LARGE SCALE GEOMETRY OF COMMUTATOR SUBGROUPS

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ABSTRACT. Let G be a finitely presented group, and G' its commutator subgroup. Let C be the Cayley graph of G' with *all commutators in G* as generators. Then C is large scale simply connected. Furthermore, if G is a torsion-free nonelementary word-hyperbolic group, C is one-ended. Hence (in this case), the asymptotic dimension of C is at least 2.

1. INTRODUCTION

Let G be a group and let $G' := [G, G]$ denote the commutator subgroup of G. The group G' has a canonical generating set S, which consists precisely of the set of commutators of pairs of elements in G. In other words,

$$S = \{[g, h] \text{ such that } g, h \in G\}$$

Let $C_S(G')$ denote the Cayley graph of G' with respect to the generating set S. This graph can be given the structure of a (path) metric space in the usual way, where edges have length 1 by fiat.

By now it is standard to expect that the large scale geometry of a Cayley graph will reveal useful information about a group. However, one usually studies finitely generated groups G and the geometry of a Cayley graph $C_T(G)$ associated to a finite generating set T. For typical infinite groups G, the set of commutators S will be infinite, and the Cayley graph $C_S(G')$ will not be locally compact. This is a significant complication. Nevertheless, $C_S(G')$ has several distinctive properties which invite careful study:

(1) The set of commutators of a group is *characteristic* (i.e. invariant under any automorphism of G), and therefore the semi-direct product $G' \rtimes \mathrm{Aut}(G)$ acts on $C_S(G')$ by isometries

(2) The metric on G' inherited as a subspace of $C_S(G')$ is both left- and right-invariant (unlike the typical Cayley graph, whose metric is merely left-invariant)

(3) Bounded cohomology in G is reflected in the geometry of G'; for instance, the translation length $\tau(g)$ of an element $g \in G'$ is the stable commutator length $\mathrm{scl}(g)$ of g in G

(4) Simplicial loops in $C_S(G')$ through the origin correspond to (marked) homotopy classes of maps of closed surfaces to a $K(G, 1)$

These properties are straightforward to establish; for details, see § 2.

This paper concerns the connectivity of $C_S(G')$ in the large for various groups G. Recall that a *thickening Y* of a metric space X is an isometric inclusion $X \to Y$ into a bigger metric space, such that the Hausdorff distance in Y between X and Y is finite. A metric space X is said to be *large scale k-connected* if for any thickening

Y of X there is another thickening Z of Y which is k-connected (i.e. $\pi_i(Z) = 0$ for $i \leq k$; also see the Definitions in § 3). Our first main theorem, proved in § 3, concerns the large scale connectivity of $C_S(G')$ where G is finitely presented:

Theorem A. *Let G be a finitely presented group. Then $C_S(G')$ is large scale simply connected.*

As well as large scale connectivity, one can study connectivity *at infinity*. In § 4 we specialize to word-hyperbolic G and prove our second main theorem, concerning the connectivity of G' at infinity:

Theorem B. *Let G be a torsion-free nonelementary word-hyperbolic group. Then $C_S(G')$ is one-ended; i.e. for any $r > 0$ there is an $R \geq r$ such that any two points in $C_S(G')$ at distance at least R from* id *can be joined by a path which does not come closer than distance r to* id.

Combined with a theorem of Fujiwara-Whyte [7], Theorem A and Theorem B together imply that for G a torsion-free nonelementary word-hyperbolic group, $C_S(G')$ has asymptotic dimension at least 2 (see § 5 for the definition of asymptotic dimension).

2. Definitions and basic properties

Throughout the rest of this paper, G will denote a group, G' will denote its commutator subgroup, and S will denote the set of (nonzero) commutators in G, thought of as a generating set for G'. Let $C_S(G')$ denote the Cayley graph of G' with respect to the generating set S. As a graph, $C_S(G')$ has one vertex for every element of G', and two elements $g, h \in G'$ are joined by an edge if and only if $g^{-1}h \in S$. Let d denote distance in $C_S(G')$ restricted to G'.

Definition 2.1. Let $g \in G'$. The *commutator length* of g, denoted $\mathrm{cl}(g)$, is the smallest number of commutators in G whose product is equal to g.

From the definition, it follows that $\mathrm{cl}(g) = d(\mathrm{id}, g)$ and $d(g, h) = \mathrm{cl}(g^{-1}h)$ for $g, h \in G'$.

