Rev. Lett. 46, 935 (1981).
[7]M. Nauenberg and J. Rudnick, Phys. Rev. B 24, 493 (1981).
[8]J. D. Farmer, Phys. Rev. Lett. 47, 179 (1981).
[9]S. Grossmann and S. Thomae, Z. Naturforsch. 32A, 1353 (1977).
[10]S. Thomae and S. Grossmann, Phys. Lett. 83A, 181 (1981).
[11]A. Wolf and J. Swift, Phys. Lett. 83A, 184 (1981).
[12]J. Mauer and A. Libchaber, J. Phys. (Paris), Lett. 40, 419 (1979); A. Libchaber and J. Mauer, J. Phys. (Paris), Colloq. 41, C3-51 (1980).
[13]J. P. Gollub and S. V. Benson, J. Fluid Mech. 100, 449 (1980); J. P. Gollub, S. Benson, and J. Steinman, Ann. N.Y. Acad. Sci. 357, 22 (1980).
[14]M. Giglio, S. Musazzi, and U. Perini, Phys. Rev. Lett. 47, 243 (1981).
[15]C. W. Smith, M. J. Tejwani, and D. A. Farris, Phys. Rev. Lett. 48, 492 (1982).
[16]P. S. Linsay, Phys. Rev. Lett. 47, 1349 (1981).
[17]J. Testa, J. Perez, and C. Jeffries, Phys. Rev. Lett. 48, 714 (1982).
[18]J. H. Magerlein, Rev. Sci. Instrum. 49, 486 (1978).
[19]W. C. Stewart, Appl. Phys. Lett. 12, 277 (1968).
[20]W. C. McCumber, J. Appl. Phys. 39, 3113 (1968).
[21]B. A. Huberman, J. P. Crutchfield, and N. H. Packard, Appl. Phys. Lett. 37, 750 (1980).
[22]N. F. Pedersen and A. Davidson, Appl. Phys. Lett. 39, 830 (1981).
[23]R. C. Kautz, J. Appl. Phys. 52, 6241 (1981).
[24]P. M. Marcus, Y. Imry, and E. Ben-Jacob, IBM Research Report No. RC8870 (to be published).
[25]M. R. Beasley and B. A. Huberman, to be published.
[26]Y. Pomeau and P. Manneville, Commun. Math. Phys. 77, 189 (1980); P. Manneville and Y. Pomeau, Physica (Utrecht) 1D, 219 (1980); J.-P. Eckmann, Rev. Mod. Phys. 53, 643 (1981).
[27]J. P. Crutchfield and B. A. Huberman, Phys. Lett. 77A, 407 (1980).
[28]E. N. Lorenz, Ann. N.Y. Acad. Sci. 357, 282 (1980).

Wavelength Selection in Systems Far from Equilibrium

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It is shown that for systems developing stationary periodic patterns there exists at most one stable wavelength state if a supercritical region is connected to a subcritical one by the imposition of a slow spatial variation of the external parameters. In nonpotential systems the selected wavelength depends on the particular combination of parameters that vary but not on the (slow) rate of spatial variation. Suitable parameter variations force the system into a dynamic state.

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During recent years there has been considerable interest in dissipative systems which develop periodic spatial structures. Such patterns have been observed among others in hydrodynamic instabilities of fluids,[1-3] crystal growth,[4] electrohydrodynamic instabilities of nematic liquid crystals,[5] and chemical-reaction–diffusion systems with autocatalytic elements.[6,7]

An open question is what determines the period in such systems and under what circumstances there exists a mechanism that brings the system into a definite wavelength state independent of initial conditions. In Taylor vortex flow a broad range of wavelength states can be obtained by appropriate experimental techniques.[8] In axisymmetric Bénard convection[1] and in directional solidification,[4] on the other hand, there appears to occur a definite wavelength for given external conditions.

Theoretical models for such systems exhibit a continuous family of *linearly stable* nonequivalent wavelength states and this property can be understood from translational and rotational symmetry. If the equations can be derived from a minimizing principle the condition of stationarity with respect to wavelength gives a selection criterion. This criterion is of little use since most pattern-forming systems do not have a generalized free energy.

Work directed towards an understanding of the wavelength selection process[3,9-14] has dealt for the most part with Bénard convection and the Taylor system in simple fluids. The treatment of Cross *et al.* and others[10,12,14] has clarified that nonperiodic boundary conditions can influence the band of allowed wave vectors but they do not, in general, select a unique wavelength. An important step was made by Pomeau and Manneville,[13] who introduced the concept of a small curvature into quasi-infinite systems to single out a specific wavelength state which remains a stationary solution in axisymmetric situations. This selection criterion coincides with free-energy minimization where the latter is applicable.

