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$D = 2 + 1$ $\mathcal{N} = 2$ Yang-Mills Theory From Wrapped Branes

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Abstract

We find a new solution of Type IIB supergravity which represents a collection of D5 branes wrapped on the topologically non-trivial $\mathbf{S}^3$ of the deformed conifold geometry $T^*\mathbf{S}^3$. The Type IIB solution is obtained by lifting a new solution of $D = 7$ $SU(2)_L \times SU(2)_R$ gauged supergravity to ten dimensions in which $SU(2)_D$ gauge fields in the diagonal subgroup are turned on. The supergravity solution describes a slice of the Coulomb branch in the large N limit of $\mathcal{N} = 2$ SYM in three dimensions.

1. Introduction

Supersymmetric gauge theories can be realized as the low energy effective field theory living on a collection of branes wrapping supersymmetric cycles [1]. The topological twisting which is required to make these gauge theories supersymmetric [2] can be incorporated in the supergravity realization of these wrapped branes by turning on non-trivial gauge fields in gauged supergravity. These gauged supergravity solutions can then be lifted to ten or eleven dimensions and give rise a host of interesting dual descriptions to the infrared dynamics of various gauge theories [3-17].

In this paper we find the supergravity solution representing a collection of N NS5-branes wrapping the supersymmetric $\mathbf{S}^3$ of the deformed conifold geometry, which admits a metric with $SU(3)$ holonomy. This supergravity solution describes the infrared dynamics of three dimensional $\mathcal{N}=2$ $SU(N)$ Super-Yang-Mills (SYM).

We find this gravity dual by finding a solution of $SO(4) = SU(2)_L \times SU(2)_R$ gauged supergravity. Supersymmetry requires that we turn on $SU(2)_D$ gauge fields of gauged supergravity. Our solution provides an example of an ansatz with $SU(2)_D = (SU(2)_L \times SU(2)_R)_{diag}$ gauge fields that solves the full $SO(4)$ gauged supergravity equations of motion, despite the fact that an $SU(2)_D$ truncation of $SO(4)$ gauged supergravity is in general inconsistent. This solution can then be lifted to ten dimensions and describes certain aspects of the gauge theory dynamics.

Previous constructions of wrapped branes representing three dimensional SYM appeared [4,18,10,16](see also [19]). SYM theory in three dimensions can include, in addition to the usual YM term, a Chern-Simons term in the action with level k. In the supergravity description based on D5 branes wrapping $\mathbf{S}^3$, this term arises when the background has a non-trivial RR field H_3^R with components in the $\mathbf{S}^3$ directions [10]. The present supergravity solutions have $H_3^R = 0$ in the $\mathbf{S}^3$ directions, so they describe 2+1 SYM without Chern-Simons term ($k = 0$).

The solutions describe a slice of the Coulomb branch of the gauge theory, which is parametrized by the $N-1$ real scalars of the vector multiplet together with the $N-1$ scalars obtained upon dualization of the photons[1]. There is a curvature singularity in the supergravity solution, related to the NS5-brane sources. We analyze the moduli space of the

[1] The real scalar of the vector multiplet arises from dimensionally reducing the four dimensional $\mathcal{N}=1$ vector multiplet.

gauge theory by sending a probe into the geometry and reproduce within supergravity the existence of a complex moduli space with a Kahler metric, as required by supersymmetry.

It is well known that monopoles in this theory generate a nonperturbative superpotential which destabilizes the Coulomb branch [20]. On the other hand, our supergravity solutions exhibits a Coulomb branch. This is not inconsistent, since the effects that induce lifting of the Coulomb branch are of order e^{-N} in the t'Hooft limit, so they are not visible in the supergravity approximation.

This paper is organized as follows. In section 2 we describe the low energy effective field theory on the wrapped branes and study the conditions under which the required $SU(2)_D$ truncation can give a solution. In section 3 we find first order equations which solve the seven dimensional gauged supergravity equations of motion. In section 4 the solution is lifted to ten dimensions, giving a one-parameter family of solutions of type II supergravity. We discuss the behavior of the solutions for different values of the parameter. Section 5 makes contact with field theory expectations. We calculate the effective action on the Coulomb branch of the gauge theory in the large N limit. Finally, in the appendix, we discuss an ansatz for NS5 branes with $SU(2)_D$ gauge fields, with the aim of describing four-dimensional $\mathcal{N} = 2$ SYM.

2. Wrapped Brane Realization of Three Dimensional N=2 $SU(N)$ SYM

This gauge theory can be realized by wrapping N NS5-branes on the $\mathbf{S}^3$ of the deformed conifold geometry $T^*\mathbf{S}^3$, which admits a metric with $SU(3)$ holonomy. In order to obtain a globally supersymmetric gauge theory one must couple the field theory on the branes to the gauge fields which couple to the R-symmetry currents of five-branes in flat space, that is, the field theory must be topologically twisted [2]. Then, supersymmetry can be realized by finding constant and "covariantly" constant spinors on $\mathbf{S}^3$ [2]

$$
\begin{aligned}
&\partial\epsilon = 0 \ , \\
&\mathcal{D}\epsilon = \left(\partial + \frac{1}{4}\omega_{ab}\Gamma^{ab} + \frac{1}{4}A_{ij}\gamma^{ij}\right)\epsilon = 0 \ ,
\end{aligned}
\tag{2.1}
$$

where a, b are tangent space indices on the $\mathbf{S}^3$ and i, j are vector indices of the $SO(4) = SU(2)_L \times SU(2)_R$ R-symmetry group of the five-branes in flat space.

