

# Radiative heat conduction and the magnetorotational instability

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## ABSTRACT

A photon or a neutrino gas, semicontained by a non-diffusive particle species through scattering, comprises a rather peculiar magnetohydrodynamic fluid where the magnetic field is truly frozen only to the comoving volume associated with the mass density. Although radiative diffusion precludes a formal adiabatic treatment of compressive perturbations, we cast the energy equation in quasi-adiabatic form by assuming a negligible rate of energy exchange among species on the time-scale of the perturbation. This leads to a simplified dispersion relation for toroidal, non-axisymmetric magnetorotational modes when the accretion disc has comparable stress contributions from diffusive and non-diffusive components. The properties of the modes of fastest growth are shown to depend strongly on the compressibility of the mode, with a reduction in growth rate consistent with the results of Blaes & Socrates for axisymmetric modes. A clumpy disc structure is anticipated on the basis of the polarization properties of the fastest-growing modes. This analysis is accurate in the near-hole region of locally cooled, hyper-accreting flows if the electron gas becomes moderately degenerate such that non-conductive, thermalizing processes with associated electron–positron release (i.e. neutrino annihilation and neutrino absorption on to nuclei) are effectively blocked by high occupation of the Fermi levels.

**Key words:** accretion, accretion discs – instabilities – MHD – neutrinos – gamma-rays: bursts.

## 1 INTRODUCTION

Understanding the magnetorotational instability (MRI) is key to the development of realistic models of accretion. Indeed, as a physical process the MRI justifies two *sine qua non* ingredients of accretion-disc theory: entropy generation from the differential shear flow and turbulent angular momentum transport. Clarifying the dynamics that leads to MRI-initiated turbulence in a few relevant astrophysical regimes is thus paramount in uncovering how the accretion process takes place.

In the context of accretion on to compact objects, the stress associated with a radiative particle species is often dynamically important; yet, our understanding of its effects on magnetohydrodynamic (MHD) processes is incomplete at best. Agol & Krolik (1998) have pointed out that compressive radiation-MHD wave modes will be strongly suppressed by diffusive loss of pressure support (i.e. Silk damping, 1968) in a range of wavenumbers  $c_{\text{ph}}/c < \tilde{k}^{-1} < 1$ , where  $\tilde{k} \equiv \ell_{\text{mfp}} k$  represents the wavenumber normalized to the mean free path and  $c_{\text{ph}}$  is the phase speed of compressive waves in the single-fluid view. However, by considering wavenumbers much larger than the inverse vertical pressure scaleheight  $k\mathcal{H} \gg 1$ , these authors

chose to ignore the effects of inertial forces and stratification. For the MRI, however, both of these effects play key roles in dictating the dynamics and polarization of the modes (see, e.g. Foglizzo & Tagger 1995).

Blaes & Socrates (2001) reported a radiation-MHD dispersion relation for axisymmetric modes that comprises the MRI under rather general circumstances. Our analysis is different from theirs mainly in the consideration of a purely toroidal field where the compressibility of non-axisymmetric modes leads to a more pronounced impact from radiative diffusive damping. Furthermore, aiming to understand ultrarelativistic flows near black holes, we emphasize toroidal mode dynamics because these modes can become predominant when shear effects begin to overwhelm other inertial accelerations leading to fastest-growing modes at large azimuthal scales (Araya-Góchez 2002). The scale and degree of intermittency in the turbulent flow bear a direct impact on the characteristic gravitational-wave signature from hyper-accreting black holes (see companion paper, Araya-Góchez 2004, in this issue). For a relativistically hot accretion disc with comparable vertical pressure support from diffusive and non-diffusive particle species (see Section 3), we solve for a dispersion relation involving Lagrangian displacements with a negligible meridional component:  $\xi_{\theta}/\xi_r \simeq k_r/k_{\theta} \rightarrow \emptyset$  [spherical coordinates ( $r, \varphi, \theta$ ) are used throughout]. This simplification affords a lowering of the rank in the dispersion

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relation to second order (in  $\omega^2$ ) which, in turn, enables a relatively straightforward albeit approximate treatment of the effects of radiative heat conduction to be achieved.

## 2 MICROPHYSICS OF NEUTRINO-COOLED ACCRETION FLOWS

Two of the most difficult tasks for the worker who aims to understand hyper-accreting, relativistic flows are ranking the relevant microphysical processes and assessing the ensuing dynamical impact on the flow structure and energy deposition. In this section we try to address briefly the former aspect of the problem whereas the rest of the paper is chiefly concerned with the latter. Because of the possibility of gravitational-wave emission from hyper-accretion (Araya-Góchez 2004), we focus on the properties of the fluid in the near-hole region.

Local cooling by neutrino emission imposes rough boundaries on the accretion rate,  $0.1 \leq \dot{M} \leq 1 M_{\odot} \text{ s}^{-1}$ , where the upper limit is consistent with the neglect of energy advection (Popham, Woosley & Fryer 1999, hereafter PWF). In this non-advective accretion regime, the innermost region of a neutrino-cooled flow near a hyper-accreting black hole will possess a density  $10^{+10} \leq \rho \leq 10^{+11} \text{ g cm}^{-3}$ , and temperature  $5 \leq T \leq 10 \text{ MeV}$ . Advection of energy will generally increase these nominal estimates (Di Matteo, Perna & Narayan 2002) but its effect on fully relativistic discs is only roughly understood. Under these conditions, the fluid is composed of an admixture of free<sup>1</sup> protons and neutrons, electrons, positrons, photons and neutrino–antineutrino pairs. Furthermore, assuming thermal and chemical equilibrium in the unperturbed flow, the electron component will possess at least a mild degree of degeneracy, leading to an effective suppression of the positron density and to a low proton–nucleon fraction  $Y_e$  (Beloborodov 2003).

