $\kappa^2 < 4$. The particular surface $\xi = \xi_o$, with ξ_o defined by

$$\xi_o^2 = \frac{a^2\zeta_o^2}{b^2}\frac{\kappa^2}{4-\kappa^2} = \frac{\zeta_o^2\kappa^2}{4(1+\zeta_o^2)-\kappa^2} < \frac{\kappa^2}{4} < 1, \tag{A2}$$

is identical to the surface of the star $\zeta = \zeta_o$. Paradoxically, the constant $\tilde{\mu}$ surfaces also satisfy the equation

$$\frac{\varpi^2}{1-\tilde{\mu}^2} + \frac{\kappa^2}{4-\kappa^2}\frac{z^2}{\tilde{\mu}^2} = b^2. \tag{A3}$$

Thus, the constant $\tilde{\mu}$ surfaces are the *same* family of spheroids as the constant ξ surfaces. Thus the coordinates ξ and $\tilde{\mu}$ constitute a bi-spheroidal coordinate system. We note that the equatorial and polar radii of the spheroid, R_e and R_p defined in Eqs. (2.5) and (2.6), are related to the constants b and ξ_o by

$$R_e^2 = b^2(1-\xi_o^2), \tag{A4}$$

$$R_p^2 = b^2\xi_o^2\frac{4-\kappa^2}{\kappa^2}, \tag{A5}$$

In order to understand how the coordinates ξ and $\tilde\mu$ cover the interior of the star it is helpful to introduce two additional coordinates s and $\tilde\theta$:

$$s^2\sin^2\tilde\theta = \frac{\varpi^2}{R_e^2} = \frac{(1-\xi^2)(1-\tilde\mu^2)}{1-\xi_o^2}, \tag{A6}$$

$$s^2\cos^2\tilde\theta = \frac{z^2}{R_p^2} = \frac{\xi^2\tilde\mu^2}{\xi_o^2}. \tag{A7}$$

The coordinate s has the value 1 on the surface of the star and 0 at its center. The coordinate $\tilde\theta$ ranges from the value 0 on the positive rotation axis, through $\pi/2$ on the equatorial plane, to π on the negative rotation axis. Thus, the coordinates s and $\tilde\theta$ map the interior of the star into the unit sphere in a natural way. It will be instructive to express the coordinates ξ and $\tilde\mu$ then in terms of s and $\tilde\theta$. These expressions can be obtained from Eqs. (A6) and (A7):

$$\xi^2 = \tfrac{1}{2}(u+v), \tag{A8}$$

$$\tilde\mu^2 = \tfrac{1}{2}(u-v), \tag{A9}$$

where

$$u = 1 - s^2 + s^2(\xi_o^2 + \cos^2\tilde\theta), \tag{A10}$$

$$v^2 = u^2 - 4s^2\xi_o^2\cos^2\tilde\theta. \tag{A11}$$

We note that while Eqs. (A6) and (A7) are symmetric in ξ and $\tilde\mu$, this symmetry has been broken in order to obtain the expressions (A8) and (A9).

We will now show that the coordinates ξ and $\tilde\mu$ also cover the interior of the star when their values are restricted to the ranges: $\xi_o \le \xi \le 1$ and $-\xi_o \le \tilde\mu \le \xi_o$. It is easy to see from the second equalities in Eqs. (A6) and (A7) that each point $(\xi, \tilde\mu)$ in the domain $\xi_o \le \xi \le 1$ and $-\xi_o \le \tilde\mu \le \xi_o$ corresponds to a point within the star (i.e. a point with $s \le 1$). Proving the converse is more difficult. We do this in three steps: First we show that the functions ξ and $\tilde\mu$ are real and finite at each point within the interior of the star. Second we show that these functions have no critical points (e.g. no maxima or minima) except on the surface of the star. Third, and last, we show that ξ and $\tilde\mu$ are confined to the ranges $\xi_o \le \xi \le 1$ and $-\xi_o \le \tilde\mu \le \xi_o$ for points on the boundary.