[2] The standard Dirac equation on $\mathbf{S}^3$ does not admit zero modes and therefore an untwisted supersymmetric theory cannot be realized on it.

Turning this background gauge field effectively changes the transformation properties of all fields under a modified tangent space symmetry group of $\mathbf{S}^3$, under which the unbroken supersymmetries transform as scalars. The topological twisting giving rise to a three dimensional $\mathcal{N}=2$ theory is obtained by identifying the $SU(2)_{S^3}$ spin connection with the $SU(2)_D=(SU(2)_L\times SU(2)_R)_{\rm diag}$ subgroup of the $SU(2)_L\times SU(2)_R$ R-symmetry group of the NS5-brane so that the supersymmetry generators are singlets under $SU(2)_{\rm diag}=(SU(2)_{S^3}\times SU(2)_D)_{\rm diag}$.[3]

To verify that the field content that follows from this twist gives rise to a three dimensional $\mathcal{N}=2$ massless vector multiplet, one decomposes the original six dimensional $\mathcal{N}=(1,1)$ vector multiplet on the five-branes under the symmetries left unbroken by the background, which are $SO(1,2)\times SU(2)_{S^3}\times SU(2)_L\times SU(2)_R$

$$\begin{aligned}\text{gauge}&:(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3},\mathbf{1},\mathbf{1})\\ \text{scalars}&:(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})\\ \text{spinors}&:(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}).\end{aligned}\tag{2.2}$$

Then it follows that the transformation properties of these modes under $SO(1,2)\times SU(2)_{\rm diag}$ are

$$\begin{aligned}\text{gauge}&:(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\\ \text{scalars}&:(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\\ \text{spinors}&:2(\mathbf{2},\mathbf{1})\oplus2(\mathbf{2},\mathbf{3}).\end{aligned}\tag{2.3}$$

Therefore, the modes singlet under $SU(2)_{\rm diag}$ fill the $\mathcal{N}=2$ vector multiplet.

In supergravity, the $SO(4)$ global R-symmetry group of a five-brane appears as a gauge symmetry of the 7-dimensional effective supergravity theory obtained by reducing Type II supergravity on the transverse $\widetilde{\mathbf{S}}^3$ which appears in the near horizon solution of the five-brane.[4] Therefore, 7-dimensional gauged supergravity is a natural arena to construct supergravity solutions in which a background R-symmetry gauge field is turned on. solution

[3] A different twisting, used previously in [3], in which one identifies the $SU(2)_{S^3}$ spin connection with the $SU(2)_L$ subgroup of the $SU(2)_L\times SU(2)_R$ R-symmetry group, gives rise to a four-dimensional $\mathcal{N}=1$ theory. In that case the supersymmetry generators are singlets under $SU(2)_{\rm diag}=(SU(2)_{S^3}\times SU(2)_L)_{\rm diag}$.

[4] We recall that the near horizon geometry of a five-brane is $\mathbf{R}^{1,5}\times\mathbf{R}\times\widetilde{\mathbf{S}}^3$ with a linear dilaton along $\mathbf{R}$.

The bosonic content of the $SO(4)$ gauged supergravity is the graviton, a two-form, $SO(4)$ gauge fields A_{ij} and ten scalars T_{ij} transforming in the two-index symmetric representation of $SO(4) = SU(2)_L \times SU(2)_R$

$$\begin{aligned} \text{gauge fields } A_{ij} &: (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \\ \text{scalars } T_{ij} &: (\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{3}) \ . \end{aligned} \tag{2.4}$$

By turning on some fields in this gauged supergravity, one can construct solutions having a space-time interpretation as wrapped branes.

The bosonic Lagrangian of $SO(4)$ gauged supergravity in the absence of the two-form is given by [21]

$$\begin{aligned} \mathcal{L} = \sqrt{g}\Big(R - \frac{5}{16} Y^{-2} \partial_\mu Y \partial^\mu Y - \frac{1}{4} \widetilde{T}^{-1}_{ij} \widetilde{T}^{-1}_{kl} D_\mu \widetilde{T}_{jk} D^\mu \widetilde{T}_{li} - \frac{1}{8} Y^{-1/2} \widetilde{T}^{-1}_{ik} \widetilde{T}^{-1}_{jl} F^{ij}_{\mu\nu} F^{\mu\nu}_{kl} \\ - \frac{1}{2} g^2 Y^{1/2} (2 \widetilde{T}_{ij} \widetilde{T}_{ij} - (\widetilde{T}_{ii})^2) \Big), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} Y &= \det(T_{ij}) \ , \\ \widetilde{T}_{ij} &= Y^{-1/4} T_{ij} \ , \\ DT_{ij} &= dT_{ij} + g[A,T]_{ij} \ . \end{aligned} \tag{2.6}$$

Any solution to the $SO(4)$ gauged supergravity equations lifts to a solution of ten dimensional Type IIB supergravity. Explicit formulas [22] for the ten dimensional fields in terms of the seven dimensional ones are given in section 4.