Lemma 2.2. *The group $G' \rtimes \mathrm{Aut}(G)$ acts on $C_S(G')$ by isometries.*

Proof. $\mathrm{Aut}(G)$ acts as permutations of S, and therefore the natural action on G extends to $C_S(G')$. Further, G' acts on $C_S(G')$ by left multiplication. $\square$

Lemma 2.3. *The metric on $C_S(G')$ restricted to G' is left- and right-invariant.*

Proof. Since the inverse of a commutator is a commutator, we have $\mathrm{cl}(g^{-1}h) = \mathrm{cl}(h^{-1}g)$. Since the conjugate of a commutator by any element is a commutator, we have $\mathrm{cl}(h^{-1}g) = \mathrm{cl}(gh^{-1})$. This completes the proof. $\square$

Definition 2.4. Given a metric space X and an isometry h of X, the *translation length* of h on X, denoted $\tau(h)$, is defined by the formula

$$\tau(h) = \lim_{n \to \infty} \frac{d(p, h^n(p))}{n}$$

where $p \in X$ is arbitrary.

By the triangle inequality, the limit does not depend on the choice of p.

For $g \in G'$ acting on $C_S(G')$ by left multiplication, we can take $p = \mathrm{id}$. Then $d(\mathrm{id}, g^n(\mathrm{id})) = \mathrm{cl}(g^n)$.

Definition 2.5. Let G be a group, and $g \in G'$. The *stable commutator length* of g is the limit

$$\operatorname{scl}(g) = \lim_{n \to \infty} \frac{\operatorname{cl}(g^n)}{n}$$

Hence we have the following:

Lemma 2.6. *Let $g \in G'$ act on $C_S(G')$ by left multiplication. There is an equality $\tau(g) = \operatorname{scl}(g)$.*

Proof. This is immediate from the definitions. $\qquad\square$

Stable commutator length is related to two-dimensional (bounded) cohomology. For an introduction to stable commutator length, see [3]; for an introduction to bounded cohomology, see [10].

If X is a metric space, and g is an isometry of X, one can obtain lower bounds on $\tau(g)$ by constructing a Lipschitz function on X which grows linearly on the orbit of a point under powers of g. One important class of Lipschitz functions on $C_S(G')$ are *quasimorphisms*:

Definition 2.7. Let G be a group. A function $\phi : G \to \mathbb{R}$ is a *quasimorphism* if there is a least positive real number $D(\phi)$ called the *defect*, such that for all $g, h \in G$ there is an inequality

$$|\phi(g) + \phi(h) - \phi(gh)| \leq D(\phi)$$

From the defining property of a quasimorphism, $|\phi(\mathrm{id})| \leq D(\phi)$ and therefore by repeated application of the triangle inequality, one can estimate

$$|\phi(f[g, h]) - \phi(f)| \leq 7D(\phi)$$

for any $f, g, h \in G$. In other words,

Lemma 2.8. *Let G be a group, and let $\phi : G \to \mathbb{R}$ be a quasimorphism with defect $D(\phi)$. Then ϕ restricted to G' is $7D(\phi)$-Lipschitz in the metric inherited from $C_S(G')$.*

Word-hyperbolic groups admit a rich family of quasimorphisms. We will exploit this fact in § 4.

3. Large scale simple connectivity

The following definitions are taken from [9], pp. 23–24.

Definition 3.1. A *thickening* Y of a metric space X is an isometric inclusion $X \to Y$ with the property that there is a constant C so that every point in Y is within distance C of some point in X.

Definition 3.2. A metric space X is *large scale k-connected* if for every thickening $X \subset Y$ there is a thickening $Y \subset Z$ which is k-connected in the usual sense; i.e. Z is path-connected, and $\pi_i(Z) = 0$ for $i \leq k$.

For G a finitely generated group with generating set T, Gromov outlines a proof ([9], $1.C_2$) that the Cayley graph $C_T(G)$ is large scale 1-connected if and only if G is finitely presented, and $C_T(G)$ is large scale k-connected if and only if there exists a proper simplicial action of G on a $(k+1)$-dimensional k-connected simplicial complex X with compact quotient X/G.

For T an infinite generating set, large scale simple connectivity is equivalent to the assertion that G admits a presentation $G = \langle T \mid R \rangle$ where all elements in R have *uniformly bounded length* as words in T; i.e. all relations in G are consequences of relations of bounded length.