Electron degeneracy has a very important consequence for neutrino transfer: electron- and positron-releasing processes such as neutrino (or antineutrino) absorption on to nuclei and neutrino annihilation into electron–positron pairs are effectively blocked by the high occupation of the Fermi levels. The blocking factor (Bahcall & Wolf 1965; Shapiro & Teukolsky 1983) is roughly  $\propto \exp(-\mu/kT)$ , where  $\mu$  ( $>kT$ ) represents the chemical potential for the electrons. Estimates of the degeneracy parameter,  $\mu/kT$ , vary in the literature, from mild degeneracy  $\mu/kT \lesssim 1$  (Beloborodov 2003), to a more strongly degenerate electron gas  $\mu/kT \simeq 2\text{--}4$  (Lee, Ramirez-Ruiz & Page 2004). Assuming efficient neutrino cooling (i.e. in the non-advective accretion regime), the degeneracy parameter may be expected to be of order a few and the chemical potential for the neutrinos,  $\mu_\nu$ , will be negligible compared to that of the electrons (Beloborodov 2003). Strictly speaking, the radiative conduction approximation used in Section 5 is only accurate when the above energy exchange processes are blocked by electron degeneracy.

The time to attain thermal equilibrium<sup>2</sup> (through neutronization) is steeply dependent on temperature:  $\propto T^5$  (see, e.g. Beloborodov 2003). This time-scale hovers on values near the dynamical time-scale,  $t_{\text{dyn}} \simeq \Omega^{-1} \simeq 200 \mu\text{s}$  (for  $M_{\text{hole}} \simeq 15 M_{\odot}$ ), when degeneracy is negligible:  $t_{\text{neu}} \simeq 20\text{--}700 \mu\text{s}$ . However, since degeneracy suppresses neutronization rates exponentially, the latter time-scale is likely to be well above the dynamical time-scale ( $\simeq t_{\text{MRI}}$ ) when

<sup>1</sup> We assume that the mass fraction of free nucleons is  $X_{\text{nuc}} = 1$ ; Beloborodov’s (2003) shaded contours in figs 1 and 2 show where composite nuclei may be expected.

<sup>2</sup> We thank the referee for stressing the importance of the thermalization time-scale and its bearing on the radiative conductive approximation.

$\mu/kT \gtrsim 2$ . Radiative heat conduction in this setting thus means that, on the time-scale of the instability, neutrinos will transport energy through a medium with no effective internal degrees of freedom, which would otherwise thermalize the local neutrino flux energy density. Note that the average neutrino energy is well below the nucleon masses so the scattering process is essentially elastic. Moreover, if  $t_{\text{neu}} \gg \Omega^{-1}$ , transport issues related to finite gradients in lepton chemical potentials (i.e. lepton diffusion and the possibility of doubly diffusive instabilities such as neutron fingers, etc.) can also be neglected. Note that it is the MRI which, in principle, sets up the entropy and chemical potential profiles which may drive convective and (doubly) lepton/entropy diffusive instabilities secondarily.

Although neutrino stress never truly dominates the dynamics, it is not unreasonable to expect the neutrinos to be sufficiently trapped to make a dynamically significant impact, much like photons in radiation-pressure-dominated discs where the optical depth to scattering greatly exceeds unity. The Rosseland mean opacity to neutrino–nucleon scattering for  $\mu_\nu = 0$  and  $Y_e = 0.5$  has been computed by Di Matteo et al. (2002). This opacity is rather insensitive to the proton fraction and equals 3/5 of the unmodified URCA absorption opacity (i.e. the opacity in the absence of electron degeneracy). Thus, electron degeneracy (and the ensuing neutrino transparency to absorption and to annihilation into electrons and positrons) has only a mild impact on the total optical depth of the disc.

The dynamical consequences for optically thick media to neutrinos,  $\tau_\nu \gg 1$ , are estimated as follows. The pressure contributions from radiation, neutrinos and electron–positron pairs, all have the same temperature dependence,  $\propto a/6 T^4$ , with relative contributions varying only by internal degrees of freedom times particle statistics factors:  $2 \times 1, 6 \times 7/8$ , and  $\approx 4 \times 7/8 \times \exp(-\mu/kT)$ , respectively (with only one helicity state per neutrino and a correction for the finite chemical potential of the pairs). Thus, the neutrino pressure is never greater than  $\simeq p_{\text{thermal}}$ , where  $p_{\text{thermal}} \simeq a [1 + 7/4 \exp(-\mu/kT)] T^4/3$  includes the pressure from photons and electron–positron pairs. Since the neutrinos constitute the lone diffusive species, their contribution to the stress tensor is identified with  $p_{\text{rad}}$  below (e.g. radiative = diffusive species throughout this paper). On the other hand, numerical estimates of the nucleon pressure at the innermost disc region ( $\dot{M} = 1 M_{\odot} \text{ s}^{-1}$ ) are consistent with an upper limit near equipartition (Di Matteo et al. 2002), or much lower (PWF) while degeneracy pressure is expected to be dynamically negligible (PWF, Di Matteo et al. 2002; Lee et al. 2004). For the sake of simplicity in the analysis below, we will subsequently assume rough equipartition among the sources of stress and an optically thick, locally cooled disc.