For the case under study, one needs to turn on $SU(2)_D$ gauge fields A_a. These can be identified as the following components of the $SO(4)$ gauge fields A_{ij}

$$A_1 \equiv A_{23} \qquad A_2 \equiv -A_{13} \qquad A_3 \equiv A_{12}. \tag{2.7}$$

The $SU(2)_D$ gauge fields may in general act as non-linear sources for the gauge fields which have been set to zero. The reason is that they are sources for the fields A_{14}, A_{24}, A_{34}. We have studied the $SO(4)$ gauged supergravity equations following from (2.5) and found sufficient conditions that have to be met in order for a solution with only $SU(2)_D$ gauge fields to be compatible with the full $SO(4)$ equations of motion. They can be summarized as follows:

• Only $SO(4)$ matter modes of gauged supergravity that are singlets under $SU(2)_D$ can be turned on. Since the non-trivial $SO(4)$ gauged supergravity modes transform under $SU(2)_D$ as

$$\begin{aligned}\text{gauge fields} &: 2\ \mathbf{3} \\ \text{scalars} &: 2\ \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5},\end{aligned} \tag{2.8}$$

then only $SU(2)_D$ gauge fields A_a, 2 real scalars, the two-form and the metric can be turned on. Moreover, invariance under $SU(2)_D$ requires the scalar matrix to take the following form[5]

$$T_{ij} = e^{y/4}\text{diag}(e^x, e^x, e^x, e^{-3x}) \ . \tag{2.9}$$

• The $SU(2)_D$ gauge fields must satisfy the following condition

$$F_a \wedge *F_b = 0 \text{ for } a \neq b. \tag{2.10}$$

Once these conditions are satisfied one can perform an $SU(2)_D$ reduction of the $SO(4)$ gauged supergravity Lagrangian (2.5), giving the $SU(2)_D$ Lagrangian

$$\begin{aligned}\mathcal{L} = \sqrt{g}\Big(R - \frac{5}{16}\partial_\mu y \partial^\mu y - 3\partial_\mu x \partial^\mu x - \frac{1}{4}e^{-2x-y/2}(F^1_{\mu\nu}F_1^{\mu\nu} + F^2_{\mu\nu}F_2^{\mu\nu} + F^3_{\mu\nu}F_3^{\mu\nu}) \\ + \frac{1}{2}g^2 e^{y/2}(3e^{2x} + 6e^{-2x} - e^{-6x})\Big).\end{aligned} \tag{2.11}$$

Any solution of this Lagrangian with gauge fields satisfying (2.10) fully solves the $SO(4)$ gauged supergravity equations, and it can be lifted as a solution of ten dimensional Type IIB supergravity.

3. The Supergravity Solution in $D = 7$

In order to construct the $\mathcal{N} = 2$ supersymmetric solution, we will consider seven dimensional $SO(4)$ gauged supergravity, in which only $SU(2)_D$ gauge fields are turned on. The worldvolume of the five-branes we are interested in is $\mathbf{R}^{1,2} \times \mathbf{S}^3$. The metric ansatz compatible with the worldvolume of the five-branes is

$$ds_7^2 = e^{2f(r)}(ds^2(\mathbf{R^{1,2}}) + dr^2) + \frac{a^2(r)}{4}w_a w_a \ , \tag{3.1}$$

[5] This can be easily understood as follows. The $SO(4)$ generators $J_{ij} = -J_{ji}$ can be divided into boost generators K_a and rotation generators J_a. The $SU(2)_D$ subgroup of $SO(4)$ is generated by J_a, and the most general four-dimensional symmetric matrix invariant under rotations is of the form (2.9).

where w_a are left invariant $SU(2)$ one-forms satisfying

$$dw_a = \frac{1}{2}\epsilon_{abc} w_b \wedge w_c \ . \tag{3.2}$$

One can describe them in Euler angle parametrization,

$$w_1 = \cos\beta d\theta + \sin\beta \sin\theta d\varphi \ , \quad w_2 = \sin\beta d\theta - \cos\beta \sin\theta d\varphi \ , \quad w_3 = d\beta + \cos\theta d\varphi \ .$$

where the angles β, θ and φ parametrize the $\mathbf{S}^3$ which the branes are wrapping.

The identification of the $SU(2)_{S^3}$ spin connection with the $SU(2)_D$ gauge fields that follows from (2.1) requires that

$$A_a = -\frac{1}{2g} w_a \ , \tag{3.3}$$

with the A_a's given in (2.7), and the corresponding curvature is given by

$$F_a = -\frac{1}{8g}\epsilon_{abc} w_b \wedge w_c \ . \tag{3.4}$$

We note that this gauge field configuration is compatible with the $SO(4)$ gauged supergravity equations since $F^a \wedge * F^b = 0$ for $a \neq b$.

The string metric for wrapped NS5-branes typically has no warp factor in the parallel, flat directions along the brane. Since the ten dimensional string frame metric is of the form [22], $ds^2_{st} = e^{2f+y/2} ds^2(\mathbf{R}^{1,2}) + \ldots$ (see below in section 5), it is natural to look for solutions with $y(r) = -4f(r)$. It is easy to show that this ansatz is compatible with the equations of motion that follow from (2.11).

We find the equations of motion by reducing the problem to that of an effective quantum mechanics for the ansatz, which together with a Hamiltonian constraint results in the gauged supergravity equations of motion. Using the ansatz (3.1), (3.3) together with $y = -4f$ the various terms in the action (2.11) take the following form

$$\begin{aligned}
\mathcal{L}_R &= e^{2f} a^3 \left(6\frac{e^{2f(r)}}{a^2} + 6\left(\frac{\dot{a}}{a}\right)^2 + 18\frac{\dot{a}}{a}\dot{f} + 6\dot{f}^2 \right) \ , \\
\mathcal{L}_{scal} &= -e^{2f} a^3 \left(5\dot{f}^2 + 3\dot{x}^2 \right) \ , \\
\mathcal{L}_{gauge} &= -\frac{1}{2g^2} e^{2f} a^3 \left(\frac{e^{4f-2x}}{a^4} \right) \ , \\
\mathcal{L}_{pot} &= \frac{1}{2} e^{2f} a^3 g^2 (3e^{2x} + 6e^{-2x} - e^{-6x}).
\end{aligned} \tag{3.5}$$

