A Probabilistic Approach to Structural Model Updating

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ABSTRACT: The problem of updating a structural model and its associated uncertainties by utilizing measured dynamic response data is addressed. A Bayesian probabilistic formulation is followed to obtain the posterior probability density function (PDF) of the uncertain model parameters for given measured data. The present paper discusses the issue of identifiability of the model parameters and reviews existing asymptotic approximations for identifiable cases. The focus of the paper is on the treatment of the general non-identifiable case where the earlier approximations are not applicable. In this case the posterior PDF of the parameters is found to be concentrated in the neighborhood of an extended and extremely complex manifold in the parameter space. The computational difficulties associated with calculating the posterior PDF in such case are discussed and an algorithm for an efficient approximate representation of the above manifold and the posterior PDF is presented. Numerical examples involving noisy data are presented to demonstrate the concepts and the proposed method.

Key words: system identification, model updating, Bayesian method, uniqueness, structural dynamics.

1 INTRODUCTION

Over the last decade or so, the challenging problem of *structural model updating* has gained much interest as finite-element modeling capabilities and modal testing have become more mature areas of structural dynamics (e.g. refs.[6,9–11]). In model updating, linearity is usually assumed and the finite-element model of a structure is adjusted so that either the calculated response time histories, frequency response functions or modal parameters "best" match the corresponding quantities measured or identified from the test data. In practice, this inverse problem is usually ill-conditioned and nonunique (e.g. refs.[4,7,8,13]) because of insufficient data relative to the desired model complexity.

The need for model updating arises because there are always errors associated with the process of constructing a theoretical model of a structure, and this leads to uncertain accuracy in the predictive response. Because of these modeling errors, model updating is best tackled as a statistical inference problem. This can be done by embedding the class of "deterministic" structural models within a class of probability models so that the structural models give a predictable ("systematic") part and the prediction error is modeled as an uncertain ("random") part (refs.[1,2]). Beck and Katafygiotis[3] presented a general Bayesian statistical system identification framework which properly handles the uncertainties and nonuniqueness associated with model updating. Furthermore, they described an asymptotic approximation for the multi-dimensional integrals that arise when calculating the updated probabilistic predictions of the structural response. This approximation is valid for identifiable cases, which usually occur when the number of the updating parameters is relatively small. In such a case the posterior PDF of the model parameters is very peaked at a finite number of optimal points, at which some positive measure-of-fit function is minimized, and it is practically concentrated in the immediate neighborhood of these optimal points. It was shown that in this case the probability distribution of the predictive response can be asymptotically

approximated by a sum of weighted Gaussian distributions centered at the predictive responses of the optimal models. In the same paper[3] the authors presented an asymptotic approximation for the posterior PDF of the parameters using a finite sum of appropriately weighted Gaussian distributions centered at the optimal points. Katafygiotis and Beck[8] presented an algorithm for resolving the problem of model identifiability[8] , that is, given an optimal model, they resolved the problem of finding all other output-equivalent optimal models.

This paper is based on the general Bayesian statistical model updating framework presented in ref.[3] Herein we address the computational difficulties associated with the implementation of this general methodology in a non-identifiable case, where the asymptotic approximations presented in ref.[3] are not valid. In this case, it is found that the posterior PDF is concentrated in the neighborhood of an extended and extremely complex manifold in the parameter space. Analytical representation of this manifold is usually not possible and, therefore, issues related to its numerical calculation and its practical representation arise. Herein an algorithm is proposed for approximating this manifold by a finite number of its points and it is shown that for practical applications the posterior PDF of the model parameters can be approximated by a finite number of weighted Dirac delta functions centered at the computed points on the manifold. Furthermore, the PDF of the uncertain predictive response can be approximated by a weighted sum of the PDFs of the predictive response corresponding to these points.

2 STATISTICAL SYSTEM IDENTIFICATION

A short overview of the Bayesian statistical system identification framework[3] upon which this paper is based and of the related asymptotic results for identifiable cases is next presented. Structural model updating is viewed as a system identification problem which involves choosing the "best" (optimal)

model(s) from a specified class of structural models $\mathcal{M}$ defined by prescribing a functional relationship $\mathbf{q}(n; \mathbf{a}, Z_1^n, \mathcal{M})$ for the input-output behavior of a structure, where $\mathbf{a} \in S(\mathbf{a}) \subset R^{N_a}$ are the "free" parameters which need to be assigned values from a region $S(\mathbf{a})$ in order to choose a particular model $M(\mathbf{a})$ in $\mathcal{M}$; $\mathbf{q}(n; \mathbf{a}) = \mathbf{q}(n; \mathbf{a}, Z_1^n, \mathcal{M}) \in R^{N_d}$ is the *model* output vector at N_d DOF (degrees of freedom) at time $t_n = n\Delta t$ where Δt is a prescribed sampling interval; and $Z_1^n = \{\mathbf{z}(m) \in R^{N_I} : m = 1, 2, \ldots, n\}$ is the *system* input up to this time.

The Bayesian statistical framework is set up by embedding the class of deterministic structural models $\mathcal{M}$ in a class of probability models $\mathcal{M}_P$ for the system output. This is done by choosing, in addition to $\mathcal{M}$, a class $\mathcal{P}$ of maximum-entropy probability models $P(\sigma)$ for the prediction-error, where σ is a prediction-error model parameter. The class of probability models $\mathcal{M}_P$, which is defined by the selection of the classes $\mathcal{M}$ and $\mathcal{P}$, is parameterized by $\boldsymbol{\alpha} = [\mathbf{a}^T, \sigma]^T \in S(\boldsymbol{\alpha}) \subset R^{N_\alpha}$ where $N_\alpha = N_a + 1$. For a given value of $\boldsymbol{\alpha}$ the PDF of the system output sequences at the observed and unobserved DOF, $Y_1^M = \{\mathbf{y}(n) \in R^{N_o} : n = 1, 2, \ldots, M\}$ and $X_1^M = \{\mathbf{x}(n) \in R^{N_d - N_o} : n = 1, 2, \ldots, M\}$ respectively, is given by:

$$p(Y_1^M, X_1^M \mid \boldsymbol{\alpha}, Z_1^M, \mathcal{M}_P) = \frac{1}{(\sqrt{2\pi}\sigma)^{MN_d}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=1}^{M} \|\mathbf{y}(n) - S_o\mathbf{q}(n; \mathbf{a})\|^2\right] \times \exp\left[-\frac{1}{2\sigma^2}\sum_{n=1}^{M} \|\mathbf{x}(n) - S_u\mathbf{q}(n; \mathbf{a})\|^2\right] \tag{1}$$

where $\|.\|$ denotes the standard Euclidean 2-norm and S_o and S_u are selection matrices for the observed and unobserved DOF respectively, which have only one non-zero element, equal to unity, in each row and column. In order to account for the uncertainty in the values for the parameters $\boldsymbol{\alpha}$, the specification of $\mathcal{M}_P$ also involves choosing an initial ("prior") PDF $\pi(\boldsymbol{\alpha})$ over the set $S(\boldsymbol{\alpha})$ of possible parameter values, that is:

$$p(\boldsymbol{\alpha} \mid \mathcal{M}_P) = \pi(\boldsymbol{\alpha}) \tag{2}$$

The choice for $\pi(\boldsymbol{\alpha})$ allows engineering judgement about the plausibilities of the different models to be incorporated. It can be chosen as a smooth slowly-varying PDF which is mathematically convenient and roughly reflects the engineer's judgement regarding the relative plausibilities of the different values of the parameters $\boldsymbol{\alpha}$.

An updated ("posterior") PDF for the system parameters can be obtained for given dynamic test data $\mathcal{D}_N$ consisting of sampled input and output time histories, $\hat{Z}_1^N = \{\hat{\mathbf{z}}(n) \in R^{N_I} : n = 1, 2, \ldots, N\}$ and $\hat{Y}_1^N = \{\hat{\mathbf{y}}(n) \in R^{N_o} : n = 1, 2, \ldots, N\}$, respectively, by applying Bayes' Theorem:

$$p(\boldsymbol{\alpha} \mid \mathcal{D}_N, \mathcal{M}_P) = c f_N(\hat{Y}_1^N; \boldsymbol{\alpha}, \hat{Z}_1^N)\, \pi(\boldsymbol{\alpha}) \tag{3}$$

where

$$f_N(\hat{Y}_1^N; \boldsymbol{\alpha}, \hat{Z}_1^N) = \frac{1}{(\sqrt{2\pi}\sigma)^{NN_o}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=1}^{N} \|\hat{\mathbf{y}}(n) - S_o\mathbf{q}(n; \mathbf{a})\|^2\right] \tag{4}$$

and c is a normalizing constant such that the integral on the right side of (3) over the space $S(\boldsymbol{\alpha})$ is equal to unity.

The posterior PDF of the structural model parameters $\mathbf{a}$ can be obtained from:

$$p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P) = \int_0^\infty c f_N(\hat{Y}_1^N; \boldsymbol{\alpha}, \hat{Z}_1^N)\, \pi(\boldsymbol{\alpha}) d\sigma \tag{5}$$

Assuming that the prior distribution $\pi(\boldsymbol{\alpha})$ is a slowly varying function of σ and that the number N of observed data is large it can be found that the integrand in the right hand side of (5) is very peaked at the value

$$\hat{\sigma}^2(\mathbf{a}) = \frac{1}{N_o N}\sum_{n=1}^{N} \|\hat{\mathbf{y}}(n) - S_o\mathbf{q}(n; \mathbf{a})\|^2 = J(\mathbf{a}) \tag{6}$$

The integral in (5) can then be calculated using the asymptotic approximation described in ref.[12] to

obtain:

$$p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P) = c_1 J(\mathbf{a})^{-N_J} \pi(\mathbf{a}, \hat{\sigma}(\mathbf{a})) \tag{7}$$

where $N_J = (NN_0 - 1)/2$, c_1 is a normalizing constant, and $\hat{\sigma}(\mathbf{a})$, $J(\mathbf{a})$ are defined in (6). Note that for a large number of observed data points N, which is usually the case in practice, the exponent N_J in (7) is a large number and, therefore, the relative posterior probabilities of the various structural model parameters $\mathbf{a}$ are very sensitive to the corresponding values $J(\mathbf{a})$. Specifically, $p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ becomes negligible everywhere, except for the region of the parameter space where the corresponding values of $J(\mathbf{a})$ are very close to the global minimum $\hat{\sigma}^2 = \min\{J(\mathbf{a}), \mathbf{a} \in S(\mathbf{a})\}$. Thus, the region of important probabilities extends around the points which globally minimize $J(\mathbf{a})$. The parameter values at which $J(\mathbf{a})$ reaches its global minimum are named *optimal parameter* values. Note that the model output $\mathbf{q}(n; \mathbf{a})$ involved in the definition of $J(\mathbf{a})$ in (6) is a nonlinear function of the parameters $\mathbf{a}$, even in the case of linear dynamic models. Therefore, multiple optimal parameter values may exist for which $J(\mathbf{a})$ might attain its global minimum. This raises the issue of (system) identifiability[3] of these parameters. It is reminded that ref.[3] defines a parameter to be (system) identifiable when there exists either only one optimal value for this parameter (global identifiability) or, in the case where more than one optimal values exist, when the distance between any two optimal values is finite (local identifiability). In a (system) identifiable case the set of optimal parameter points is discrete. Furthermore, assuming that the parameter domain $S(\boldsymbol{a})$ is bounded, identifiability implies the existence of a finite number of optimal points $\hat{\mathbf{a}}^{(k)}, k = 1, \ldots, K$, satisfying:

$$J(\hat{\mathbf{a}}^{(k)}) = \min_{\mathbf{a} \in S(\mathbf{a})} J(\mathbf{a}) = \hat{\sigma}^2 \tag{8}$$

The task of finding all global minima of the nonconvex function $J(\mathbf{a})$ is nontrivial. Katafygiotis and Beck[8] presented an algorithm for identifiable cases which finds all output-equivalent optimal parameters

once one of them is given.