## 3 RADIATIVE VISCOSITY VERSUS DIFFUSION

In a standard radiation-pressure-dominated  $\alpha$ -disc, the optical depth to photon propagation is dominated by electron scattering. The radiative flux from the surface is roughly  $F_{\text{rad}} \simeq c p_{\text{rad}}/\tau_{\text{disc}}$ , while the energy generated between the mid-plane and the surface is  $Q \simeq -2\hat{A}\Omega\alpha p_{\text{tot}}\mathcal{H}$ . Here  $\hat{A} \equiv 1/2d_{\text{ln}r} \ln \Omega$  represents Oort’s  $A$  parameter normalized to the local rotation rate and  $\mathcal{H}$  is the meridional pressure scaleheight:  $\mathcal{H}^{-1} \equiv r^{-1}\partial_\theta \ln p_{\text{rad}}$ . Assuming a stationary, locally cooled disc with  $p_{\text{tot}} \approx p_{\text{rad}}$ , we obtain

$$\tau_{\text{disc}} = -\frac{c}{2\alpha\hat{A}\Omega\mathcal{H}}. \quad (1)$$

This result is explicitly sensitive only to the local nature of the cooling although it is implicitly subject to a suitable gradient of heat deposition (see, e.g. Krolik 1999). In a neutrino-cooled disc, a replacement in the source of opacity by neutrino scattering and a similar procedure leads to an optical depth that is slightly lower than that in equation (1) by the ratio of neutrino-to-total pressures. None the less, as long as  $\tau_{\text{disc}} \gg 1$ , such a reduction in optical depth has relatively little bearing in the phenomenology of radiative heat conduction as discussed below.

On the other hand, the length-scale of the fastest-growing MRI eddies in the direction parallel to the field is  $l_{\text{eddy}} = k_{\text{MRI}}^{-1} \simeq v_{\text{Alf}}/\Omega$ . The optical depth through these eddies is related to the depth through the disc by  $\tau_{\text{eddy}} \simeq \tau_{\text{disc}} v_{\text{Alf}}/c_{\mathcal{H}}$ , where  $c_{\mathcal{H}} = \Omega \mathcal{H} (c_{\mathcal{H}}^2 \equiv p_{\text{tot}}/\rho)$ . Thus, for magnetic angular momentum transport such that  $\alpha \simeq (v_{\text{Alf}}/c_{\mathcal{H}})^2$ , the *isotropic* diffusion time through these eddies is  $t_{\text{diff}} = \tau_{\text{eddy}}/c k_{\text{MRI}} \simeq (-\dot{\Lambda}\Omega)^{-1}$ . Surprisingly, this time also corresponds to the time-scale for the fastest-growing MRI modes to develop (see Section 6). The optical depth through these eddies is

$$\tau_{\text{eddy}} \simeq -\frac{1}{2\dot{\Lambda}\sqrt{\alpha}} \frac{c}{c_{\mathcal{H}}} \gg 1, \quad (2)$$

so we are clearly in the diffusive limit. Furthermore, the damping rate due the radiative viscosity is roughly equal to the fraction of the bulk momentum carried by the diffusive particles times  $k^2$  times the viscosity ( $\simeq$  particle speed times the mean free path). Thus, on dimensional grounds, the diffusive drag rate on the eddies goes as  $t_{\text{drag}}^{-1} \simeq (u_{\text{rad}}/\rho c^2) k_{\text{MRI}}^2 \ell_{\text{mfp}} c \approx (c_{\mathcal{H}}/c)^2 \dot{\Lambda}\Omega \ll \dot{\Lambda}\Omega$  (recall  $\tau_{\text{disc}} = H/\ell$ ), so we can safely ignore the viscous drag in the eddy dynamics of a locally cooled, optically thick disc.

#### 4 A LAGRANGIAN FORMULATION OF THE MRI IN SOFT MEDIA

We have adopted spherical coordinates in accordance with a slim-disc approach. Field curvature and entropy gradients are neglected (thus excluding convective modes) while a simple meridional pressure profile sets the physical length-scale of the problem:  $\mathcal{H}^{-1} = r^{-1} d_{\theta} \ln p_{\text{tot}} = -g_{\theta}/(c_{\mathcal{H}}^2 + 2v_{\text{Alf}}^2)$ , where  $p_{\text{tot}}$  is the total pressure,  $c_{\mathcal{H}}$  is the effective sound speed of the medium,  $v_{\text{Alf}}$  represents the Alfvén speed, and where the meridional component of the inertial acceleration,  $g_{\theta}$ , is taken to be constant.<sup>3</sup> Although photon (and neutrino) bubble modes may be important, we exclude these modes here since for dynamically interesting values of the Shakura & Sunyaev ‘ $\alpha$ ’ parameter, one expects MRI modes to grow at larger scales than do the photon bubble modes (which incidentally also originate from the slow MHD mode). More importantly, the fundamental source of free energy is the differential shear which feeds the MRI primarily. The ensuing energy deposition profile may feed photon or neutrino bubble modes as a secondary process in spite of their generically larger growth rates.

We take a semilocal approach by computing instantaneous (co-moving) growth rates in the limit  $k_r/k_{\theta} \rightarrow 0$  and referring to these as the fastest-growing modes on the implicit understanding of their transient nature [since  $k_r(t) = k_{0r} - (d_{\ln r}\Omega) k_{\phi} t$ ]. In terms of a local, comoving observer, azimuthal wavenumbers are no longer discrete and consequently neither are the comoving frequencies (Ogilvie & Pringle 1996). Furthermore, we discard radial gradient terms. The

<sup>3</sup> Note that the component of effective gravity along the meridional direction,  $\mathbf{1}_{\theta}$ , is  $\propto \sin^2(\pi/2 - \theta)$  while gravity along the vertical direction,  $\mathbf{1}_z$ , is  $\propto \sin(\pi/2 - \theta)$ , so these differ fractionally by  $\sin(\pi/2 - \theta)$  (Abramowicz, Lanza & Percival 1997). This does not affect the dispersion relation (12) since we only consider a constant effective gravity case and rotation on spherical shells.