Defining $e^{2h} = e^{-2f}a^2$ and $e^{2\alpha} = e^{2f}a^3 = e^{5f+3h}$, we arrive at the following effective Lagrangian

$$\mathcal{L}_{eff} = e^{2\alpha}(T - V) \ , \tag{3.6}$$

where

$$\begin{aligned} T &= 4\dot{\alpha}^2 - 3\dot{h}^2 - 3\dot{x}^2 \ , \\ V &= \frac{3}{2g^2}e^{-4h-2x} - 6e^{-2h} - \frac{1}{2}g^2(3e^{2x} + 6e^{-2x} - e^{-6x}) \ , \end{aligned} \tag{3.7}$$

where $\dot{\alpha} \equiv \frac{d\alpha}{dr}$, etc.

This effective Lagrangian together with the constraint $H = T + V = 0$ arising from diffeomorphism invariance in the radial coordinate determine the second order equations of motion for $\alpha(r),\ h(r),\ x(r)$. For supersymmetric configurations it is always possible to find a set of first order differential equations whose solutions solve the equations of motion. This can be done in the present case. The Hamilton-Jacobi linear equations are $p_i = \frac{\partial F}{\partial \varphi_i}$, where $\varphi_i = \{\alpha, h, x\}$. Taking as principal function $F = e^{2\alpha}W(h,x)$ and if the potential of the effective quantum mechanics can be written in terms of a superpotential W as follows:

$$V = \frac{1}{12}(\partial_h W)^2 + \frac{1}{12}(\partial_x W)^2 - \frac{1}{4}W^2 \ , \tag{3.8}$$

then one can find first order equations which solve the equations of motion. The solution for (3.8) is

$$W = 3ge^x + ge^{-3x} + \frac{3}{g}e^{-2h-x} \ . \tag{3.9}$$

Thus the first-order (BPS) equations are:

$$\dot{\alpha} = \frac{1}{4}W = \frac{3g}{4}e^x + \frac{g}{4}e^{-3x} + \frac{3}{4g}e^{-2h-x} \ , \tag{3.10}$$

$$\dot{h} = -\frac{1}{6}\partial_h W = \frac{1}{g}e^{-2h-x} \ , \tag{3.11}$$

$$\dot{x} = -\frac{1}{6}\partial_x W = -\frac{g}{2}e^x + \frac{g}{2}e^{-3x} + \frac{1}{2g}e^{-2h-x} \ . \tag{3.12}$$

By a change of radial coordinate $r \to \rho(r) = \frac{g^2}{2}e^{2h(r)}$, and defining $u = e^{2x}$, eqs. (3.10)-(3.12) take the form

$$\frac{d\alpha}{d\rho} = \frac{1}{4}(3u + u^{-1}) + \frac{3}{8\rho} \ , \qquad u^{1/2}(\rho)\ d\rho = g dr \ , \tag{3.13}$$

$$\frac{du}{d\rho} - \frac{1}{2\rho}\ u = 1 - u^2 \ . \tag{3.14}$$

The general solution to eq. (3.14) is given by

$$u(\rho) = e^{2x(\rho)} = \frac{c_0 I_{1/4}(\rho) + (1-c_0) I_{-1/4}(\rho)}{c_0 I_{-3/4}(\rho) + (1-c_0) I_{3/4}(\rho)} \ , \tag{3.15}$$

where we are using the standard notation $I_\nu(z)$ for the modified Bessel functions and c_0 is a parameter of the solution. Then $\alpha(\rho)$ is determined by direct integration of (3.13). These functions have a simple behavior, which will be described below.

4. Ten-dimensional solution

Since we have fully solved the $SO(4)$ gauged supergravity equations one can write the corresponding Type IIB supergravity solution in ten dimensions. The expression of the ten dimensional Einstein frame metric, the dilaton and the NS-NS three-form field strength are given in terms of the gauged supergravity fields by

$$\begin{aligned}
ds_E^2 &= Y^{1/8}\left(\Delta^{1/4} ds_7^2 + g^{-2}\Delta^{-3/4} T_{ij}^{-1} D\mu^i D\mu^j\right) \\
e^{2\phi} &= \frac{Y^{3/2}}{\Delta} \\
H &= \frac{1}{6g^2\Delta^2}\epsilon_{i_1 i_2 i_3 i_4}\Big(U\ D\mu^{i_1}\wedge D\mu^{i_2}\wedge D\mu^{i_3}\mu^{i_4} - 3D\mu^{i_1}\wedge D\mu^{i_2}\wedge DT_{i_3 j} T_{i_4 k}\mu^j\mu^k \\
&\quad - 3g\Delta\ F^{i_1 i_2}\wedge D\mu^{i_3} T_{i_4 j}\mu^j\Big),
\end{aligned} \tag{4.1}$$

where the μ^i's parametrise the transverse $\widetilde{\mathbf{S}}^3$ ($\mu^i\mu^i = 1$) and

$$\begin{aligned}
\Delta &= T_{ij}\mu^i\mu^j \ , \\
D\mu^i &= d\mu^i + gA^{ij}\mu^j \ , \\
U &= 2T_{ik}T_{jk}\mu^i\mu^j - \Delta T_{ii} \ , \\
DT_{ij} &= dT_{ij} + gA^{ik}T_{kj} + gA^{jk}T_{ik} \ .
\end{aligned} \tag{4.2}$$

The metric in the string frame is therefore

$$ds_{st}^2 = Y^{1/2}\left(ds_7^2 + g^{-2}\Delta^{-1}T_{ij}^{-1} D\mu^i D\mu^j\right) . \tag{4.3}$$

Using (2.9), (3.1), we note that the string metric has no warped factor in the $\mathbf{R}^{1,2}$ part, with the identification we made earlier $y(r) = -4f(r)$.