The values $\hat{\boldsymbol{\alpha}}^{(k)} = [\hat{\mathbf{a}}^{(k)}, \hat{\sigma}]^T, k = 1, \ldots, K$ are referred to as the optimal values for the parameters $\boldsymbol{\alpha}$. Assuming a large number of data N and a relatively slowly varying prior $\pi(\boldsymbol{\alpha})$ it can be shown that at the optimal values $\hat{\boldsymbol{\alpha}}^{(k)}$ the function f_N in (4) reaches its global maximum. The optimal prediction-error parameter $\hat{\sigma}$ is always globally identifiable[3] and is given by (8). Therefore, the issue of identifiability concerns only the structural model parameters $\mathbf{a}$. These parameters could be either (globally or locally) identifiable or non-identifiable. The larger the dimension N_a of the parameter vector $\mathbf{a}$, the more likely it is to encounter non-identifiability of $\mathbf{a}$.

Beck and Katafygiotis[3] derived an asymptotic approximation for the posterior PDF of the structural model parameters $\mathbf{a}$ which is valid for identifiable cases and for a large number N of sampling points:

$$p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P) \approx \sum_{k=1}^{K} w_k N(\hat{\mathbf{a}}^{(k)}, A_N^{-1}(\hat{\mathbf{a}}^{(k)})) \tag{9}$$

where $N(\boldsymbol{\mu}, \Sigma)$, denotes a multivariate Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix Σ. Equation (9) approximates the posterior PDF of the parameters $\mathbf{a}$ with a finite sum of weighted Gaussian distributions centered at the optimal values $\hat{\mathbf{a}}^{(k)}, k = 1, \ldots, K$. The matrix $A_N(\hat{\mathbf{a}})$ in (9) can be shown to be equal to the Hessian of the function $g(\mathbf{a}) = N_J \ln J(\mathbf{a})$ evaluated at $\hat{\mathbf{a}}$. Finally the weighting coefficients w_k are given by:

$$w_k = \frac{w_k'}{\sum_{k=1}^{K} w_k'} \quad \text{where} \quad w_k' = \pi(\hat{\mathbf{a}}^{(k)}) \left| A_N(\hat{\mathbf{a}}^{(k)}) \right|^{-1/2} \tag{10}$$

and account for the total probability in the neighborhood of each optimal point.

The updated predictive PDF at the unobserved DOFs for the given input $\hat{Z}_1^N$ as well as the updated predictive PDF at both the observed and unobserved DOFs for a prescribed future input $\hat{Z}_{N+1}^M$ is given

by the Total Probability Theorem as:

$$p(Y_{N+1}^M, X_1^M \mid \mathcal{D}_N, \hat{Z}_{N+1}^M, \mathcal{M}_P) = \int_{S(\boldsymbol{\alpha})} p(Y_{N+1}^M, X_1^M \mid \boldsymbol{\alpha}, \hat{Z}_1^M, \mathcal{M}_P) p(\boldsymbol{\alpha} \mid \mathcal{D}_N, \mathcal{M}_P)\, d\boldsymbol{\alpha} \tag{11}$$

The updated predictive PDF in (11) using the whole class $\mathcal{M}_P$ and the data $\mathcal{D}_N$ can be viewed as a weighted average of the predictive PDFs for each model in $\mathcal{M}_P$, where the weights are given by the updated probabilities for each model.

To avoid the computationally prohibitive problem of performing numerical integration of (11) when the dimension of the parameter space $S(\boldsymbol{\alpha})$ is not small, Beck and Katafygiotis[3] derived an asymptotic approximation for the updated predictive PDF which is valid for identifiable cases and for a large number N of sampling points:

$$p(Y_{N+1}^M, X_1^M \mid \mathcal{D}_N, \hat{Z}_{N+1}^M, \mathcal{M}_P) \approx \sum_{k=1}^{K} w_k\, p(Y_{N+1}^M, X_1^M \mid \hat{\boldsymbol{\alpha}}^{(k)}, \hat{Z}_1^M, \mathcal{M}_P) \tag{12}$$

Expression (12) states that under the aforementioned assumptions, the updated predictive PDF can be approximated by a weighted average of the predictive PDFs of only the finite number of optimal models.

3 NON-IDENTIFIABLE CASE

The asymptotic results (9) and (12) presented in the previous section assume that the posterior PDF of the parameters is concentrated in the close neighborhood of a finite number of optimal points. This assumption is satisfied only under the following two conditions: 1) the structural model parameter vector $\mathbf{a}$ is (system) identifiable under $\mathcal{D}_N$ and 2) the number of measured data N is sufficiently large. It must be noted that identifiability of $\mathbf{a}$ implies identifiability of $\boldsymbol{\alpha}$ because the prediction-error parameter σ is always globally identifiable at $\hat{\sigma}$.

The condition of identifiability ensures that the set of optimal parameters consists of discrete points.

Furthermore, assuming that the parameter domain $S(\mathbf{a})$ is bounded, system identifiability implies the existence of a finite number of optimal points. On the contrary, in a non-identifiable case there exists an infinite number of optimal points and the set of these points is not discrete but continuous. A simple example of a non-identifiable case is the case of a SDOF oscillator subjected to ground motion when both the mass and stiffness are included in the parameters to be updated. It is easy to show in this case that given an optimal model all other models with proportional mass and stiffness are output-equivalent and, therefore, also optimal. In this case the set of optimal points corresponds to a straight line in the parameter space and the posterior PDF of the parameters is concentrated in the neighborhood of this line, instead of being concentrated in the neighborhood of a discrete set of points as assumed in the aforementioned approximations. In the general non-identifiable case the set of optimal points forms a manifold $\mathfrak{S} \subset S(\mathbf{a})$ with dimension $1 \leq N_{\mathfrak{S}} \leq N_a$. In the above SDOF example $N_{\mathfrak{S}} = 1$.