We here analyze wavelength selection for systems with periodicity in one dimension and slowly varying external parameters. Of special interest are such spatial variations ("ramps") which connect a homogeneous supercritical region to a subcritical region. Such ramps lead to a unique final state and therefore act as selecting boundaries. The results are confirmed by numerical simulation of simple model equations which do not have a potential.

The analysis is demonstrated most easily for a set of simple reaction-diffusion equations,

$$\partial_t u_i = D_{ij}\nabla^2 u_j + f_i(u_1, \ldots, u_n), \tag{1}$$

$i=1,\ldots,n$ (sum over repeated indices is always implied). As a simple example in one spatial dimension consider

$$\begin{aligned}\partial_t u_1 &= D_1\,\partial_x{}^2 u_1 + a_1 u_1(1-u_1{}^2) - b_1 u_2,\\ \partial_t u_2 &= D_2\,\partial_x{}^2 u_2 - a_2 u_2(1-u_2{}^2) + b_2 u_1.\end{aligned} \tag{2}$$

It has a soft-mode instability at finite $k=2\pi/\lambda$ (λ = wavelength) typical for many models. The instability occurs at $k^2=(a_1D_2-a_2D_1)/2D_1D_2$ when b_1b_2 becomes smaller than $(a_1D_2+a_2D_1)^2/4D_1D_2$ while the inequalities $1<a_2/a_1<D_2/D_1$ hold. Beyond the instability Eqs. (2) have a continuous family of linearly stable periodic states.

We now suppose that the functions f_i vary on a slow scale $X=\epsilon x$, with $\epsilon \ll 1$, and look for a solution based on an expansion in ϵ. This sort of problem may be treated by the WKB-like approach for nonlinear equations.[15,16] Thus we seek a solution of the form

$$u_i = u_i{}^{(0)}(\eta, X) + \epsilon u_i{}^{(1)}(\eta, X) + \ldots \tag{3}$$

with $\partial_x\eta = O(1)$ and $u_i{}^{(l)}$ periodic in η with period 2π. Under the assumption that $\partial_x \eta$ varies slowly in space it is useful to introduce a slowly varying phase θ by

$$\eta = \epsilon^{-1}\theta(X, T), \tag{4}$$

where $T=\epsilon^2 t$ will turn out to be the appropriate slow time scale. Then

$$\partial_x\eta = \partial_X\theta = k(X) \tag{5}$$

defines the local wave number k. With use of

$$\partial_x = k\partial_\eta + \epsilon\partial_X \tag{6}$$

equations may be developed at each order in ϵ.

At order ϵ^0 the equations define periodic solutions which are the equilibrium solutions $u_i{}^{(0)}$ for the local values of f_i, with wave number k undetermined at this order. The required result is obtained at order ϵ^1. We find

$$V_i\,\partial_T\theta = D_{ij}[(\partial_X k) + 2k\,\partial_X]V_j + L_{ij}u_j{}^{(1)}, \tag{7}$$

where $V_i = \partial_\eta u_i{}^{(0)}$ was introduced. The linear operator

$$L_{ij} = D_{ij}k^2\,\partial_\eta{}^2 + \partial f_i/\partial u_j{}^{(0)}$$

is singular since the translation mode V_i satisfies $L_{ij}V_j=0$. One then obtains

$$\partial_T\theta = (\partial_X k)\langle V_i{}^\dagger D_{ij}V_j\rangle + 2k\langle V_i{}^\dagger D_{ij}\,\partial_X V_j\rangle \tag{8}$$

as a solubility condition, where $V_i{}^\dagger$ is the periodic zero-eigenvalue left-eigenfunction of L_{ij} and $\langle\cdots\rangle = \int_0^{2\pi} d\eta$. The normalization of $V_i{}^\dagger$ is chosen such that $\langle V_i{}^\dagger V_i\rangle = 1$.