Let us now write explicitly this new Type IIB supergravity solution. First, we note that the seven dimensional gauge coupling is related to the brane charge N via the relation $1/g^2 = N$. Using (4.3), (3.1), (2.9) the metric in the string frame and the dilaton are given by ($\alpha' = 1$)

$$ds^2_{st} = ds^2(\mathbf{R}^{1,2}) + N\left[e^{2x}d\rho^2 + \frac{\rho}{2}w_a w_a + \frac{1}{G}\left(e^{-2x}\left(D\mu_1^2 + D\mu_2^2 + D\mu_3^2\right) + e^{2x}d\mu_4^2\right)\right] \quad (4.4)$$

$$e^{2\phi} = e^{2\phi_0}\rho e^{-2\alpha - x}G^{-1}, \quad (4.5)$$

where

$$G \equiv e^{-x-y/4}\Delta = \sin^2\psi + e^{-4x}\cos^2\psi\ , \quad (4.6)$$

and ϕ_0 arises as the integration constant in eq. (3.13) for α. This is in the parametrization of $\widetilde{\mathbf{S}}^3$ given by

$$\mu_1 = \sin\psi\cos\phi_1,\quad \mu_2 = \sin\psi\sin\phi_1\cos\phi_2,\quad \mu_3 = \sin\psi\sin\phi_1\sin\phi_2,\quad \mu_4 = \cos\psi.$$

Taking into account the $SU(2)_D$ truncation (2.7), one has the following covariant derivatives

$$D\mu_1 = d\mu_1 - \tfrac{1}{2}w_3\mu_2 + \tfrac{1}{2}w_2\mu_3,\ \ D\mu_2 = d\mu_2 + \tfrac{1}{2}w_3\mu_1 - \tfrac{1}{2}w_1\mu_3,\ \ D\mu_3 = d\mu_3 - \tfrac{1}{2}w_2\mu_1 + \tfrac{1}{2}w_1\mu_2.$$

where we have used the gauge fields in the ansatz (3.3) that implement the required topological twisting. One can also write down the three-form flux using (4.1) , the curvature F_{ij} given in (3.4) and

$$dT_{ij} = \frac{dT_{ij}}{d\rho}d\rho\ ,\qquad T_{ij} = e^{y/4}\mathrm{diag}\left(e^x, e^x, e^x, e^{-3x}\right)\ ,$$

$$U = \tfrac{1}{2}e^{-6x+\frac{y}{2}}\left(1 - 4e^{4x} - e^{8x} + (e^{4x}-1)^2\cos 2\psi\right)\ ,\quad y = \frac{4}{5}(3h - 2\alpha),\quad e^{2h} = 2N\rho\ .$$

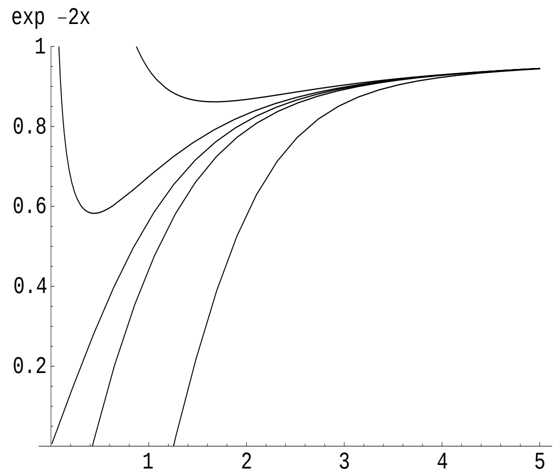

Fig. 1: The function $e^{-2x(\rho)}$ vanishes at some $\rho_0 > 0$ for $c_0 < 0$. The figure shows the different curves for the values $c_0 = -6,\ -0.5,\ 0,\ 0.4,\ 1.2$.

Let us now examine the behavior of the solution as a function of the c_0 integration constant (see plot of e^{-2x} in fig.1). There are three different regimes:

a) For $c_0 < 0$, the function $e^{-2x} = u^{-1}$ becomes negative at $\rho < \rho_0$. Therefore, the space terminates at $\rho = \rho_0$ (since the scalar field x is real). The value of ρ_0 depends on c_0 and it decreases towards zero as $c_0 \to 0^-$. The behavior near $\rho = \rho_0$ is as follows: [6]

$$e^{-2x} = \tilde{\rho} + O\big(\tilde{\rho}^2\big) \ , \quad \alpha \cong \frac{3}{4} \log \tilde{\rho} \ ,$$

where $\tilde{\rho} = \rho - \rho_0$. The metric and dilaton near the singularity are then given by

$$\begin{aligned} ds^2(\text{NS5}) &\cong ds^2(\mathbf{R}^{1,2}) + \frac{N\rho_0}{2} w_a w_a + \frac{N}{\tilde{\rho}} \Big(d\tilde{\rho}^2 + \tilde{\rho}^2 \big(d\phi_1^2 + \sin^2\phi_1 d\phi_2^2\big) + d\psi^2 \Big) \ , \\ e^{-2\phi_{\text{NS5}}} &\cong e^{-2\phi_0} \sin^2(\psi) \frac{\tilde{\rho}}{\rho_0} . \end{aligned} \tag{4.7}$$