To show that $C_S(G')$ is large scale 1-connected, it suffices to show that there is a constant K so that for every simplicial loop γ in $C_S(G')$ there are a sequence of loops $\gamma = \gamma_0, \gamma_1, \cdots, \gamma_n$ where γ_n is the trivial loop, and each γ_i is obtained from γ_{i-1} by cutting out a subpath $\sigma_{i-1} \subset \gamma_{i-1}$ and replacing it by a subpath $\sigma_i \subset \gamma_i$ with the same endpoints, so that $|\sigma_{i-1}| + |\sigma_i| \le K$.

More generally, we call the operation of cutting out a subpath σ and replacing it by a subpath σ' with the same endpoints where $|\sigma| + |\sigma'| \le K$ a K-*move*.

Definition 3.3. Two loops γ and γ' are K-*equivalent* if there is a finite sequence of K-moves which begins at γ, and ends at γ'.

K-equivalence is (as the name suggests) an equivalence relation. The statement that $C_S(G')$ is large scale 1-connected is equivalent to the statement that there is a constant K such that every two loops in $C_S(G')$ are K-equivalent.

First we establish large scale simple connectivity in the case of a free group.

Lemma 3.4. *Let F be a finitely generated free group. Then $C_S(F')$ is large scale simply connected.*

Proof. Let γ be a loop in $C_S(F')$. After acting on γ by left translation, we may assume that γ passes through id, so we may think of γ as a simplicial path in $C_S(F')$ which starts and ends at id. If $s_i \in S$ corresponds to the ith segment of γ, we obtain an expression

$$s_1 s_2 \cdots s_n = \mathrm{id}$$

in F, where each s_i is a commutator. For each i, let $a_i, b_i \in F$ be elements with $[a_i, b_i] = s_i$ (note that a_i, b_i with this property are not necessarily unique). Let Σ be a surface of genus n, and let α_i, β_i for $i \le n$ be a standard basis for $\pi_1(\Sigma)$; see Figure 1.
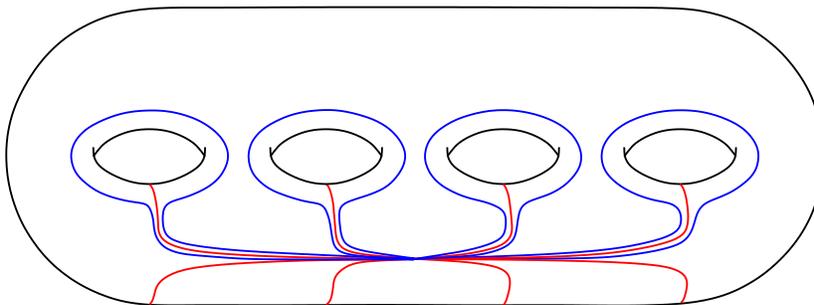
FIGURE 1. A standard basis for $\pi_1(\Sigma)$ where Σ has genus 4. The α_i curves are in red, and the β_i curves are in blue.

Let X be a wedge of circles corresponding to free generators for F, so that $\pi_1(X) = F$. We can construct a basepoint preserving map $f : \Sigma \to X$ with $f_*(\alpha_i) = a_i$ and $f_*(\beta_i) = b_i$ for each i. Since X is a $K(F, 1)$, the homotopy class of f is uniquely determined by the a_i, b_i. Informally, we could say that loops in

$C_S(F')$ correspond to based homotopy classes of maps of marked oriented surfaces into X (up to the ambiguity indicated above).

Let ϕ be a (basepoint preserving) self-homeomorphism of Σ. The map $f \circ \phi : \Sigma \to X$ determines a new loop in $C_S(F')$ (also passing through id) which we denote $\phi_*(\gamma)$ (despite the notation, this image does not depend only on γ, but on the choice of elements a_i, b_i as above).

Sublemma 3.5. *There is a universal constant K independent of γ or of ϕ (or even of F) so that after composing ϕ by an inner automorphism of $\pi_1(\Sigma)$ if necessary, γ and $\phi_*(\gamma)$ as above are K-equivalent.*

Proof. Suppose we can express ϕ as a product of (basepoint preserving) automorphisms

$$\phi = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_1$$

such that if α_i^j, β_i^j denote the images of α_i, β_i under $\phi_j \circ \phi_{j-1} \circ \cdots \circ \phi_1$, then ϕ_{j+1} fixes all but K consecutive pairs α_i^j, β_i^j up to (basepoint preserving) homotopy. Let $s_i^j = [f_*\alpha_i^j, f_*\beta_i^j]$, and let γ^j be the loop in $C_S(F')$ corresponding to the identity $s_1^j s_2^j \cdots s_n^j = \mathrm{id}$ in F.