The total dependence of $u_i{}^{(0)}$ (and therefore V_i) on X may be divided into a dependence on $k(X)$ and an explicit dependence on the spatially varying parameters of f_i which we will call $\alpha_l(X)$. Thus (8) can also be written in the form

$$\begin{aligned}\partial_T\theta = (\partial_X k)\langle V_i{}^\dagger D_{ij}(1+2k\,\partial_k)V_j\rangle \\ + 2k(\partial_X\alpha_l)\langle V_i{}^\dagger D_{ij}\,\partial_{\alpha_l}V_j\rangle.\end{aligned} \tag{9}$$

The integrals in (9) are functions of k and α_l, determined through the zero-order problem. Since $k=\partial_X\theta$, Eq. (9) constitutes a phase diffusion equation[11,16] with drift from the inhomogeneities.

We now focus attention on static situations where the right-hand side of (9) must vanish. This constitutes a first-order differential equation for $k(X)$ and thus determines the local wave number if it is prescribed at one point. Specifically, if the variation of the parameters α_l is such that the system becomes (slowly) subcritical on one side then the wave number is determined everywhere because it is unique in the threshold region.[17]

In general the wave number selected by such a ramp is not a unique function of the local parameters α_l but depends on the full shape of the ramp. It is, however, clear that all ramps that can be transformed into each other by a (linear or nonlinear) transformation of the spatial variable lead to the same $k(\alpha_l)$ (in the slow-variation limit). Therefore, if only one parameter (or parameter combination) is varied, the selected wavelength is uniquely defined.

Equation (9) also determines stability of static solutions with respect to long-wavelength fluctuations. In a homogeneous region the quantity $\langle V_i^\dagger D_{ij}(1+2k)V_j\rangle$, which corresponds to the longitudinal phase diffusion constant, must be positive. The limits of this criterion give the Eckhaus instability.[18]

We now turn to the example (2) where one may choose $V_1^\dagger = -b_2 V_1$, $V_2^\dagger = b_1 V_2$ (up to normalization) and consider the situation where only the parameters b_1 and b_2 depend on X. Then the static version of (8) reduces to

$$D_1 b_2(X)\,\partial_X[k\langle V_1^2\rangle] - D_2 b_1(X)\,\partial_X[k\langle V_2^2\rangle] = 0. \qquad (10)$$

Specifically one may consider situations where b_2/b_1 is independent of X. Then (10) goes over into

$$k[D_1(b_2/b_1)\langle V_1^2\rangle - D_2\langle V_2^2\rangle] = \text{const}. \qquad (11)$$

Matching to a subcritical region (where $V_i \to 0$) leads to the requirement that the constant in Eq. (11) is zero. This condition determines uniquely the wave number as a function of b_1 and b_2. Curve 2 of Fig. 1 shows this wavelength as a function of $b = b_1 = b_2$ for $D_1 = 1$, $D_2 = 4$, $a_1 = 1$, and $a_2 = 1.2$ whereas curve 1 exhibits the Eckhaus stability limits.

To test these results numerical simulations of Eqs. (2) were performed. An implicit discretization scheme was used to integrate (2) on a strip of length L with various boundary and initial conditions. Boundary conditions that are consistent with periodicity led in the homogeneous case for $t \to \infty$ to stationary periodic states in the Eckhaus-stable region. Some types of nonperiodic boundary conditions reduced the accessible range but still left a band of states in agreement with previous results.[10,12,14]

The picture changed drastically when a slow monotonous spatial variation on b_1 and/or b_2 was imposed so that the system became sub-

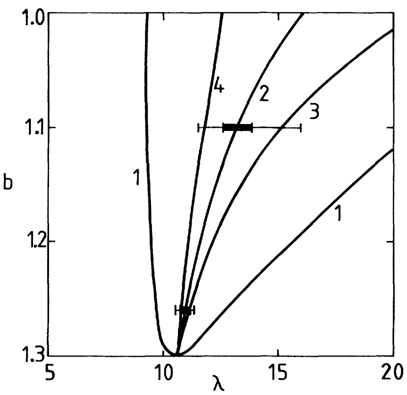

FIG. 1. Curve 1: Linear stability limits for periodic solutions with wavelength λ for $D_2/D_1 = 4$, $a_1 = 1$, $a_2 = 1.2$ as a function of $b = b_1 = b_2$. Curve 2: Wavelength selected by a slow ramp with $b_1 = b_2 = b$. The bars give the approximate width of the remaining band for a steep ramp with $db/dx = 0.015$ (inner bars) and a step change in b to 1.5 (outer bars). Curve 3: Selected wavelength for ramps with $b_1 = b = $ const and b_2 increasing from b to a subcritical value. Curve 4: b_1 and b_2 interchanged.

critical before the boundary was reached. To determine the wavelength with sufficient precision we kept a homogeneous region adjacent to the ramp. Under such conditions a unique final state was reached independent of the initial conditions. When b_1 was chosen equal to b_2 ($=b$) everywhere (including the ramp region) the final wavelength fell onto curve 2 in agreement with the theory. A selection within an accuracy of 1% was found when b changed from its bulk value to threshold over a length of at least about four wavelengths. The remaining bandwidth for a steeper ramp and step is indicated in Fig. 1.