We note that in this infrared limit the transverse $\widetilde{\mathbf{S}}^3$ degenerates to an $\widetilde{\mathbf{S}}^1 \times \widetilde{\mathbf{S}}^2$. The metric has a curvature singularity at $\rho = \rho_0$. There the dilaton e^{ϕ} goes to ∞. The applicability of holography in the presence of singularities was discussed in [1],[23]. According to the criterium of [1], the singularity is a good one, since $g_{00}^E = -e^{-\phi}$ does not increase as we approach the singularity. Here $g_{00}^E \cong -(\rho - \rho_0)^{1/2} \to 0$ as $\rho \to \rho_0$. In the next section

[6] For $c_0 = 0$ we have near $\rho = 0$ that $e^{-2x} \cong \frac{2}{3}\ \rho$, $e^{2\alpha} \cong \rho^3$, $e^{2\phi} \sim \rho^{-3/2}$ which results in a singular solution, which is an acceptable one according to the criterium proposed by [1].

we will see that this singularity is due to a brane source and it corresponds to a Coulomb branch configuration of the dual three-dimensional $\mathcal{N}=2$ SYM theory.

b) For $0 < c_0 \leq 1$, e^{-2x} goes to $+\infty$ as $\rho \to 0$. For a generic value of the integration constant $1 > c_0 > 0$, we have near $\rho = 0$: [7]

$$e^{-2x} \sim \rho^{-1/2} \ , \qquad e^{2\phi} \sim \rho \ , \qquad e^{2\alpha} \sim \rho^{3/4} \ . \tag{4.8}$$

We see that the metric has a bad singularity, with g_{00}^E diverging at $\rho = 0$ and we discard it as a possible gravity description of the infrared dynamics of a gauge theory.

c) For $c_0 > 1$, the factor e^{-2x} blows up at some ρ_1, $e^{2x} = (\rho - \rho_1) + O\big((\rho-\rho_1)^2\big)$ and $e^{2\phi} = \text{const.}(\rho - \rho_1)$. Thus $e^{-\phi} \to \infty$, so the singularity is again a bad one and we will discard these solutions.

In some respects, the structure of the solution resembles the solution found in [14,15], and the solution of [24], describing $\mathcal{N}=2$ 3+1 SYM. The parameter c_0 is the counterpart of the parameters k and γ in [14], [24], [25].

The UV behavior near $\rho = \infty$ is, for any c_0, given by $u = e^{2x} \cong 1 + \frac{1}{4\rho}$ (i.e. $x \to 0$), $\phi \sim -\rho$, $\alpha = \rho + \frac{1}{2}\log\rho$. This is in fact the same leading behavior as the solution in [10] (but here there is no subleading term of the form $\log\rho$ in ϕ) . The metric is

$$ds^2_{st} = ds^2(\mathbf{R}^{1,2}) + N\left(\frac{1}{4}d\rho^2 + \frac{\rho}{2}w_a w_a + D\mu_1^2 + D\mu_2^2 + D\mu_3^2 + d\mu_4^2\right) \tag{4.9}$$

This describes the near horizon limit of the NS5 brane wrapped on 3-sphere, with a twist.

In conclusion, we expect that the present supergravity solution with $c_0 < 0$ is dual to three dimensional $\mathcal{N}=2$ $SU(N)$ SYM. Below we discuss this correspondence in more detail.

5. Field theory from the supergravity description

We will now show that the present $\mathcal{N}=2$ model based on the one-parameter family of solutions with $c_0 < 0$ is a gravitational dual of the Coulomb branch of $\mathcal{N}=2$ SYM theory in three space-time dimensions.

---

[7] If $c_0 = 1$, the behavior near $\rho = 0$ is $e^{-2x} \cong (2\rho)^{-1}$, $e^{2\phi} \sim \rho^{3/2}$, $e^{2\alpha} \sim \rho$. The metric with this singularity at $\rho = 0$ is deemed to be unacceptable as a dual description of a field theory [1].

To properly describe the infrared dynamics of the gauge theory on the NS5-branes one has to perform an S-duality transformation and consider D5-branes [26]. The corresponding D5-brane solution is given by

$$ds^2(\mathrm{D5}) = e^{-\phi_{\mathrm{NS}}} ds^2(\mathrm{NS5}) \ , \quad \phi_{\mathrm{D5}} = -\phi_{\mathrm{NS}} \tag{5.1}$$

$$C^R_{(2)} = -B^{NS}_{(2)} \ , \qquad C^R_{(6)} = -B^{NS}_{(6)}$$

where $B^{NS}_{(6)}$ is dual to $B^{NS}_{(2)}$ and $C^R_{(6)}$ is dual to $C^R_{(2)}$

$$dB^{NS}_{(6)} = e^{-2\phi_{\mathrm{NS5}}} * H^{NS}. \tag{5.2}$$

Since there is no flux of H^R through the $\mathbf{S}^3$ which the D5-branes wrap, there is no induced Chern-Simons term and the supergravity solution describes the large N limit of $D = 2+1$ $SU(N)$ SYM theory with no Chern-Simons term. This field theory is classically related to the dimensional reduction of $\mathcal{N} = 1$ SYM in four dimensions. Perturbatively this theory has a moduli space of vacua parametrized by the scalars in the vector multiplet together with the dual photons. An interesting property of this gauge theory is that BPS monopole configurations induce a non-perturbative superpotential which lifts the Coulomb branch [20]. In the 't Hooft limit these effects are of order e^{-N} and are not seen in the supergravity approximation. The supergravity analysis exhibits the existence, in perturbation theory, of a Coulomb branch of vacua.