For each j, let supp_{j+1} denote the *support* of ϕ_{j+1}; i.e. the set of indices i such that $\phi_{j+1}(\alpha_i^j) \neq \alpha_i^j$ or $\phi_{j+1}(\beta_i^j) \neq \beta_i^j$. By hypothesis, supp_{j+1} consists of at most K indices for each j.

Because it is just the marking on Σ which has been changed and not the map f, if $k \leq i \leq k + K - 1$ is a maximal consecutive string of indices in supp_{j+1}, then there is an equality of products

$$s_k^j s_{k+1}^j \cdots s_{k+K-1}^j = s_k^{j+1} s_{k+1}^{j+1} \cdots s_{k+K-1}^{j+1}$$

as elements of F. This can be seen geometrically as follows. The expression on the left is the image under f_* of an element represented by a certain embedded based loop in Σ, while the expression on the right is its image under $f_* \circ \phi_{j+1}$. The automorphism ϕ_{j+1} is represented by a homeomorphism of Σ whose support is contained in regions bounded by such loops. Hence the expressions are equal. It follows that γ^j and γ^{j+1} are $2K$-equivalent.

So to prove the Sublemma it suffices to show that any automorphism of S can be expressed (up to inner automorphism) as a product of automorphisms ϕ_i with the property above.

The hypothesis that we may compose ϕ by an inner automorphism means that we need only consider the image of ϕ in the mapping class group of Σ. It is well-known since Dehn [5] that the mapping class group of a closed oriented surface Σ of genus g is generated by twists in a finite standard set of curves, each of which intersects at most two of the α_i, β_i essentially; see Figure 2.

So write $\phi = \tau_1 \tau_2 \cdots \tau_m$ where the τ_i are all standard generators. Now define

$$\phi_j = \tau_1 \tau_2 \cdots \tau_{j-1} \tau_j \tau_{j-1}^{-1} \cdots \tau_1^{-1}$$

We have

$$\phi_j \phi_{j-1} \cdots \phi_1 = \tau_1 \tau_2 \cdots \tau_j$$

Moreover, each ϕ_j is a Dehn twist in a curve which is the image of a standard curve under $\phi_{j-1} \cdots \phi_1$, and therefore intersects $\alpha_i^{j-1}, \beta_i^{j-1}$ essentially for at most 2 (consecutive) indices i. This completes the proof of the Sublemma (and shows, in fact, that we can take $K = 4$). $\square$
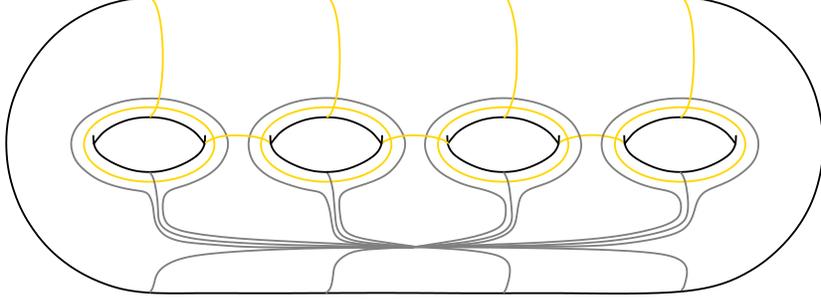
FIGURE 2. A standard set of $3g-1$ simple curves, in yellow. Dehn twists in these curves generate the mapping class group of Σ.

We now complete the proof of the Lemma. As observed by Stallings (see e.g. [14]), a nontrivial map $f : \Sigma \to X$ from a closed, oriented surface to a wedge of circles factors (up to homotopy) through a *pinch* in the following sense. Make f transverse to some edge e of X, and look at the preimage Γ of a regular value of f in e. After homotoping inessential loops of Γ off e, we may assume that for some edge e and some regular value, the preimage Γ contains an embedded essential loop δ.

There are two cases to consider. In the first case, δ is nonseparating. In this case, let ϕ be an automorphism which takes α_1 to the free homotopy class of δ. Then γ and $\phi_*(\gamma)$ are K-equivalent by the Sublemma. However, since $f(\delta)$ is homotopically trivial in X, there is an identity $[\phi_*\alpha_1, \phi_*\beta_1] = \text{id}$ and therefore $\phi_*(\gamma)$ has length 1 shorter than γ.