The condition of having a sufficiently large number of data N, or equivalently a sufficiently large exponent N_J in (7), is needed in order to guarantee that $p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ decays rapidly in all directions around any optimal point. Assuming an identifiable case, this condition ensures that the posterior PDF is concentrated in a number of disjoint regions, each corresponding to the close neighborhood of an optimal point. If the decay of $p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ is not sufficiently rapid, the neighborhoods of significant probability corresponding to two optimal points may be overlapping. In addition, rapid decay of $p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ ensures that the assumed Gaussian approximation of $p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ is sufficiently accurate in the neighborhood of each optimal point where probabilities are significant. Slow decay in certain directions implies that the PDF is concentrated in the close neighborhood of a manifold in the parameter space, denoted as earlier by $\mathfrak{S}$, extending along the directions of slow decay. The dimension of the manifold $N_{\mathfrak{S}}$ is equal to the number of directions of slow decay, and satisfies $1 \leq N_{\mathfrak{S}} \leq N_a$.

The points on this manifold, other than the optimal points, have corresponding values of $J(\mathbf{a})$ slightly larger than the global minimum value $\hat{\sigma}^2$ and, therefore, have significant probability without necessarily belonging in the close neighborhood of an optimal point. In this case the finite number of optimal points does not suffice to describe the posterior PDFs of the parameters and the predictive response. This case, although being identifiable according to the definitions in ref.[3] , appears to have many similarities to a non-identifiable case, and from a practical standpoint should be classified as non-identifiable with the points on the manifold behaving as "almost" optimal points.

The above discussion leads us to introduce a new definition for identifiability, to which we will refer to as *system identifiability of order* R, where $R \geq 0$.

Definition: The parameters $\mathbf{a}$ are said to be *system identifiable of order* R for the class $\mathcal{M}_P$ and the data $\mathcal{D}_N$ if for any optimal parameter $\hat{\mathbf{a}}$, that is, any value such that: $J(\hat{\mathbf{a}}) = \min_{\mathbf{a}\in S(\mathbf{a})} J(\mathbf{a})$, the following condition is satisfied:

$$\min_{1\leq i\leq N_a} \lambda_i(A_N(\hat{\mathbf{a}})) > R \geq 0 \tag{13}$$

where $A_N(\hat{\mathbf{a}})$ is the matrix appearing in (9), namely the Hessian matrix of the function $g(\mathbf{a}) = N_J \ln J(\mathbf{a})$ evaluated at $\hat{\mathbf{a}}$, and $\lambda_i(A_N(\hat{\mathbf{a}}))$, $i = 1, \ldots, N_a$ are the eigenvalues of this matrix.

The above definition ensures that the set of optimal points is discrete because it requires the Hessian of $J(\mathbf{a})$ to be a positive definite matrix as can be seen from the definition (13). Furthermore, assuming a slowly varying prior distribution, it ensures that the curvature of the function $\ln p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ calculated in any direction and at any of the optimal points is larger than $R > 0$. This guarantees that if one moves away from any optimal point a distance x in any direction, the PDF will decay faster than $\exp(-Rx^2/2)$. Let us introduce a threshold value ε, where $0 < \varepsilon < 1$, such that only points with relative PDF values larger than ε are considered as having significant probability. One may introduce the

following alternative definition for identifiability (involving two parameters r and ε): "The parameters $\mathbf{a}$ are said to be identifiable if the neighborhood of significant probabilities (specified by ε) around any optimal point $\hat{\mathbf{a}}$ is contained within a sphere of radius r centered at $\hat{\mathbf{a}}$". The latter definition seems more natural but involves the specification of two parameters. By choosing R so that, $\varepsilon = \exp(-Rr^2/2)$ the two definitions become equivalent. The latter observation is useful for selecting the desired value for the parameter R. Clearly the larger R, the stronger the above definition of identifiability becomes. It must be noted that system identifiability of the parameters $\mathbf{a}$ according to the definition in ref.[3] corresponds to system identifiability of order $R = 0$ under the current definition.

Based on the above definition, non-identifiability receives a less restrictive interpretation. Specifically, the strict definition of non-identifiability of ref.[3] , corresponds to the particular case of non-identifiability of order $R = 0$. In this case there exist one or more directions along which $J(\mathbf{a})$ remains flat as one moves away from an optimal point, which implies the existence of a non-discrete (infinite) set of optimal solutions. Non-identifiability of order $R > 0$ implies either a non-discrete (infinite) set of optimal solutions, or a finite number of optimal solutions but with unsatisfactory decay rate. In a non-identifiable case of order R, the region of significant probabilities is contained in the neighborhood of a manifold $\mathfrak{S}$, with dimension $N_{\mathfrak{S}}$ determined by the number of the eigenvalues $\lambda_i(A_N(\hat{\mathbf{a}}))$ which satisfy the condition $\lambda_i(A_N(\hat{\mathbf{a}})) \leq R$. The region of significant probabilities and the neighborhood of the manifold are assumed here to be specified as earlier through the parameters ε and r, where $\varepsilon = \exp(-Rr^2/2)$; the neighborhood of the manifold refers to the points of the parameter space which lie at distance less or equal to r from some point on the manifold. Clearly in a non-identifiable case $1 \leq N_{\mathfrak{S}} \leq N_a$. Note that the parameter ε, used to specify the region of points with significant probability, can be also used to define the boundaries of $\mathfrak{S}$. That is, $\mathfrak{S}$ extends around the optimal points in the directions where $J(\mathbf{a})$ is flat or almost flat and until the value of $J(\mathbf{a})$ reduces to the level where the corresponding relative

probabilities become less than ε. Finally, note that $\mathfrak{S}$ may be a connected or disconnected, that is, it may consist of several disjoint regions each containing one or more optimal points.

One can extend the concept of the manifold to identifiable cases by considering in this case the manifold to be the discrete set of optimal points. Clearly, in this case the dimension of the manifold is $N_{\mathfrak{S}} = 0$. Based on this generalized interpretation of the manifold $N_{\mathfrak{S}}$, one can state that always, in both identifiable and non-identifiable cases, the posterior PDF of the parameters is concentrated in the neighborhood of the manifold $\mathfrak{S}$. Note that here, and throughout the remaining of this paper unless explicitly otherwise mentioned, the terms identifiability and non-identifiability are used in the context of the new definition introduced above, that is, assuming a chosen value for R.