The fact that the supergravity solution behaves as $g^E_{00} \cong -(\rho - \rho_0)^{1/2} \to 0$ as $\rho \to \rho_0$, means that, in this region, excitations in the background of fixed proper energy correspond to very small energy excitations in the dual field theory. Therefore, despite the metric being singular at $\rho = \rho_0$, the singularity type can capture information about infrared physics of the gauge theory [1].

The background exhibits also an interesting property. In the UV region, we have seen that it reduces essentially to the near horizon limit of NS5-branes wrapped on $\mathbf{S}^3$, with a twist. In the IR region, the metric and dilaton are given by eq. (4.7). It describes NS5-branes with a flux on $\widetilde{\mathbf{S}}^2 \times \widetilde{\mathbf{S}}^1$, i.e. smeared on a ring parametrized by ψ, and wrapped on $\mathbf{S}^3$. A similar feature was observed in the solution of [14,15].

In order to study the moduli space, we follow [27,25,28] and consider the effective action of a probe D5 brane, which corresponds to breaking $SU(N) \to SU(N-1) \times U(1)$

(see also [29], [30] for other probe calculations with four supercharges). In the present D5 brane solution, the NS two-form vanishes, and the effective action is given by

$$S = -\frac{1}{(2\pi)^5 \alpha'^3} \int \left(d^6y \; e^{-\phi_{\rm D5}} \sqrt{-\det\left[G_{ab} + 2\pi\alpha' F_{ab}\right]} - \; C_{(6)} \right) , \tag{5.3}$$

where F_{ab} is the world-volume abelian gauge field. We fix the static gauge where the world-volume coordinates are identified with $x_{0,1,2}, \theta, \beta, \varphi$, so that the four scalar fields $\rho, \psi, \phi_1, \phi_2$ are functions of these coordinates. The effective action contains kinetic terms and a potential for these scalar fields. The potential can be determined by setting to constants the scalar fields $\rho, \psi, \phi_1, \phi_2$ and $F_{ab} = 0$. The locus where the potential vanishes determines the moduli space. For the present background, the potential has a complicated form, but it is easy to see that it vanishes at $\rho = \rho_0$, $c_0 < 0$. The reason is as follows. For $\rho \cong \rho_0$, the metric and dilaton take the simple form as in eq. (4.7), where the five brane is smeared on $\mathbf{S}^1$ parametrized by ψ. We find

$$e^{-\phi_{\rm D5}} \sqrt{-\det\left[G_{ab}\right]} = e^{-2\phi_0} (\frac{N}{2})^{3/2} \rho_0^{1/2} \tilde{\rho} \sin\theta \sin^2\psi \ .$$

Using

$$U \cong -e^{2x+y/2} \sin^2\psi \ , \quad \Delta \cong e^{x+y/4} \sin^2\psi \ ,$$

one can see that in this region $\rho \cong \rho_0$ the three-form field strength reduces to

$$H^{NS} = -N \sin\phi_1 \; d\phi_1 \wedge d\phi_2 \wedge d\psi \tag{5.4}$$

As expected, this is the 3-form corresponding to the five brane (4.7) with flux on $\widetilde{\mathbf{S}}^2 \times \widetilde{\mathbf{S}}^1$. The dual 6-form is then

$$C^R_{(6)} = e^{-2\phi_0} (\frac{N}{2})^{3/2} \rho_0^{1/2} \tilde{\rho} \sin\theta \sin^2\psi \; dx^0 \wedge dx^1 \wedge dx^2 \wedge d\varphi \wedge d\theta \wedge d\beta \ . \tag{5.5}$$

Therefore

$$S_{\rm pot}(\rho = \rho_0) = 0 \ , \quad \text{for } c_0 < 0 \ .$$

Thus the moduli space is at $\rho = \rho_0$, $c_0 < 0$.

The dimension of the moduli space is determined by the number of massless scalar fluctuations at $S_{\rm pot} = 0$. Thus we are to look at the Born-Infeld lagrangian for the solutions with $c_0 < 0$ in the limit $\rho \to \rho_0$. We get the following kinetic terms for the bosonic fields

$$S_{\rm kin} = -\int d^3x \; \left(L^2 \sin^2\psi \partial_i\psi \partial_i\psi + 4g_3^2 \partial_i A \partial_i A \right) \ , \tag{5.6}$$

$$L^2 = \frac{Ne^{-2\phi_0}}{8g_3^2\pi^2\rho_0} \ , \qquad g_3^2 = \frac{(2N\rho_0)^{3/2}}{4\pi} \ ,$$

where we have dualized the gauge field into a compact scalar A of period 2π. It is is easy to see that there is no mass term for ψ at the locus of $S_{\rm pot} = 0$, so ψ and A are moduli. Thus we find a two-dimensional moduli space, as expected for an $\mathcal{N} = 2$ the gauge theory in three space-time dimensions, which gives further evidence of the supersymmetry of the solution. Defining $\eta = \cos\psi$, the moduli space metric takes the form

$$ds^2_{\rm Mod} = L^2 d\eta^2 + 4g_3^2 dA^2 \ , \qquad |\eta| \leq 1 \ , \tag{5.7}$$

which describes a flat closed strip. It is a complex moduli space with a Kahler metric as expected by supersymmetry. [8]

In conclusion, the supergravity solution captures the expected features of the large N limit of three dimensional $\mathcal{N} = 2$ SYM theory in the Coulomb branch.

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[8] A Kahler metric for four dimensional $\mathcal{N} = 4$ broken to $\mathcal{N} = 1$ SYM via mass terms was found in [31].