first-order relation between the Lagrangian,  $\Delta$ , and the Eulerian,  $\delta$ , variations (see, e.g. Chandrasekhar & Lebovitz 1964; Lynden-Bell & Ostriker 1967) is

$$\Delta = \delta + \boldsymbol{\xi} \cdot \nabla. \quad (3)$$

The Euler velocity perturbation,  $\mathbf{v} \equiv \delta \mathbf{V}$ , is related to the Lagrangian displacement,  $\boldsymbol{\xi}$ , through

$$\mathbf{v} = (\partial_t + \mathbf{V} \cdot \nabla) \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla) \mathbf{V} \rightarrow i\sigma \boldsymbol{\xi} - \xi^r \Omega_r \mathbf{1}_{\phi}. \quad (4)$$

The algebraic relation follows from the assumption of differential rotation and from writing  $\exp[i(\omega t + m\phi + k_z z)]$  dependencies for  $\boldsymbol{\xi}$ , while  $\sigma \equiv \omega + m\Omega$  denotes the comoving frequency of the perturbations.

These geometrical equations have their more traditional equivalents in the so-called shearing sheet approximation where a comoving, ‘locally Cartesian frame’  $(\hat{r}, \hat{\phi}, \hat{\theta}) \rightarrow (x, y, z)$ , is used along with the linearized shear velocity field,  $\mathbf{V}(x) = [d_{\ln r}\Omega] x \mathbf{1}_y$ , to treat the problem locally while introducing the Coriolis terms by hand. (The reader is invited to examine Araya-Góchez 2002, where the inertial terms are induced in a fully covariant, geometric way.)

The linearized equations of motion for the displacement vector correspond to the Hill equations (Chandrasekhar 1961; Balbus & Hawley 1992)

$$\ddot{\xi}_r - 2\Omega \dot{\xi}_{\phi} = -4A_{\text{Oort}} \Omega \xi_r + \frac{f_r}{\rho}, \quad \ddot{\xi}_{\phi} + 2\Omega \dot{\xi}_r = \frac{f_{\phi}}{\rho},$$

and  $\ddot{\xi}_{\theta} = \frac{f_{\theta}}{\rho}$ , (5)

where each over-dot stands for a Lagrangian time derivative and  $\mathbf{f}$  stands for the sum of body forces.

The Lagrangian perturbation of mass density and the Eulerian perturbation of the field follow straightforwardly from equation (3), and from mass and magnetic flux conservations (notwithstanding the peculiar nature of the fluid under study)

$$\frac{\Delta \rho}{\rho} = -\nabla \cdot \boldsymbol{\xi}, \quad \delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}). \quad (6)$$

Using the latter relation, the Lorentz force in the comoving frame is readily laid out

$$\begin{aligned} \delta \frac{1}{4\pi\rho} \mathbf{J} \times \mathbf{B} &= v_{\text{Alf}}^2 \times [\nabla(\nabla \cdot \boldsymbol{\xi}) \\ &+ \nabla_{\mathbf{B}}^2 \boldsymbol{\xi} - \nabla_{\mathbf{B}} \nabla(\mathbf{1}_{\mathbf{B}} \cdot \boldsymbol{\xi}) - \mathbf{1}_{\mathbf{B}} \nabla_{\mathbf{B}}(\nabla \cdot \boldsymbol{\xi})] \\ &\xrightarrow{\nabla \rightarrow i\mathbf{k}} -v_{\text{Alf}}^2 \times [(k_i \xi_i - k_{\mathbf{B}} \xi_{\mathbf{B}}) \mathbf{k} + k_{\mathbf{B}}^2 \boldsymbol{\xi} - \mathbf{1}_{\mathbf{B}} k_{\mathbf{B}} k_i \xi_i]. \end{aligned} \quad (7)$$

where  $\mathbf{1}_{\mathbf{B}}$  is a unit vector in the direction of the unperturbed field and  $\nabla_{\mathbf{B}} \equiv \mathbf{1}_{\mathbf{B}} \cdot \nabla \rightarrow k_{\mathbf{B}} \equiv \mathbf{1}_{\mathbf{B}} \cdot \mathbf{k}$ . Note that the term  $(k_i \xi_i - k_{\mathbf{B}} \xi_{\mathbf{B}}) \equiv k_{\perp} \xi_{\perp}$  may be interpreted as a restoring force due to the compression of field lines (as opposed to field-line bending, Foglizzo & Tagger 1995).

Next, we look in detail at the non-magnetic stress terms.

In the single-fluid view, the Lagrangian variation of the specific pressure gradient,  $\Delta(\rho^{-1} \nabla p_{\text{r+g}})$ , contains two terms (see, e.g. Lynden-Bell & Ostriker 1967): one  $\propto \Delta \rho^{-1}$ , and another  $\propto \Delta \nabla p_{\text{r+g}}$ . In terms of the displacement vector, the first term is proportional to the equilibrium value of  $\nabla p_{\text{r+g}}$  which is negligible in the local treatment ( $\propto$  radial gradient when  $\xi_{\theta}/\xi_r \rightarrow 0$ ). For the same reason, the Eulerian and Lagrangian variations of the specific pressure gradient are nearly identical. The ‘thermodynamic’ pressure term is thus given by

$$-\delta \left( \frac{1}{\rho} \nabla p_{\text{r+g}} \right) = \Gamma \frac{p_{\text{r+g}}}{\rho} \nabla(\nabla \cdot \boldsymbol{\xi}) \xrightarrow{\nabla \rightarrow i\mathbf{k}} c_s^2 \mathbf{k} k_i \xi_i, \quad (8)$$

where, for a heterogeneous fluid,  $\Gamma$  ( $\equiv d_{[\ln \rho]} \ln p$ ) represents a generalized adiabatic index (see, e.g. Chandrasekhar 1939; Mihalas & Mihalas 1984).