It follows from the above discussion that the distinctive difference between identifiable and non-identifiable cases lies in the dimension of the corresponding manifold $\mathfrak{S}$. In an identifiable case the manifold $\mathfrak{S}$ has dimension zero and can be expressed as the finite set $\mathfrak{S} = \{\mathbf{a}^{(k)}, k = 1, \ldots, K\}$. This allows for the model updating problem to be reduced to finding the finite number of optimal points comprising $\mathfrak{S}$. Refs.[3,14] provide the tools to resolve this problem. On the other hand, in the non-identifiable case the dimension of the manifold $\mathfrak{S}$ is $N_{\mathfrak{S}} \geq 1$; this makes the model updating problem much more complicated since all the points on the manifold, an infinite non-discrete set of points in this case, have significant probabilities and should be accounted for when attempting an approximate representation of the posterior PDF of the parameters or the posterior PDF of the predictive response.

If the manifold $\mathfrak{S}$ is known then asymptotic results similar to the ones presented for the identifiable case can be obtained accounting for the rapid decay of $p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ in any direction perpendicular to $\mathfrak{S}$.

An approximation analogous to (12) can be obtained to simplify the evaluation of (3) as follows:

$$p(Y_{N+1}^M, X_1^M \mid \mathcal{D}_N, Z_{N+1}^M, \mathcal{M}_P) \simeq \int_{\mathbf{a}\in\mathfrak{S}} w(\mathbf{a})\, p(Y_{N+1}^M, X_1^M \mid \mathbf{a}, \hat{\sigma}(\mathbf{a}), \mathcal{D}_N, Z_{N+1}^M, \mathcal{M}_P)\, d\mathbf{a} \tag{14}$$

The weighting function $w(\mathbf{a})$ in (14) is specified for all points $\mathbf{a} \in \mathfrak{S}$ by:

$$w(\mathbf{a}) = c_2 J(\mathbf{a})^{-N_J} |B_N(\mathbf{a})|^{-1/2} \pi(\mathbf{a}, \hat{\sigma}(\mathbf{a})) \tag{15}$$

where c_2 is a normalizing constant to ensure that the integral of $w(\mathbf{a})$ over the manifold $\mathfrak{S}$ is equal to unity and $B_N(\mathbf{a})$ is a matrix of dimension $(N_a - N_{\mathfrak{S}})$ corresponding to the Hessian matrix of the function $g(\mathbf{a}) = N_J \ln J(\mathbf{a})$ considered in the subspace of $S(\mathbf{a})$ which is perpendicular to the manifold at an arbitrary point $\mathbf{a} \in \mathfrak{S}$. Note that the subspace of $S(\mathbf{a})$ which is tangent to the manifold $\mathfrak{S}$ at the point $\mathbf{a} \in \mathfrak{S}$ has the same dimension as $\mathfrak{S}$, that is, $N_{\mathfrak{S}}$; therefore, its orthogonal completion, defined above as the perpendicular subspace and used to calculate $B_N(\mathbf{a})$, has dimension $(N_a - N_{\mathfrak{S}})$. The weightings $w(\mathbf{a})$ account for the volume of the PDF in the neighborhood of each point $\mathbf{a} \in \mathfrak{S}$ in the direction perpendicular to the manifold.

Equation (14) allows for the $(N_a + 1)$-dimensional integral in (11) to be replaced by an $N_{\mathfrak{S}}$-dimensional integral. It expresses the fact that the updated predictive PDF can be approximated using a continuous weighted average of the predictive PDFs of all models in $\mathfrak{S}$. Notice that although all models in $\mathfrak{S}$ give an almost equally good fit to the data at the observed DOFs, their responses at the unobserved DOFs can differ significantly. Clearly, the applicability of (14) assumes the knowledge of the manifold $\mathfrak{S}$. Furthermore, it is clear from (14) that all models in $\mathfrak{S}$ are important and must be considered in the model updating; therefore, the computation of the manifold $\mathfrak{S}$ is of major importance for resolving the model updating problem.

4 COMPUTATION OF THE MANIFOLD $\mathfrak{S}$

The manifold $\mathfrak{S}$ is usually very complicated and generally there are no analytical expressions available for its calculation even in relatively simple cases. In light of that, the issue of calculation and meaningful representation of $\mathfrak{S}$ becomes a challenging problem which needs to be resolved numerically.

The main difficulty in developing a computational method for calculating the manifold $\mathfrak{S}$ is due to the fact that $\mathfrak{S}$ is, in general, of much smaller dimension than the dimension of the parameter space $S(\mathbf{a})$. Thus, the manifold $\mathfrak{S}$, together with its close neighborhood of significant probabilities, corresponds to only a very small subset of $S(\mathbf{a})$. It can be shown that a systematic search of the parameter space $S(\mathbf{a})$ for finding points on, or very close to, $\mathfrak{S}$ based on calculating the relative PDF values at a uniform grid of points is not successful even when the grid used is relatively fine. This is because the PDF calculated in this way usually leads to a few isolated peaks corresponding to grid points which happened to be located closest to $\mathfrak{S}$; note that since the decay of the probability density in the direction perpendicular to $\mathfrak{S}$ is very sharp, only points which happened to be located approximately equally close to $\mathfrak{S}$ are controlling the resulting estimation of the updated PDF. This is demonstrated later in this paper with a numerical example. For such an approach to be even partially successful one needs to select a grid size which is of order at least equal (or smaller) to the smallest thickness of the region of significant probabilities; the thickness here is measured in any of the directions perpendicular to $\mathfrak{S}$. However, since the updated PDF decays extremely fast in any direction perpendicular to $\mathfrak{S}$, this would imply an extremely fine grid which would make computations prohibitive.