First we show that ξ and $\tilde\mu$ are real and finite for each point within the star. The quantity u defined in Eq. (A10) is positive for points within the star (i.e. points with $s \le 1$). Next we show that v (defined as the positive root in Eq. A11) is real and thus positive for points within the star. We do this by re-writing v^2 as

$$\begin{aligned} v^2 = {} & (1 + s\xi_o\cos\tilde\theta + s\sqrt{1-\xi_o^2}\sin\tilde\theta) \\ & \times(1 + s\xi_o\cos\tilde\theta - s\sqrt{1-\xi_o^2}\sin\tilde\theta) \\ & \times(1 - s\xi_o\cos\tilde\theta + s\sqrt{1-\xi_o^2}\sin\tilde\theta) \\ & \times(1 + s\xi_o\cos\tilde\theta - s\sqrt{1-\xi_o^2}\sin\tilde\theta). \end{aligned} \tag{A12}$$

Each of the terms on the right side of Eq. (A12) is positive, since each is 1 plus the inner product of a pair of unit vectors multiplied by s. Thus $v^2 \ge 0$ and so v is real and positive. Further, $v \le u$ and so ξ^2 and $\tilde\mu^2$ are positive. Thus ξ and $\tilde\mu$ are finite and real for each point of the interior of the star.

Second we wish to show that the transformation between the bi-spheroidal coordinates $(\xi, \tilde\mu)$ and the standard cylindrical coordinates (ϖ, z) is non-singular everywhere within the star. We do this by evaluating the Jacobian matrix (i.e. the matrix of partial derivatives) of the transformation:

$$\frac{\partial\xi}{\partial z} = \frac{\kappa\tilde\mu(1-\xi^2)}{b(\tilde\mu^2-\xi^2)\sqrt{4-\kappa^2}}, \tag{A13}$$

$$\frac{\partial\xi}{\partial\varpi} = \frac{\xi\sqrt{(1-\xi^2)(1-\tilde\mu^2)}}{b(\tilde\mu^2-\xi^2)}, \tag{A14}$$

$$\frac{\partial\tilde\mu}{\partial z} = -\frac{\kappa\xi(1-\tilde\mu^2)}{b(\tilde\mu^2-\xi^2)\sqrt{4-\kappa^2}}, \tag{A15}$$

$$\frac{\partial\tilde\mu}{\partial\varpi} = -\frac{\tilde\mu\sqrt{(1-\xi^2)(1-\tilde\mu^2)}}{b(\tilde\mu^2-\xi^2)}. \tag{A16}$$

These expressions show that the Jacobian matrix, and hence the coordinate transformation between $(\xi, \tilde\mu)$ and (ϖ, z), is non-singular except for the points within the star where $\xi = 1$, or $\xi^2 = \tilde\mu^2 = \xi_o^2$. The transformation is also singular at $\tilde\mu^2 = 1$, however, these points are not in the range of interest to us here. The non-singularity of the Jacobian matrix proves that $\nabla_a\xi$ and $\nabla_a\tilde\mu$ are non-vanishing everywhere within the star. Thus, the maximum and minimum values of ξ and $\tilde\mu$ will only occur on the surface or the rotation axis of the star.

Third, and finally, we explore the values of the coordinates $(\xi, \tilde\mu)$ at specific physical locations in the star, e.g. the surface of the star, the rotation axis, etc. We begin first with the rotation axis. In the $(s, \tilde\theta)$ coordinates defined in Eqs. (A6) and (A7), the rotation axis corresponds to the points where $\sin\tilde\theta = 0$. If follows then that these points correspond to $\xi = 1$ and $\tilde\mu^2 = s^2\xi_o^2$. Thus, the rotation axis is the surface $\xi = 1$, a singular surface of the coordinate transformation. The coordinate $\tilde\mu$ takes on its entire range, $-\xi_o \le \tilde\mu \le \xi_o$, for points along this axis. The equatorial plane of the star, $\cos\tilde\theta = 0$, corresponds to the coordinate surface $\tilde\mu = 0$. The coordinate $\xi^2 = 1 - s^2 + s^2\xi_o^2$ ranges from $\xi = 1$ (on the rotation axis) to the value $\xi = \xi_o$ on the surface of the star.