When a ramp with $b_1 \neq b_2$ was constructed then the final state was still unique but the wavelength differed from the one found above in agreement with the theory. In curve 3 of Fig. 1 the final wavelength is plotted for a ramp where $b_1 = b$ is kept constant while b_2 increases from b (in the homogeneous part) to a subcritical value. Curve 4 shows the selected wavelength for the case where b_1 and b_2 are interchanged.[19]

We believe that the concept introduced here is applicable quite generally and propose an experimental test by constructing a selecting boundary in a way analogous to what we did for the numerical simulation. In the Taylor system one can decrease the gap by making one of the cylinders conical at one end.[20] For Bénard convection this could involve reducing the heating gradually at one end of a long rectangular cell or decreasing the height of the fluid layer.[21] The two types of ramps may well select (slightly) different wavelengths. An elegant way to observe this effect would be to have different types of ramps on the two ends. A dynamic situation presumably would then develop where the pattern moves steadily from one side to the other.

We point out that in potential systems, i.e., systems where the right-hand side of Eqs. (1) with the spatial variations included can be obtained as variational equations of a minimizing functional, all ramps select the minimizing wavelength.

It is in principle straightforward to generalize the analysis presented here to other types of models. It appears especially interesting to consider higher-dimensional patterns with ramp-type boundaries.

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[1]E. L. Koschmieder, Adv. Chem. Phys. 32, 109 (1975).

[2]J. P. Gollub, in "Nonlinear Dynamics and Turbulence," edited by D. Joseph and G. Iooss (Pitman, New York, to be published).

[3]F. H. Busse, in *Hydrodynamic Instabilities and the Transition to Turbulence* (Springer-Verlag, Berlin, 1981).

[4]J. S. Langer, Rev. Mod. Phys. 52, 1 (1980).

[5]R. W. H. Kozlowski and E. F. Carr, Mol. Cryst. Liq. Cryst. 64, 299 (1981).

[6]M. Flicker and J. Ross, J. Chem. Phys. 60, 3458 (1974); D. Feinn, P. Ortoleva, W. Scalf, S. Schmidt, and M. Wolff, J. Chem. Phys. 69, 27 (1978).

[7]P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems* (Springer-Verlag, Berlin, 1979).

[8]R. C. DiPrima and H. L. Swinney, in *Hydrodynamic Instabilities and the Transition to Turbulence* (Springer-Verlag, Berlin, 1981); G. Ahlers, D. S. Cannell, and M. A. Dominquez-Lerma, private communication.

[9]Y. Pomeau and P. Manneville, J. Phys. (Paris), Lett. 40, 609 (1979).

[10]M. C. Cross, P. G. Daniels, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. Lett. 45, 898 (1980).

[11]Y. Pomeau and P. Manneville, Phys. Lett. 75A, 296 (1980).

[12]Y. Pomeau and S. Zaleski, J. Phys. (Paris) 42, 515 (1981).

[13]Y. Pomeau and P. Manneville, J. Phys. (Paris) 42, 1067 (1981).

[14]M. C. Cross, P. G. Daniels, P. C. Hohenberg, and E. D. Siggia, to be published.

[15]G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[16]A. C. Newell and M. C. Cross, unpublished.

[17]The threshold region may be treated with the conventional "amplitude equation" as in Ref. 10. Equation (9) then reduces to $\partial_X(|A|^2k) = 0$ with A the complex amplitude in this limit.

[18]See A. C. Newell and J. C. Whitehead, J. Fluid Mech. 38, 279 (1969).

[19]Based on an earlier version of this work Y. Pomeau and S. Zaleski (unpublished) developed an analytic theory for one particular variation of b_1 and b_2 near threshold.

[20]Recent experiments by G. Ahlers, D. S. Cannell, and M. A. Dominquez-Lerma (private communication) have shown selection under such conditions.

[21]An essentially equivalent problem was studied near threshold by P. M. Eagles [Proc. Roy. Soc. London, Ser. A 371, 359 (1980)] who finds that a stationary solution exists only at the selected wavelength.