Because of the above considerations a successful methodology for calculating the manifold $\mathfrak{S}$ must employ a minimization algorithm for minimizing $J(\mathbf{a})$, to ensure that points on the manifold $\mathfrak{S}$ can be found. It must be noted that in an identifiable case use of an unconstrained minimization algorithm

will yield one out of the finite number of optimal points. However, in a non-identifiable case such algorithm will most likely converge to some arbitrary (depending on the starting value) point on the manifold $\mathfrak{S}$ and then stop; this is because the function $J(\mathbf{a})$ is extremely flat within $\mathfrak{S}$ and therefore, after reaching a point on $\mathfrak{S}$ the algorithm usually satisfies the prescribed convergence criteria and stops without necessarily reaching a global minimum. The latter observation could serve as a strong argument against identification methods which yield a unique solution, since the resulting solution will depend on the starting point used in the optimization.

In the absence of any analytical solutions, the calculation and meaningful representation of $\mathfrak{S}$ can be achieved by generating a finite set of points $\{\mathbf{a}^{(l)}, l = 1, \ldots, L\}$ on this manifold. In order to obtain this set of points a series of minimizations could be employed with different starting points. However, a systematic approach is needed in order to ensure that the points found are distributed over the whole manifold, so that the whole manifold is adequately represented. Furthermore, in order to increase the accuracy of this representation for a given number of generated points, these points should be distributed in an approximately uniform manner over the manifold.

Next, we present a new adaptive algorithm for generating such a set of points $\{\mathbf{a}^{(l)}, l = 1, \ldots, L\}$. The total number of generated points L depends on a spacing parameter d which needs to be specified at the beginning and which controls the order of the distance between the various generated neighboring points. The algorithm begins with calculating one point on $\mathfrak{S}$ using an unconstrained minimization algorithm for $J(\mathbf{a})$ starting from an initial guess point, which is usually selected as the most probable point in the parameter space based on the prior information. Considering this point as the center of an N_a-dimensional cube of side length d, the algorithm performs a constrained minimization, minimizing $J(\mathbf{a})$ within all neighboring cubes. The minimum point reached within each cube is checked to determine

whether its corresponding relative PDF value, normalized with respect to the maximum value encountered among all minimum points calculated so far, is larger than the specified threshold ε of significant probabilities. If it is, then the algorithm proceeds with performing constrained minimizations within all new neighboring cubes; otherwise the algorithm stops expanding around that cube.

In order to account for the possibility that the manifold is disconnected, the algorithm does not stop when all possible expansions have been carried out. Instead, starting from the point with the smallest J value found up to this time, and employing the algorithm of ref.[3] , the algorithm continues to search the entire parameter space $S(\mathbf{a})$ to find other possible points giving identical response at the observed DOFs. Such points, if any, have an equally small corresponding value of J, or equivalently, an equally large probability. For every additional point found the algorithm checks whether it belongs in the region of the parameter space where optimizations have already been performed. If it is found that it does not belong in any of the cubes searched up to that point the algorithm continues with another series of cube expansions around this new point.

The proposed algorithm is very efficient in the sense that it is adaptive and searches only a minimal region of the parameter space containing the manifold to be determined. Furthermore, it allows for disconnected manifolds to be handled. The set of points generated on the manifold are almost uniformly distributed and their density can be controlled through the prescribed cube side parameter d.

Once a set of points $\{\mathbf{a}^{(l)}, l = 1, \ldots, L\}$ is found, the updated model can be approximated using only the finite number of models in $\mathcal{M}_P$ with parameters $\boldsymbol{\alpha}^{(l)} = [\mathbf{a}^{(l)}, \hat{\sigma}(\mathbf{a}^{(l)})]^T, l = 1, \ldots, L$, each model having a corresponding weighting $w_l, l = 1, \ldots, L$; that is, the updated PDF $p(\mathbf{a} \mid \mathcal{D}_N, \mathcal{M}_P)$ can be expressed as a weighted sum of Dirac delta functions centered at the points $\mathbf{a}^{(l)}$ with corresponding weighting w_l. Using this finite set of points, the integral over the manifold $\mathfrak{S}$ in equation (14) for the predictive PDF

can be approximated by a sum as follows:

$$p(Y_{N+1}^M, X_1^M \mid \mathcal{D}_N, Z_{N+1}^M, \mathcal{M}_P) \simeq \sum_{l=1}^{L} w_l \, p(Y_{N+1}^M, X_1^M \mid \mathbf{a}^{(l)}, \hat{\sigma}(\mathbf{a}^{(l)}), \mathcal{D}_N, Z_{N+1}^M, \mathcal{M}_P) \qquad (16)$$

that is, the updated predictive PDF can be approximated as a weighted average of the predictive PDFs corresponding to the models with parameters $\mathbf{a}^{(l)}, l = 1, \ldots, L$. The weighting coefficient w_l in (16) is given by:

$$w_l = c_2 I(\mathbf{a}^{(l)}) J(\mathbf{a}^{(l)})^{-N_J} |B_N(\mathbf{a}^{(l)})|^{-1/2} \pi(\boldsymbol{\alpha}^{(l)}) \qquad (17)$$

where c_2 is a normalizing coefficient such that $\sum_{l=1}^{L} w_l = 1$. The coefficient $I(\mathbf{a}^{(l)})$ accounts for the nonuniform distribution of the generated points on $\mathfrak{S}$. The l-th term in the sum in (16) approximates the contribution to the integral in (14) of a small subregion of $\mathfrak{S}$ located around the point $\mathbf{a}^{(l)}$; considering $\mathfrak{S}$ as being comprised of such disjoint subregions, the coefficient $I(\mathbf{a}^{(l)})$ accounts for the relative area of the subregion corresponding to the point $\mathbf{a}^{(l)}$. The larger the relative density of the generated points $\mathbf{a}^{(l)}, l = 1, \ldots, L$, in the vicinity of the point $\mathbf{a}^{(l)}$, the smaller is the effective area of this point and, therefore, the smaller the corresponding coefficient $I(\mathbf{a}^{(l)})$ will be. In the case where the set of the generated points $\mathbf{a}^{(l)}, l = 1, \ldots, L$ is almost uniformly distributed over $\mathfrak{S}$ this coefficient should be taken equal to unity. The remaining coefficients contributing to the weightings w_l in (17) are the same as in (15) and were discussed earlier. The evaluation of $I(\mathbf{a}^{(l)})$ as well as other computational aspects are demonstrated next with two numerical examples involving one-dimensional manifolds.