Assembling  $f$  from equations (7) and (8), and plugging this form into equations (5) yields

$$\begin{aligned} \ddot{\xi} + 2\Omega \times \dot{\xi} = [c_s^2 \mathbf{k} + v_{\text{Alf}}^2 (\mathbf{k} - k_{\hat{\mathbf{B}}} \mathbf{1}_{\hat{\mathbf{B}}})](\mathbf{k} \cdot \xi) \\ + v_{\text{Alf}}^2 (k_{\hat{\mathbf{B}}}^2 \xi - k_{\hat{\mathbf{B}}} \xi_{\hat{\mathbf{B}}} \mathbf{k}) - 4A\Omega \xi_r \mathbf{1}_r. \end{aligned} \quad (9)$$

Note that this equation is general and independent of any specific magnetic field topology. Equation (9) agrees with the matrix decomposition of Foglizzo & Tagger (1995) for  $\Gamma = 1$ . These authors have stressed the importance of the vertical pressure gradient and rotation (inertial accelerations) in determining the polarization of the slow magnetosonic (e.g. MRI) mode (see, e.g. Section C of Foglizzo & Tagger 1995). When  $k_r/k_\theta \rightarrow 0$

$$\frac{\xi_\theta}{\xi_r} \simeq \mathcal{O}\left(\frac{k_r}{k_\theta}\right) \quad (10)$$

which corresponds to fundamentally horizontal displacements.

Equations (9) and (10) impose the following anisotropy constraint on the components of the Lagrangian displacement vector field

$$\mathbf{k}_\perp \cdot \xi_\perp = -\frac{\Gamma}{\Gamma + 2\Theta} k_{\parallel} \xi_{\parallel} \equiv -\Lambda k_{\parallel} \xi_{\parallel}, \quad (11)$$

where  $\Theta \equiv \text{phspace-2pt}_B/p_{r+g}$ , and  $\Lambda$  parametrizes the degree of anisotropy of the modes. Note that  $p_{r+g}$  represents the combined pressure contribution from the gas and radiative component.

In such a horizontal regime, with  $k_\theta \xi_\theta$  finite, the dispersion relation derived from equations (9) and (11) reads

$$\omega^4 - \{(\Lambda + 1)\hat{q}_{\hat{\mathbf{B}}}^2 + \hat{\chi}^2\} \omega^2 + \Lambda \hat{q}_{\hat{\mathbf{B}}}^2 \{\hat{q}_{\hat{\mathbf{B}}}^2 + 4\hat{\Lambda}\} = 0, \quad (12)$$

where all frequencies are normalized to the local rotation rate,  $\hat{\chi}^2 \equiv 4(1 + \hat{\Lambda})$  is the square of the epicyclic frequency, and  $\hat{q}_{\hat{\mathbf{B}}} \equiv (\mathbf{k} \cdot \mathbf{v}_{\text{Alf}})/\Omega$  is a frequency related to the component of the wavevector along the field (in velocity units). This expression matches the form given by Foglizzo (1995).

In terms of the function,  $\Lambda$  ( $\Gamma$ ,  $\Theta$ ), we find the fastest-growing wavenumbers to be given by

$$\begin{aligned} \hat{q}_{\hat{\mathbf{B}}}^2 = -2\hat{\Lambda} + \left(\frac{1 + \Lambda}{2\Lambda}\right) \times \left\{ -\frac{2\Lambda\hat{\Lambda}^2}{\mathcal{D}} \right\}, \\ \text{where } \mathcal{D} \equiv 1 + \left(\frac{1 - \Lambda}{2}\right) \hat{\Lambda} + \sqrt{1 + (1 - \Lambda)\hat{\Lambda}}, \end{aligned} \quad (13)$$

and where the expression in the curly brackets corresponds to the square of the growth rate. Note that the compressibility of non-axisymmetric, MRI modes is imprinted on deviations of  $\Lambda$  from unity:

$$\frac{\Delta\rho}{\rho} = (1 - \Lambda)(ik_{\parallel})\xi_{\parallel}, \quad (14)$$

e.g. the degree of compression gets stronger with field strength and, naturally, with a softer equation of state.

## 5 A QUASI-ADIABATIC INDEX FOR WAVE PHENOMENA

In a fluid composed of a neutral baryonic plasma (with ideal gas properties) plus a neutrino and/or photon component, ideal MHD dictates that the field lines remain frozen to the charged massive component while pressure perturbations associated with such material gas must track density perturbations on all scales. On the other hand, collisional coupling between the gas and the diffusive species

translates into a scale-dependent containment of the pressure perturbation associated with the radiative species. Consequently, acoustic wave modes partially supported by radiative pressure will be damped from the non-adiabatic loss of pressure support when the diffusive and wave time-scales are comparable (Agol & Krolik 1998). Such phenomenology can be quantitatively implemented by considering the mathematical equivalent to the equation of state for the diffusive component of the fluid.

The comoving frame, frequency-integrated, radiative transfer equation corrected to first order in the motion of the fluid,  $\beta$ , and with isotropic, elastic scattering as the lone source of opacity is

$$\begin{aligned} \tilde{\partial}_t I(\mathbf{n}) + \mathbf{n} \cdot \tilde{\nabla} I(\mathbf{n}) = (1 + 3\beta \cdot \mathbf{n})J - 2\beta \cdot \mathbf{H} \\ -(1 - \mathbf{n} \cdot \beta)I(\mathbf{n}). \end{aligned} \quad (15)$$

The tilde here means normalization to either the mean free path,  $\ell_{\text{mfp}}$ , or the mean crossing time,  $\ell_{\text{mfp}}/c$ , while  $I_\nu(\mathbf{n})$  is the radiative intensity in the direction of the unit vector  $\mathbf{n}$ , and  $J$  and  $\mathbf{H}$  are the first two moments of the intensity.