In the second case, ϕ is separating, and we can let ϕ be an automorphism which takes the free homotopy class of $[\alpha_1, \beta_1] \cdots [\alpha_j, \beta_j]$ to δ. Again, by the Sublemma, γ and $\phi_*(\gamma)$ are K-equivalent. But now $\phi_*(\gamma)$ contains a subarc of length j with both endpoints at id, so we may write it as a product of two loops at id, each of length shorter than that of γ.

By induction, γ is K-equivalent to the trivial loop, and we are done. $\square$

We are now in a position to prove our first main theorem.

Theorem A. *Let G be a finitely presented group. Then $C_S(G')$ is large scale simply connected.*

Proof. Let W be a smooth 4-manifold (with boundary) satisfying $\pi_1(W) = G$. If $G = \langle T \mid R \rangle$ is a finite presentation, we can build W as a handlebody, with one 0-handle, one 1-handle for every generator in T, and one 2-handle for every relation in R. If $r_i \in R$ is a relation, let D_i be the cocore of the corresponding 2-handle, so that D_i is a properly embedded disk in W. Let $V \subset W$ be the union of the 0-handle and the 1-handles. Topologically, V is homotopy equivalent to a wedge of circles. By the definition of cocores, the complement of $\cup_i D_i$ in W deformation retracts to V. See e.g. [12], Chapter 1 for an introduction to handle decompositions of 4-manifolds.

Given γ a loop in $C_S(G')$, translate it by left multiplication so that it passes through id. As before, let Σ be a closed oriented marked surface, and $f : \Sigma \to W$ a map representing γ.

Since G is finitely presented, $H_2(G; \mathbb{Z})$ is finitely generated. Choose finitely many closed oriented surfaces $S_1, \cdots, S_r$ in W which generate $H_2(G; \mathbb{Z})$. Let K' be the supremum of the genus of the S_i. We can choose a basepoint on each S_i, and maps to W which are basepoint preserving. By tubing Σ repeatedly to copies of the S_i with either orientation, we obtain a new surface and map $f' : \Sigma' \to W$ representing a loop γ' such that $f'(\Sigma')$ is null-homologous in W, and γ' is K'-equivalent to γ (note that K' depends on G but not on γ).

Put f' in general position with respect to the D_i by a homotopy. Since $f'(\Sigma')$ is null-homologous, for each proper disk D_i, the signed intersection number vanishes: $D_i \cap f'(\Sigma') = 0$. Hence $f'(\Sigma) \cap D_i = P_i$ is a finite, even number of points which can be partitioned into two sets of equal size corresponding to the local intersection number of $f'(\Sigma')$ with D_i at $p \in P_i$.

Let $p, q \in P_i$ have opposite signs, and let μ be an embedded path in D_i from $f'(p)$ to $f'(q)$. Identifying p and q implicitly with their preimages in Σ', let α and β be arcs in Σ' from the basepoint to $(f')^{-1}p$ and $(f')^{-1}q$. Since μ is contractible, there is a neighborhood of μ in D_i on which the normal bundle is trivializable. Hence, since $f'(\Sigma')$ and D_i are transverse, we can find a neighborhood U of μ in W disjoint from the other D_j, and co-ordinates on U satisfying

(1) $D_i \cap U$ is the plane $(x, y, 0, 0)$
(2) $\mu \cap U$ is the interval $(t, 0, 0, 0)$ for $t \in [0, 1]$
(3) $f'(\Sigma') \cap U$ is the union of the planes $(0, 0, z, w)$ and $(1, 0, z, w)$

Let A be the annulus consisting of points $(t, 0, \cos(\theta), \sin(\theta))$ where $t \in [0, 1]$. Then A is disjoint from D_i and all the other D_j, and we can tube $f'(\Sigma')$ with A to reduce the number of intersection points of $f'(\Sigma')$ with $\cup_i D_i$, at the cost of raising the genus by 1. Technically, we remove the disks $(f')^{-1}(0, 0, s\cos(\theta), s\sin(\theta))$ and $(f')^{-1}(1, 0, s\cos(\theta), s\sin(\theta))$ for $s \in [0, 1]$ from Σ', and sew in a new annulus which we map homeomorphically to A. The result is $f'' : \Sigma'' \to W$ with two fewer intersection points with $\cup_i D_i$. This has the effect of adding a new (trivial) edge to the start of γ', which is the commutator of the elements represented by the core of A and the loop $f'(\alpha) * \mu * f'(\beta)$. Let γ'' denote this resulting loop, and observe that γ'' is 1-equivalent to γ'. After finitely many operations of this kind, we obtain $f''' : \Sigma''' \to W$ corresponding to a loop γ''' which is $\max(1, K')$-equivalent to γ, such that $f'''(\Sigma''')$ is disjoint from $\cup_i D_i$.