The surface of the star, $s = 1$, is unexpectedly complicated in the $(\xi, \tilde{\mu})$ coordinate system. For points of the stellar surface near the equator, $\cos^2\tilde{\theta} \leq \xi_o^2$, the function v defined in Eq. (A11) has the value: $v = \xi_o^2 - \cos^2\tilde{\theta}$. Thus, the surface of the star in this region has $\xi = \xi_o$ while $\tilde{\mu} = \cos\tilde{\theta}$. In the regions of the stellar surface near the rotation axis, $\cos^2\tilde{\theta} \geq \xi_o^2$, the function v (defined as the *positive* root in Eq. A11) has the value: $v = \cos^2\tilde{\theta} - \xi_o^2$. Thus the stellar surface in these regions have $\tilde{\mu} = \pm\xi_o$ and $\xi^2 = \cos^2\tilde{\theta}$. The role of ξ and $\tilde{\mu}$ as "radial" and "angular" coordinates are reversed therefore in different regions of the star.

In summary then, we have shown that the maximum and minimum values of ξ are 1 and ξ_o respectively, while the maximum and minimum values of $\tilde{\mu}$ are $\pm\xi_o$. This concludes our demonstration that the coordinates $(\xi, \tilde{\mu})$ when confined to the ranges $\xi_o \leq \xi \leq 1$ and $-\xi_o \leq \tilde{\mu} \leq \xi_o$ do form a non-singular coordinate system that covers the interior of the star. The transformation between these and the usual cylindrical coordinates is singular only on the rotation axis, $\xi = 1$, and at the singular point $\xi^2 = \tilde{\mu}^2 = \xi_o^2$ on the surface of the star.

Finally, it will be useful to work out the relationship between the surface values of the $(\xi, \tilde{\mu})$ coordinates with those of the oblate spheroidal coordinates (ζ, μ). Using the definitions of the $(s, \tilde{\theta})$ coordinates introduced in Eqs. (A6) and (A7), and the definitions of the oblate spheroidal coordinates (ζ, μ) from Eqs. (4.1), (4.2), and (4.3), it is straightforward to show that

$$s^2 \sin^2\tilde{\theta} = \frac{(\zeta^2+1)(1-\mu^2)}{\zeta_o^2+1}, \tag{A17}$$

$$s^2 \cos^2\tilde{\theta} = \frac{\zeta^2\mu^2}{\zeta_o^2}. \tag{A18}$$

Thus on the surface of the star, $\zeta = \zeta_o$ and $s = 1$, we have $\cos\tilde{\theta} = \mu$. We have also shown that $\tilde{\mu} = \cos\tilde{\theta}$ on the portion of the surface of the star where $\cos^2\tilde{\theta} \leq \xi_o^2$, and that $\xi = \pm\cos\tilde{\theta}$ on the portion of the surface where $\cos^2\tilde{\theta} \geq \xi_o^2$. Thus we find that $\tilde{\mu} = \mu$ on the portion of the surface where $\xi = \xi_o$, and that $\xi = \pm\mu$ on the portion of the surface where $\tilde{\mu} = \pm\xi_o$.

---

[1] N. Andersson, Astrophys. J. , in press, gr-qc/9706075.

[2] J. L. Friedman, and S. M. Morsink, Astrophys. J. , in press gr-qc/9706073.

[3] L. Lindblom, B. J. Owen, and S. M. Morsink, Phys. Rev. Lett. , **80**, 4843 (1998).

[4] B. J. Owen, L. Lindblom, C. Cutler, B. F. Schutz, A. Vecchio, and N. Andersson, Phys. Rev. D , in press (1998).

[5] G. H. Bryan, *Phil. Trans. Roy. Soc. London*, **A180**, 187 (1889).

[6] The term $r-$mode traditionally refers to modes whose velocity perturbations are proportional to $\vec{r}\times\vec{\nabla}Y_{lm}$, J. Papaloizou and J. E. Pringle, Mon. Not. R. Astron. Soc. **182**, 423 (1978). Such modes were called $r-$modes because they are analogous to the Rossby waves of atmospheric physics. The modes studied here do not fit neatly into the traditional classification of the $r-$ and $g-$modes of slowly rotating stars. The velocity perturbations of our modes are not in general purely axial (as are the traditional $r-$modes) nor purely polar (as are the traditional $g-$modes) in the small angular velocity limit. Thus we have extended the traditional definition of $r-$mode by defining it in terms of the physical process that governs the mode, rotation, rather than the symmetry of its velocity perturbation. Our generalized $r-$modes include all modes considered $r-$modes under the traditional definition.