Assuming that multipoles of  $I(\mathbf{n})$  higher than quadrupole vanish, Agol & Krolik (1998) found an otherwise exact form for the mean intensity perturbation,  $\delta J \propto \tilde{\omega}^{-1} \tilde{\mathbf{k}} \cdot \delta\beta$ , in a background with  $J = I$ ,  $\mathbf{H} = 0$  (e.g. such that  $\delta = \Delta$ ). Tacitly conforming with the premise of isotropy, we rewrite their equation (23) by taking each of the perturbed diagonal components of the radiative field stress tensor,  $\delta \mathbf{K} = (1/4\pi)\delta \int d\nu d\Omega \mathbf{nn} I_\nu(\mathbf{n})$ , to be identical and proportional to the perturbed radiative pressure  $\delta p_{\text{rad}}$ . Thus  $\delta K_{ii} = 1/3\delta J$  and upon insertion of  $-\delta \ln V$  in lieu of  $\tilde{\omega}^{-1} \tilde{\mathbf{k}} \cdot \delta\beta$ , the following relation for quasi-adiabatic perturbations of the radiative component of the fluid ensues

$$\frac{\delta p_{\text{rad}}}{p_{\text{rad}}} = -\frac{4}{3} \left[ (1 - i\tilde{\omega}) + \frac{i}{15} \frac{\tilde{k}^2}{\tilde{\omega}} \frac{5 - 9i\tilde{\omega}}{1 - i\tilde{\omega}} \right]^{-1} \frac{\delta V}{V}. \quad (16)$$

This equation may be interpreted as a form of energy equation characterizing radiative ‘heat conduction’ out of compressive wave modes [Agol & Krolik 1998, compare equation (18), below, with equation (51.12) of Mihalas & Mihalas 1984].

Let us inspect the natural parametrization of this expression. When the non-diffusive pressure is negligible, this index represents a truly adiabatic processes for wave modes only if  $\tilde{\omega}$ ,  $\tilde{k}^2$  and  $\tilde{k}^2/\tilde{\omega} \ll 1$ . Individually, both  $\tilde{\omega} = \omega(\ell_{\text{mfp}}/c) \simeq |\hat{\Lambda}|(c_{\mathcal{H}}/c)\tau_{\text{disc}}^{-1} \simeq 2\alpha\hat{\Lambda}^2(c_{\mathcal{H}}/c)^2$ , and  $\tilde{k} \simeq k\mathcal{H}\tau_{\text{disc}}^{-1} \simeq 2\sqrt{\alpha}|\hat{\Lambda}|(c_{\mathcal{H}}/c)$ , are expected to be small for the scales of interest in the magnetorotational instability problem. However, the ratio  $\tilde{k}^2/\tilde{\omega}$  is not small and this has a non-trivial interpretation:

$$\left\{ \frac{k_{\parallel}^2}{k^2} \right\} \frac{\tilde{k}^2}{\tilde{\omega}} \doteq -i \frac{\ell_{\text{mfp}}^2/l_{\text{eddy}}^2}{\ell_{\text{mfp}}|\omega_{\text{MRI}}|/c} \simeq -\frac{i}{|\hat{\Lambda}|\Omega} \frac{(c/\tau_{\text{eddy}})}{l_{\text{eddy}}} \sim \mathcal{O}(2i) \quad (17)$$

which identifies with the ratio of eddy diffusion rate to MRI growth rate.  $\alpha_4\varphi\nu$

To zeroth order, from equation (16) one thus has

$$\tilde{\Gamma} \simeq \frac{4}{3} \left(1 + \frac{1}{3} \frac{\tilde{k}^2}{\tilde{\omega}}\right)^{-1} \quad (18)$$

which indicates that acoustic wave perturbations (i.e. with  $\omega$  real) on the scale of the MRI are strongly damped if radiative pressure is the predominant source of stress.

When there is more parity between the diffusive and non-diffusive sources of stress, the optical depths are slightly lower (see Section 3) but to place an upper limit on the quasi-adiabatic index this is less important than the uncertainty in  $(k_{\parallel}/k)^2$  (Section 6). Note that the value of the *generalized* quasi-adiabatic index as computed below is rather insensitive to small changes in the ratios  $\tilde{k}^2/\tilde{\omega}$  and

$(k_{\parallel}/k)^2$ . Such an index is calculated by defining a thermodynamic ‘quasi-volume’,  $\tilde{V} \equiv V^{\tilde{\eta}}$ , through equation (16) as follows

$$\frac{dp_{\text{rad}}}{p_{\text{rad}}} = -\frac{4}{3} \tilde{\eta} \frac{dV}{V} \equiv -\frac{4}{3} \frac{d\tilde{V}}{\tilde{V}}, \quad (19)$$

while computing the diffusive stress contribution to thermodynamic processes in terms of the logarithmic differential of this quasi-volume:  $d \ln \tilde{V} = \tilde{\eta} d \ln V$ . This yields the effective volume filled by the radiative, ‘leaky’ component of the fluid. Indeed, the scale-dependent function  $\tilde{\eta}$  is nothing but the logarithmic differential ratio between the effective and actual volumes occupied by the diffusive component.

We consider two cases in turn: (1) an admixture of photon radiation plus an ideal gas (with  $p_{\text{gas}} \simeq p_{\text{rad}}$ ) and (2) an admixture of neutrinos ( $p_{\text{rad}} = 7/8aT^4$ ) plus gas/radiation/pairs ( $p_{\text{gas}} = 11/12aT^4$ ) where the relatively cool nucleons make a negligible contribution to the total pressure (this assumption is consistent with the results of PWF but not with those of Di Matteo et al. 2002). An expression that includes the nucleon pressure contribution is readily derivable from the procedure described below). Further, when the degeneracy parameter inhibits the electron–positron pair density (Section 2) one should simply use  $p_{\text{gas}} \simeq a [1 + 7/4 \exp(-\mu/kT)] T^4/3$ . [Note that in general degeneracy pressure is negligible (PWF; Di Matteo et al. 2002)].