5 NUMERICAL EXAMPLES

5.1 Example 1

An example using a two-DOF linear planar shear building shown in Figure 1 is used to demonstrate the difficulties encountered in model updating in a non-identifiable case. Consider the class of classically-damped models with uniform mass distribution $m_1 = m_2 = m_0$, damping ratios $\zeta_1 = \zeta_2 = \zeta_0$, and interstory stiffnesses $k_1 = \theta_1 k_0$ and $k_2 = \theta_2 k_0$. Here the values of the parameters m_0, ζ_0, and k_0 are assumed to be: $m_0 = 1$ kg, $\zeta_0 = 0.02$, and $k_0 = 1000$ N/m. In this example, only the nondimensional stiffness parameters $\theta_1, \theta_2 > 0$ are considered to be uncertain and need to be updated; thus, $\mathbf{a} = [\theta_1, \theta_2]^T$. No prior information on the distribution of these parameters is assumed, that is, $\pi(\mathbf{a})$ is taken to be uniform (noninformative prior). This example, involving only two uncertain model parameters, permits us to clearly illustrate the concepts introduced earlier in a graphical manner. For example, it allows the PDF to be visualized as a function of the model parameters in a three-dimensional plot.

The structure is assumed to be subjected to ground motion given by the 1940 El Centro NS earthquake record. It is assumed that the measured dynamic data consist of 30 sec of acceleration histories at a sampling interval of 0.02 sec, corresponding to the base (input) and the roof (output). The response data corresponding to the roof were simulated by calculating the response of the model with parameters $\theta_1 = \theta_2 = 1$ and adding to it white noise in order to simulate the measurement noise and model error encountered in practice. The level of the added noise is expressed as the percentage ratio of its standard deviation over the standard deviation of the model response.

Assuming a noise level of 20%, Figures 2-a, b and c, show the posterior PDF obtained by calculating its values at a grid of points in the parameter space, using a coarse, medium and fine grid, respectively. The

values shown have been normalized, so that the maximum calculated value is unity. It is seen that such an approach for calculating the posterior PDF fails to identify the regions of significant PDF values even in the case where a relatively fine grid is used. This is because the grid points that happened to be located closest to the manifold $\mathfrak{S}$ end up dominating the results. Figure 3 depicts the manifold $\mathfrak{S}$ calculated using the proposed algorithm and considering values of normalized PDF less than $\varepsilon = 0.001$ as non-significant. A smaller threshold will cause the boundary of $\mathfrak{S}$ to extend further. Figure 3 also depicts the location of the peaks encountered when calculating the posterior PDF in the previous Figures 2-a, b and c. It is worth noting that the point $(0.9, 1.44)$ corresponding to a peak in both Figures 2-a and 2-b does not belong in the depicted manifold; this implies that the normalized PDF of this point is less than the threshold value of 0.001. At the same time from Figure 2-b we see that the point $(1.7, 0.54)$ has a corresponding PDF value about two times larger than that of the point $(0.9, 1.44)$, implying that the point $(1.7, 0.54)$ also has a very small PDF value. This means that although the point $(1.7, 0.54)$ appears to lie on the manifold, in reality it is not. Furthermore, this implies that the neighborhood of the manifold with significant PDF values has an extremely small thickness, or equivalently, the values of the PDF decay extremely fast as one moves away from the manifold in its perpendicular direction.

Figures 4-a and 4-b show the PDF values obtained for noise levels of 20% and 40%, respectively, plotted along the corresponding manifold in each case. It is clear that as the noise level increases the PDF becomes more spread and the length of the corresponding one-dimensional manifold increases. As the noise level becomes smaller the PDF becomes more concentrated around the points $(1.0, 1.0)$ and $(2.0, 0.5)$. These two points are the optimal points in the case of zero noise level and are output-equivalent under ground excitation when only the second floor is measured. Note that at 20% noise level the manifold $\mathfrak{S}$ is disconnected, while at the 40% level it becomes connected. Calculation of the volume of the PDF under the two regions A and B in the case of the disconnected manifold of Figure 4-a shows

that these volumes are equal. This is not obvious by just looking at these figures; although the PDF values in the regions A and B are similar, the length of the manifold in region A is smaller than the length of the manifold in region B. Thus, at first it appears as if the volume of the PDF in region A is smaller than that in region B. However, in order to obtain correct results, one also needs to consider the thickness of the neighborhood of significant probabilities along these parts of the manifold, instead of considering only the values of the PDF along the manifold. Figure 5 depicts two curves corresponding to the peaks A and B of Figure 4-a, which show the distribution of the normalized PDF as a function of the distance perpendicular to the manifold. It is clear from this figure that the variation of the PDF in the direction perpendicular to the manifold is extremely sharp. Also, it can be seen that the distribution of the PDF under the peak A is broader than the one under B. This explains the result mentioned earlier, namely that the total probability in the neighborhood of regions A and B are equal.

Figure 6 shows the relative weightings w_l corresponding to the points found on the manifold by the proposed method; it also shows the corresponding values of the normalized PDF. It is clear that the weightings in region A are larger because of the broader distribution in the direction perpendicular to the manifold. Specifically the difference is due to the term $|B_N(\mathbf{a}^{(l)})|^{-1/2}$ in (17). For each point $\mathbf{a}^{(l)}$ on the manifold the term $I(\mathbf{a}^{(l)})$ in (17) was taken to be equal to the average distance of $\mathbf{a}^{(l)}$ from its two neighboring points and accounts for the nonuniformity in the spacing of the points $\mathbf{a}^{(l)}$.