[7] The Maclaurin spheroids are a special case of barotropic stellar models: stars in which the density of the perturbed fluid depends only on the pressure. In non-barotropic stars there also exist buoyancy forces that dominate the behavior of a class of modes called $g-$modes. The non-barotropic analogues of the generalized $r-$modes discussed here will almost certainly be influenced at some level by buoyancy forces. At sufficiently small angular velocities (i.e. at angular velocities smaller than the Brunt-Väisälä frequency) buoyancy forces could well dominate the dynamics of some of these modes; and such modes might then be called generalized $g-$modes in non-barotropic models. By analogy some might prefer to call these modes generalized $g-$modes even in the barotropic case. Since we do not at present know which (if any) of these modes might be dominated by buoyancy forces in the non-barotropic case, we prefer to refer to them here as $r-$modes. In the barotropic case the dynamics of these modes are dominated by rotational forces.

[8] J. Papaloizou and J. E. Pringle, Mon. Not. R. Astron. Soc. **182**, 423 (1978).

[9] It has been traditional in the literature to discuss the $r-$modes in terms of a vector-spherical-harmonic decomposition of the Lagrangian displacement. In terms of that traditional decomposition, the 'classical' $r-$modes correspond to the case $l = m$. The analysis presented here uses a certain scalar potential δU from which the Lagrangian displacement is determined. The analytic solutions that we find for δU are particular spheroidal harmonic functions. The classical $r-$modes correspond to the case $l = m + 1$ in terms of these spheroidal harmonics. The potential δU for the classical $r-$modes can also be expressed as a spherical harmonic with $l = m + 1$. In general, however, the modes discussed here do not have simple finite representations in terms of spherical harmonics.

[10] J. R. Ipser, and L. Lindblom, Astrophys. J. , **355**, 226 (1990).

[11] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Yale University Press, New Haven, 1969).

[12] H. Bateman, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953) Vol. I.

[13] The Legendre polynomial P_l has l distinct roots, all of

which lie in the interval $(-1,1)$ [12]. It follows that $dP_l/d\xi$ has $l-1$ roots which interleave the roots of P_l, while $d^mP_l/d\xi^m$ has $l-m$ roots which interleave those of $d^{m-1}P_l/d\xi^{m-1}$, etc. For the case $m=0$ then, the roots of Eq. (6.5) are simply the roots of $dP_l/d\xi$. Thus, there are $l-1$ distinct roots for κ_o in the $m=0$ case, and these lie in the interval $-2<\kappa_o<2$. For the case $m>0$ the first term in Eq. (6.5) has $l-m$ distinct zeros, while the second term has $l-m$ (including the one at $\kappa_o=2$) zeros that interleave those of the first. Evaluating the right side of Eq. (6.5) at these zeros of its second term, we find that its sign alternates. Thus, the right side of Eq. (6.5) has $l-m$ zeros that interleave those of its second term. Thus, these zeros lie in the range $-2<\kappa_o<2$. These are all of the zeros of the right side of Eq. (6.5) because it is a polynomial of degree $l-m$.

[14] We point out that Eq. (6.10) does not agree with the second-order expressions for the frequency of the classical $r-$modes given by J. Provost, G. Berthomieu, and A. Rocca, Astron. Astrophys., **94**, 126 (1981), and more recently by K. D. Kokkotas and N. Stergioulas, Astron. & Astrophys., in press (1998). This is not entirely surprising since the Provost, et al. calculation uses the Cowling approximation, and self-gravitational effects do influence the frequencies of the $r-$modes at the Ω^3 order. Kokkotas and Stergioulas only claim that their expression is an approximation.

[15] J. Provost, G. Berthomieu, and A. Rocca, Astron. Astrophys., **94**, 126 (1981).