The standard lore (see, e.g. Chandrasekhar 1939; Mihalas & Mihalas 1984) of computation of the generalized adiabatic exponent,  $\Gamma_1$ , for an ideal gas plus a radiative component involves setting  $dQ = dU + dW \doteq \emptyset$  while working out expressions for  $dU$  and  $dW$  in terms of two logarithmic differentials, say,  $d \ln T$  and  $d \ln V$ . Using  $dQ \doteq \emptyset$  is clearly artificial but this is an adequate artefact when the main concern is with the non-elastic properties of the fluid and not with the amount of heat loss. Thus, we set  $dQ_{\text{tot}} = dQ_{\text{gas}} + d\tilde{Q}_{\text{rad}} = \emptyset$  with both  $dQ_{\text{gas}} = \emptyset$  and  $d\tilde{Q}_{\text{rad}} = \emptyset$  (recall that the interactions between the two species are assumed to be entirely elastic). Computation of the specific heat contribution from the gas component is straightforward:  $dQ_{\text{gas}}/V = p_{\text{gas}}/(\Gamma_{\text{gas}} - 1) d \ln T + p_{\text{gas}} d \ln V$ , while use of the quasi-volume for the radiative component yields

$$\begin{aligned} d\tilde{Q}_{\text{rad}}/\tilde{V} &= 12 p_{\text{rad}} d \ln T + 4 p_{\text{rad}} d \ln \tilde{V} \\ &= 12 p_{\text{rad}} d \ln T + 4\tilde{\eta} p_{\text{rad}} d \ln V. \end{aligned} \quad (20)$$

Elimination of  $d \ln T$  in favour of  $d \ln p$  is achieved by computing  $dp$  ( $d \ln T$ ,  $d \ln V$ ) for the combined gas. Furthermore, utilizing  $\Gamma_{\text{gas}} = 5/3$  and the standard definition of  $\Gamma_1 \equiv -d_{[\ln V]} \ln p$  yields

$$\Gamma_1 = \frac{(4\tilde{\eta}\beta + 1)(4\beta + 1) + 12\beta + 3/2}{(1 + \beta)(12\beta + 3/2)}, \quad (21)$$

where  $\beta \equiv p_{\text{rad}}/p_{\text{gas}}$ . Naturally, the  $\Re(\Gamma_1)$  agrees with the standard result (Chandrasekhar 1939; Mihalas & Mihalas 1984). Moreover, it is relatively straightforward to use the  $\Im(\Gamma_1)$  to find the decay rate of acoustic waves partially supported by radiative stress (aside from viscosity).

For neutrino-cooled accretion flows, equation (20) is still valid under the understanding that  $p_{\text{rad}} = 7/8aT^4$  corresponds to the neutrino pressure, while  $p_{\text{gas}} = 11/12aT^4$  (or  $p_{\text{gas}} \simeq a [1 + 7/4 \exp(-\mu/kT)] T^4/3$  when the electron component becomes degenerate) corresponds to photons and pairs and  $d\tilde{Q}_{\text{rad}}/V = 12 p_{\text{gas}} d \ln T + 4 p_{\text{gas}} d \ln V$ . With these substitutions, an identical procedure to the one above yields

$$\Gamma_1 = \frac{4(1 + \tilde{\eta}\beta)}{3(1 + \beta)}. \quad (22)$$

## 6 IMPACT OF RADIATIVE DIFFUSION AT MRI SCALES

We have worked out solutions of a simplified version of the dispersion relation for non-axisymmetric MRI modes in the limit of purely ‘horizontal’ fluid displacements (e.g. non-meridional). Since such displacements maximize the efficiency of free-energy tapping from the differential shear flow, this regime generally encompasses the unstable modes of fastest growth. The ensuing dispersion relation, equation (12), shows that the MRI is sensitive to a combination of two fluid properties – the ‘softness’ of the fluid and the ratio of gas to magnetic pressure – through a single parameter,  $\Lambda$ . Such a parametrization, in turn, bears a direct connection to the compressibility of the modes (cf. equation 14). Some physical insight may be gained by rewriting the compressibility of the mode,  $1 - \Lambda$  ( $\propto \Delta \rho/\rho$ ), as follows

$$1 - \Lambda = \left( 1 + \frac{\Gamma_1 p_{\text{r+g}}}{2 p_{\text{B}}} \right)^{-1},$$

while noting that ‘2’ represents the effective index of a magnetic field to perpendicular compression (see, e.g. Shu 1992). MRI modes thus involve a large degree of compression when the Alfvén speed is close to the effective sound speed of the material medium.

On the other hand, in Section 5 (cf. equation 17) we have demonstrated that when the disc is very hot and optically thick such that a radiative, diffusive species constitutes a significant source of stress, radiative heat conduction precludes a treatment of the fluid as a single fluid for the scales of relevance to the MRI; that is, compression at these scales becomes intrinsically non-adiabatic. Indeed, the ordering of wavenumber components associated with the horizontal regime,  $k_{\phi}, k_r \ll k_{\theta}$ , implies that the smallest length-scale of the eddies is meridional; i.e. that the eddies are very nearly flat. The brisk loss of pressure support from meridional radiative diffusion, in turn, means that these modes are strongly damped; i.e. the horizontal regime is inaccessible when radiative pressure dominates the dynamics.