5.2 Example 2

In this example we consider the updating of a two-dimensional 10-DOF finite element model of a single-span elastically supported bridge shown in Figure 7. The rotational and translational springs at the supports represent the equivalent corresponding stiffnesses of the soil. Here the stiffness matrix of the

model is parameterized and updated with the help of the following nondimensional parameters: θ_1, θ_2 scaling the translational spring constants k_{yA} and k_{yE}, respectively, θ_3 scaling a rotational spring constant k_θ such that $k_{\theta A} = k_{\theta E} = \theta_3 k_\theta$, and θ_4 scaling the bending rigidity EI of the deck. The following nominal values were chosen (corresponding to a unit $\boldsymbol{\theta}$ vector): $k_{yA} = k_{yE} = 10^7$N/m, $k_\theta = 10^5$Nm, $EI = 10^6$Nm^2. The following lumped masses were considered and assumed to be deterministic: $m_A = m_E = 8 \times 10^3$kg, and $m_B = m_C = m_D = 16 \times 10^3$kg. Next, the bridge was assumed to be subjected to a ground motion $\ddot{y}(t)$, as shown in Figure 7, given by the NS 1940 El Centro earthquake record. It was assumed that only one DOF, the translational DOF at the midspan C, was measured. The measured response was simulated by first calculating the acceleration response at C of a nominal structure, and then adding a 10% white noise. The nominal structure was selected having the following properties: translational spring constants $k_{yA} = 1.1 \times 10^7$ N/m and $k_{yE} = 0.9 \times 10^7$ N/m, respectively; rotational spring constants $k_{\theta A} = 1.2 \times 10^5$ Nm and $k_{\theta E} = 0.85 \times 10^5$ Nm, respectively; bending rigidities of elements AB, BC, CD and DE equal to $0.95EI$, $1.05EI$, $0.9EI$ and $0.95EI$, respectively. The selection of this nominal structure allows for some modeling error while the added noise was introduced to model additional modeling and measurement errors. Twenty seconds of data with sampling interval $\Delta t = 0.02$ sec were used, that is, $N = 1000$ points were used.

It was found that for this example the parameters $\mathbf{a} = \boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3, \theta_4]^T$ are non-identifiable and $\mathfrak{S}$ is a one-dimensional manifold in the four-dimensional parameter space. Here a value of $R = 400$ was used. Assuming relative probabilities of less than $\varepsilon = 0.001$ to be negligible, this value of R implies that the region of significant probability is contained within a neighborhood of radius $r = 0.18$ of the manifold. It also implies that the manifold has length at least equal to $2r$. Figure 8 shows the set of points $\boldsymbol{\theta}^{(l)}$ on the manifold calculated by the proposed algorithm and plotted in the $\theta_1 - \theta_2 - \theta_3$ subspace. In Figure 9 the various terms contributing to the weighting coefficient w_l of the identified points $\boldsymbol{\theta}^{(l)}$, as they appear in

equation (17) are plotted. Figure 9-a shows the variation of the spacing term $I(\boldsymbol{\theta}^{(l)})$ (normalized) which is taken to be proportional to the average distance of $\boldsymbol{\theta}^{(l)}$ from its two neighboring points. Figure 9-b shows the variation of the term $|B_N(\boldsymbol{\theta}^{(l)})|^{-1/2}$ (normalized) which reflects the contribution of the "thickness" of the manifold to the weightings of the various points. Figure 9-c shows the variation of the term $J(\boldsymbol{\theta}^{(l)})^{-N_J}$ (normalized) reflecting the relative values of the posterior PDF at the calculated points on the manifold. Finally, Figure 9-d shows the variation of the overall weighting coefficients w_l.

Using the calculated points $\boldsymbol{\theta}^{(l)}$ and their corresponding weightings w_l one can proceed with the calculation of the statistics of various stiffness parameters or various response quantities of interest. Figure 10 shows the marginal cumulative distributions of the stiffness parameters θ_i, $i = 1, \ldots, 4$. It can be seen that the degree of uncertainty after updating, best expressed by the coefficient of variation (COV), ranges from 0.26 for θ_2 to 0.06 for θ_4. Thus, the value of the rotational spring is much more uncertain than the value of the bending rigidity of the deck. Figures 11-a, b, and c show the cumulative distribution for the maximum displacement response at points A, B, and C respectively. It can be seen that the uncertainty at point C, where the response was assumed to be measured, is very small, while at point A the corresponding COV is 0.18. Similarly, Figures 12-a, and b show the cumulative distribution for the maximum rotation and the bending moment, respectively, at point A. Although the COV for the maximum rotation is only 0.02, the corresponding COV for the moment is 0.25. The reason for this is that the moment at A can be expressed as a product of the rotation at this point times the stiffness of the rotational spring at A which is proportional to θ_3 and which as can be seen from Figure 10-c has a large COV of about 0.25.

It is clearly seen that although the uncertainties in the model parameters have little effect in the predictions at measured DOFs, they can have significant effect on predictions at unmeasured DOFs. The methodology presented here provides the tools for calculating the uncertainties in the model parameters

and for calculating more reliably the response predictions and the uncertainties in these predictions.

6 CONCLUSION

The general Bayesian statistical framework presented in Beck and Katafygiotis[3] has been extended to treat the general non-identifiable case arising when the number of parameters to be identified is relatively large considering the amount of information that can be extracted from the measured data. In this case the updated PDF of the model parameters is concentrated in a small neighborhood of a manifold of dimension larger than zero. For large number of measured data, it is asymptotically correct to represent the updated model using only the models with corresponding parameters on this manifold. Analytical expressions for calculation of this manifold are not available. A computationally efficient numerical algorithm has been presented to search the parameter space for a finite set of approximately uniformly distributed points on the manifold, which can be used for its representation. It is shown that the full Bayesian predictive probability distribution for the structural response, originally involving computationally prohibitive high-dimensional integrations, can be approximated by a weighted sum over the predictive probability distributions for the models corresponding to the set of points on the manifold computed by the proposed algorithm.

7 ACKNOWLEDGMENT

This work is based upon work partly supported by the Hong Kong Research Grant Council under grants HKUST 639/95E and HKUST 6041/97E. Valuable discussions with Professor James L. Beck at the California Institute of Technology are sincerely appreciated.

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