After composing with a deformation retraction, we may assume f''' maps Σ''' into V. Let $F = \pi_1(V)$, and let $\rho : F \to G$ be the homomorphism induced by the inclusion $V \to W$. There is a loop γ^F in $C_S(F')$ corresponding to f''' such that $\rho_*(\gamma^F) = \gamma'''$ under the obvious simplicial map $\rho_* : C_S(F') \to C_S(G')$. By Lemma 3.4, the loop γ^F is K-equivalent to a trivial loop in $C_S(F')$. Pushing forward the sequence of intermediate loops by ρ_* shows that γ''' is K-equivalent to a trivial loop in $C_S(G')$. Since γ was arbitrary, we are done. $\qquad\square$

Remark 3.6. A similar, though perhaps more combinatorial argument could be made working directly with 2-complexes in place of 4-manifolds.

In words, Theorem A says that for G a finitely presented group, all relations amongst the commutators of G are consequences of relations involving only boundedly many commutators.

The next example shows that the size of this bound depends on G:

Example 3.7. Let Σ be a closed surface of genus g, and $G = \pi_1(\Sigma)$. If γ is a loop in $C_S(G)$ through the origin, and $f : \Sigma' \to \Sigma$ is a corresponding map of a closed surface, then the homology class of Σ' is trivial unless the genus of Σ' is at least as big as that of Σ. Hence the loop in $C_S(G)$ of length g corresponding to the relation in the "standard" presentation of $\pi_1(\Sigma)$ is not K-equivalent to the trivial loop whenever $K < g$.

In light of Theorem A, it is natural to ask the following question:

Question 3.8. *Let G be a finitely presented group. Is $C_S(G')$ large scale k-connected for all k?*

Remark 3.9. Laurent Bartholdi has pointed out that for F a finitely generated free group, there is a confluent, Noetherian rewriting system for F', with rules of bounded length, which puts every word in F' over generators S into normal form (with respect to a "standard" free generating set for F'). By results of Groves ([11]) this should imply that $C_S(F')$ is large scale k-connected for all k, but we have not verified this implication carefully. In any case, it gives another more algebraic proof of Lemma 3.4.

4. Word-hyperbolic groups

In this section we specialize to the class of *word-hyperbolic groups*. See [8] for more details.

Definition 4.1. A path metric space X is *δ-hyperbolic* for some $\delta \geq 0$ if for every geodesic triangle abc, and every point p on the edge ab, there is $q \in ac \cup bc$ with $d_X(p,q) \leq \delta$. In other words, the δ neighborhood of any two sides of a geodesic triangle contains the third side.

Definition 4.2. A group G is *word-hyperbolic* if there is a finite generating set T for G such that $C_T(G)$ is δ-hyperbolic as a path metric space, for some δ.

Example 4.3. Finitely generated free groups are word-hyperbolic. The fundamental group of a closed surface with negative Euler characteristic is word-hyperbolic. Discrete cocompact groups of isometries of hyperbolic n-space are word-hyperbolic.

To rule out some trivial examples, one makes the following:

Definition 4.4. A word-hyperbolic group is *elementary* if it has a cyclic subgroup of finite index, and *nonelementary* otherwise.

The main theorem we prove in this section concerns the geometry of $C_S(G')$ at infinity, where G is a nonelementary word-hyperbolic group. For the sake of brevity we restrict attention to torsion-free G, though this restriction is not logically necessary; see Remark 4.9.

Theorem B. *Let G be a torsion-free nonelementary word-hyperbolic group. Then $C_S(G')$ is one-ended; i.e. for any $r > 0$ there is an $R \geq r$ such that any two points in $C_S(G')$ at distance at least R from* id *can be joined by a path which does not come closer than distance r to* id.

We will estimate distance to id in $C_S(G')$ using quasimorphisms, as indicated in § 2. Hyperbolic groups admit a rich family of quasimorphisms. Of particular

interest to us are the *Epstein-Fujiwara counting quasimorphisms*, introduced in [6], generalizing a construction due to Brooks [2] for free groups.