Appendix A. NS5 branes with $SU(2)_D$ gauge fields for 3+1 SYM

As explained in sect. 2, the natural $SU(2)$ truncation that can lead to a non-abelian solution with eight supercharges is when one truncates the $SO(4)$ gauge fields to $SU(2)_D = (SU(2)_L \times SU(2)_R)_{diag}$. Supersymmetry alone fixes the required $SU(2)$ truncation to be the $SU(2)_D$ one.

Having a solution of wrapped five branes describing 2 + 1 super Yang-Mills with $\mathcal{N} = 2$ supersymmetry, it is natural to look for a solution with $SU(2)_D$ gauge fields dual to $\mathcal{N} = 2$ 3+1 super Yang-Mills. In this appendix we write down the ansatz for a collection of wrapped NS5-branes on the $\mathbf{S}^2$ of the Eguchi-Hanson geometry. The seven dimensional metric has the following form

$$ds_7^2 = e^{2f(r)}(ds^2(\mathbf{R}^{1,3}) + dr^2) + a(r)^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{A.1}$$

The non-abelian ansatz we take for the gauge fields is

$$\begin{aligned} gA^1 &= A(r)d\theta \ , \\ gA^2 &= -A(r)\sin\theta d\phi \ , \\ gA^3 &= \cos\theta d\phi \ , \end{aligned} \tag{A.2}$$

which has the following field strengths

$$\begin{aligned} gF^1 &= \dot{A}(r)dr \wedge d\theta \ , \\ gF^2 &= -\dot{A}(r)\sin\theta dr \wedge d\phi \ , \\ gF^3 &= \sin\theta(A(r)^2 - 1)d\theta \wedge d\phi \ , \end{aligned} \tag{A.3}$$

where $\dot{A} = \frac{dA}{dr}$. These gauge fields preserve the $SU(2)$ symmetry of the $\mathbf{S}^2$ and have the property that they are pure gauge when $A = 1$. Note that the gauge fields satisfy the condition (2.10) $F_a \wedge *F_b = 0$, $a \neq b$. Lastly, we will take the two scalars to depend on the radial coordinate r, so we also have $x(r)$ and $y(r)$.

One way of solving the equations of motion is to insert the ansatz into the Lagrangian (2.11) and solve the equations of motion of the effective quantum mechanical model with r being interpreted as time. One must also impose the zero energy condition on the system.

Plugging into the Lagrangian our ansatz one gets the following contributions

$$
\begin{aligned}
\mathcal{L}_R &= e^{3f}a^2\left(2\frac{e^{2f}}{a^2}+2\left(\frac{\dot{a}}{a}\right)^2+16\frac{\dot{a}}{a}\dot{f}+12\dot{f}^2\right)\\
\mathcal{L}_{scal} &= -e^{3f}a^2\left(\frac{5}{16}\dot{y}^2+3\dot{x}^2\right)\\
\mathcal{L}_{gauge} &= -\frac{1}{2g^2}e^{3f}a^2e^{-2x-y/2}\left(2\frac{\dot{A}^2}{a^2}+\frac{(A^2-1)^2}{a^4}e^{2f}\right)\\
\mathcal{L}_{pot} &= \frac{1}{2}e^{5f}a^2g^2e^{y/2}(3e^{2x}+6e^{-2x}-e^{-6x})
\end{aligned} . \tag{A.4}
$$

The ten dimensional string frame metric is of the form $ds^2_{st}=e^{2f+y/2}ds^2(\mathbf{R}^{1,3})+\ldots$ Since we are interested in NS five brane solutions, it is natural (as in the solution of sect. 3) to look for solutions with no warp factor in the parallel directions, i.e. $y=-4f$. This is consistent with the equations of motion. Indeed, consider the equation of motion for f following from (A.4). It contains a term having second derivatives of a. Using the equation for a, one can express that term in terms of first derivatives. The resulting equation is the same as the equation for y, after setting $y=-4f$ and using the Hamiltonian constraint

$$
\begin{aligned}
0=H=a^2e^{3f}&\left(2\frac{\dot{a}^2}{a^2}+16\frac{\dot{a}}{a}\dot{f}+12\dot{f}^2-\frac{1}{g^2a^2}e^{-2x-y/2}\dot{A}^2-3\dot{x}^2-\frac{5}{16}\dot{y}^2\right)\\
&-2e^{5f}+\frac{1}{2g^2}e^{5f-2x-y/2}\frac{(A^2-1)^2}{a^2}-\frac{1}{2}a^2g^2e^{5f+y/2}(3e^{2x}+6e^{-2x}-e^{-6x})\ .
\end{aligned}
$$

Setting $y=-4f$, and defining

$$
\begin{aligned}
e^{2\alpha} &= e^{3f}a^2\\
e^{2h} &= e^{-2f}a^2,
\end{aligned} \tag{A.5}
$$

one obtains the following Lagrangian

$$
\mathcal{L}_{eff}=e^{2\alpha}(T-V), \tag{A.6}
$$

with

$$
T=4\dot{\alpha}^2-2\dot{h}^2-3\dot{x}^2-\frac{1}{g^2}e^{-2h-2x}\dot{A}^2 \tag{A.7}
$$

and

$$
V=-2e^{-2h}+\frac{1}{2g^2}e^{-4h-2x}(A^2-1)^2-\frac{1}{2}g^2(3e^{2x}+6e^{-2x}-e^{-6x}). \tag{A.8}
$$

It would be interesting to search for BPS equations using this effective Lagrangian.

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