Fix a word-hyperbolic group G and a finite generating set T. Let $C_T(G)$ denote the Cayley graph of G with respect to T. Let σ be an oriented simplicial path in $C_T(G)$. A *copy* of σ is a translate $g \cdot \sigma$ for some $g \in G$. If γ is an oriented simplicial path in $C_T(G)$, let $|\gamma|_\sigma$ denote the maximal number of disjoint copies of σ contained in γ. For $g \in G$, define

$$c_\sigma(g) = d(\mathrm{id}, g) - \inf_\gamma (\mathrm{length}(\gamma) - |\gamma|_\sigma)$$

where the infimum is taken over all directed paths γ in $C_T(G)$ from id to g, and $d(\cdot, \cdot)$ denotes distance in $C_T(G)$.

Definition 4.5 (Epstein-Fujiwara). A *counting quasimorphism* on G is a function of the form

$$h_\sigma(g) := c_\sigma(g) - c_{\sigma^{-1}}(g)$$

where σ^{-1} denotes the same simplicial path as σ with the opposite orientation.

Since $|\gamma|_\sigma$ takes discrete values, the infimum is realized in the definition of c_σ. A path γ for which

$$c_\sigma(g) = d(\mathrm{id}, g) - \mathrm{length}(\gamma) + |\gamma|_\sigma$$

is called a *realizing path* for g. Realizing paths exist, and satisfy the following geometric property:

Lemma 4.6 (Epstein-Fujiwara, [6] Prop. 2.2). *Any realizing path for g is a K, ϵ-quasigeodesic in $C_T(G)$, where*

$$K = \frac{\mathrm{length}(\sigma)}{\mathrm{length}(\sigma) - 1}, \quad and \quad \epsilon = \frac{2 \cdot \mathrm{length}(\sigma)}{\mathrm{length}(\sigma) - 1}$$

Moreover,

Lemma 4.7 (Epstein-Fujiwara, [6] Prop. 2.13). *Let σ be a path in $C_T(G)$ of length at least 2. Then there is a constant $K(\delta)$ (where T is such that $C_T(G)$ is δ-hyperbolic as a metric space) such that $D(h_\sigma) \le K(\delta)$.*

Counting quasimorphisms are very versatile, as the following lemma shows:

Lemma 4.8. *Let G be a torsion-free, nonelementary word-hyperbolic group. Let g_i be a finite collection of elements of G. There is a commutator $s \in G'$ and a quasimorphism ϕ on G with the following properties:*

(1) $|\phi(g_i)| = 0$ *for all i*
(2) $|\phi(s^n) - n| \le K_1$ *for all n, where K_1 is a constant which depends only on G.*
(3) $D(\phi) \le K_2$ *where K_2 is a constant which depends only on G*

Proof. Fix a finite generating set T so that $C_T(G)$ is δ-hyperbolic. There is a constant N such that for any nonzero $g \in G$, the power g^N fixes an axis L_g ([8]). Since G is nonelementary, it contains quasigeodesically embedded copies of free groups, of any fixed rank. So we can find a commutator s whose translation length (in $C_T(G)$) is as big as desired. In particular, given $g_1, \cdots, g_j$ we choose s with $\tau(s) \gg \tau(g_i)$ for all i. Let L be a geodesic axis for s^N, and let σ be a fundamental domain for the action of s^N on L. Since $|\sigma| = N\tau(s) \gg \tau(g_i)$, Lemma 4.6 implies

that there are no copies of σ or σ^{-1} in a realizing path for any g_i. Hence $h_\sigma(g_i) = 0$ for all i. By Lemma 4.7, $D(h_\sigma) \leq K(\delta)$. It remains to estimate $h_\sigma(s^n)$.

In fact, the argument of [4] Theorem A$'$ (which establishes explicitly an estimate that is implicit in [6]) shows that for N sufficiently large (depending only on G and not on s) no copies of σ^{-1} are contained in any realizing path for s^n with n positive, and therefore $|h_\sigma(s^n) - \lfloor n/N \rfloor|$ is bounded by a constant depending only on G. The quasimorphism $\phi = N \cdot h_\sigma$ has the desired properties. $\square$

Remark 4.9. The hypothesis that G is torsion-free is included only to ensure that s is not conjugate to s^{-1}. It is possible to remove this hypothesis by taking slightly more care in the definition of s, using the methods of the proof of Proposition 2 from [1]. We are grateful to the referee for pointing this out.

We now give the proof of Theorem B:

Proof. Let $g, h \in G'$ have commutator length at least R. Let $g = s_1 s_2 \cdots s_n$ and $h = t_1 t_2 \cdots t_m$ where $n, m \geq R$ are equal to the commutator lengths of g and h respectively, and each s_i, t_i is a commutator in G. Let s be a commutator with the properties described in Lemma 4.8 with respect to the elements g, h; that is, we want s for which there is a quasimorphism ϕ with $\phi(g) = \phi(h) = 0$, with $|\phi(s^n) - n| \leq K_1$ for all n, and with $D(\phi) \leq K_2$. Let $N \gg R$ be very large. We build a path in $C_S(G')$ from g to h out of four segments, none of which come too close to id.

The first segment is

$$g, gs, gs^2, gs^3, \cdots, gs^N$$

Since s is a commutator, $d(gs^i, \mathrm{id}) \geq R - i$ for any i. On the other hand,

$$\phi(gs^i) \geq \phi(g) + \phi(s^i) - D(\phi) \geq i - K_2 - K_1$$

where K_1, K_2 are as in Lemma 4.8 (and do not depend on g, h, s). From Lemma 2.8 we can estimate

$$d(gs^i, \mathrm{id}) \geq \frac{\phi(gs^i)}{7D(\phi)} \geq \frac{i - K_2 - K_1}{7K_2}$$

Hence $d(gs^i, \mathrm{id}) \geq R/14K_2 - (K_1 + K_2)/7K_2$ for all i, so providing $R \gg K_1, K_2$, the path gs^i never gets too close to id.

The second segment is

$$gs^N = s_1 s_2 \cdots s_n s^N, s_2 \cdots s_n s^N, \cdots, s^N$$

Note that consecutive elements in this segment are distance 1 apart in $C_S(G')$, by Lemma 2.3. Since $d(gs^N, \mathrm{id}) \geq (N - K_2 - K_1)/7K_2 \gg R$ for N sufficiently large, we have

$$d(s_i \cdots s_n s^N, \mathrm{id}) \gg R$$

for all i.

The third segment is

$$s^N, t_m s^N, t_{m-1} t_m s^N, \cdots, t_1 t_2 \cdots t_m s^N = hs^N$$

and the fourth is

$$hs^N, hs^{N-1}, \cdots, hs, h$$

For the same reason as above, neither of these segments gets too close to id. This completes the proof of the theorem, taking $r = R/14K_2 - (K_1 + K_2)/7K_2$. $\square$

5. Asymptotic dimension

The main point of this section is to make the observation that G' for G as above is not a quasi-tree, and to restate this observation in terms of asymptotic dimension. We think it is worth making this restatement explicitly. The notion of asymptotic dimension is introduced in [9], p. 32.

Definition 5.1. Let X be a metric space, and $X = \cup_i U_i$ a covering by subsets. For given $D \geq 0$, the *D-multiplicity* of the covering is at most n if for any $x \in X$, the closed D-ball centered at x intersects at most n of the U_i.

A metric space X has *asymptotic dimension at most n* if for every $D \geq 0$ there is a covering $X = \cup_i U_i$ for which the diameters of the U_i are uniformly bounded, and the D-multiplicity of the covering is at most $n+1$. The least such n is the *asymptotic dimension* of X, and we write

$$\text{as dim}(X) = n$$

If X is a metric space, we say $H_1(X)$ is *uniformly generated* if there is a constant L such that $H_1(X)$ is generated by loops of length at most L. It is clear that if X is large scale 1-connected, then $H_1(X)$ is uniformly generated. Fujiwara-Whyte [7] prove the following theorem:

Theorem 5.2 (Fujiwara-Whyte, [7], Thm. 0.1). *Let X be a geodesic metric space with $H_1(X)$ uniformly generated. X has as $\dim(X) = 1$ if and only if X is quasi-isometric to an unbounded tree.*

A group whose Cayley graph is quasi-isometric to an unbounded tree has more than one end (see e.g. Manning [13], especially § 2.1 and § 2.2). Hence Theorem A and Theorem B together imply the following:

Corollary 5.3. *Let G be a nonelementary torsion-free word-hyperbolic group. Then*

$$\text{as dim}(C_S(G')) \geq 2$$

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