Nevertheless, we are looking to read the gross properties of the fastest-growth modes from the 2D dispersion relation (12) by requiring parity among diffusive and non-diffusive sources of stress while finding an expression for the *generalized quasi-adiabatic index* of the combined fluid, equations (21) and (22). It is noteworthy that such parity among sources of stress may be expected in neutrino-cooled accretion flows such as those involved in gamma-ray burst progenitor models.

To place a zeroth-order [recall  $|\tilde{\omega}| \simeq \tilde{k}^2 \ll 1$ ,  $\tilde{\omega}/\tilde{k}^2 \sim \mathcal{O}(2i)$ ] upper limit on the quasi-adiabatic index, recall that for wave phase  $\propto i(\mathbf{k} \cdot \mathbf{x} - \omega t)$ , the MRI growth rate corresponds to a purely imaginary wave frequency (i.e. since the wave phase is stationary at the corotation radius):  $\omega_{\text{MRI}} = +i|\omega_{\text{MRI}}|$ . Thus, combining equations (17) and (18) yields

$$\tilde{\Gamma} = \frac{4}{3} \frac{1}{1 + \frac{1}{3} \tilde{k}^2 / |\tilde{\omega}_{\text{MRI}}|} \rightarrow \frac{4}{3} \left[ 1 + \frac{2}{3} \left( \frac{k^2}{k_{\parallel}^2} \right) \times \frac{\hat{q}_{\text{B}}^2}{|\tilde{\omega}_{\text{MRI}}|} \right]^{-1}. \quad (23)$$

An accurate estimate of the ratio  $\hat{q}_{\text{B}}^2/|\tilde{\omega}_{\text{MRI}}|$  for arbitrary values of  $\Lambda$  ( $\Gamma_1, \Theta$ ) may be found through the iterative use of equation (13) (since  $\tilde{\eta}$  and thus  $\Gamma_1$  itself depends on  $\hat{q}_{\text{B}}^2/|\tilde{\omega}_{\text{MRI}}|$ ), but the value of the *generalized* index is rather insensitive to small changes in this ratio.

For Keplerian rotation and in the weak-field (i.e. incompressible) limit, we have  $\hat{q}_B^2/|\hat{\omega}_{\text{MRI}}| \xrightarrow{\Theta \rightarrow 0} 5/4$ . On the other hand,  $k^2/k_{\parallel}^2 \gg 1$  for 2D motions so it should be clear that by setting  $k^2 \geq 2k_{\parallel}^2$  we are merely imposing a rough upper limit to the quasi-adiabatic index:

$$\tilde{\Gamma} \leq \frac{4}{3} \tilde{\eta}_{\text{max}} \xrightarrow{\Theta \rightarrow 0} \frac{1}{2} \quad (24)$$

(again, the result we seek is rather insensitive to the precise value of  $\tilde{\eta}_{\text{max}}$ ).

Using  $\tilde{\eta}_{\text{max}} = 3/8$ , and  $p_{\text{rad}} = p_{\text{gas}}$ , in equation (21), the generalized quasi-adiabatic index for an ideal gas plus photon radiation works out to be

$$\Gamma_1 = \frac{26}{27} \rightarrow 1^- \quad (25)$$

On the other hand, using  $\tilde{\eta}_{\text{max}} = 3/8$ ,  $p_{\text{rad}} = 7/8aT^4$  and  $p_{\text{gas}} = 11/12aT^4$  in equation (22), yields

$$\Gamma_1 = \frac{239}{258} \rightarrow 1^- \quad (26)$$

for a neutrino-cooled accretion flow. In either case, the fluid behaviour is close to isothermal when the quasi-adiabatic index associated with the diffusive species drops below the isothermal value  $\tilde{\Gamma} \leq 1$ .

## 7 DISCUSSION

Our discussion is focused on two broad dynamical properties of the instability on the presumption that these help clarify the phenomenology of the turbulence in the non-linear stage. Thus, the growth rate is identified in order of magnitude with the inverse of the correlation time of the turbulence and the geometrical regime of fastest growth, with the shape of the most frequent turbulent eddies. We also pay particular attention to the energy deposition process at the outer scale since a major portion of this paper is centred on the issue of entropy generation through non-adiabatic compression and heat conduction.

When a diffusive species represents the major contributor to the stress tensor, one may connect the sluggish growth rate of the MRI (discovered by Blaes & Socrates 2001 for axisymmetric modes) to the adiabatic inaccessibility of the horizontal regime associated with fastest growth in standard fluids. For non-axisymmetric modes in a toroidal field geometry, this regime becomes intrinsically non-adiabatic because of the brisk rate of vertical (meridional) heat transport out of nearly flat, compressive perturbations. Indeed, while in an ideal fluid the dominant eddies tend to have  $k_{\theta}$  several times larger than  $k_r$  and  $k_{\varphi}$ , adding a radiative diffusive species induces conductive losses that move mostly along  $\mathbf{1}_{\theta}$ . This, in turn, means that slow MHD modes with high  $k_{\theta}$ , i.e. horizontal regime modes, are preferentially damped. The magnitude of  $k_{\theta}$  for the fastest-growing modes is then set by a marginal damping condition. Since the length-scale for marginal damping is of order  $k_{\parallel}$  (cf. equation 17), it follows that MRI eddies should be more nearly isotropic in such an environment, i.e.  $k_{\theta} \approx k_{\parallel}$ , which goes along with the modest reduction in growth rate. The ultimate shape of the eddies is determined by the necessity of the unstable modes to draw free energy from the differential shear flow (i.e. with comoving fluid excursions *perpendicular* to the local meridian) against the tendency of the eddies to avoid strong damping from meridional conductive losses. Thus, the fastest-growing eddies in a fluid with diffusive properties should still be somewhat flattened but more nearly isotropic than those of a standard fluid.

Similar arguments may be made when there is more parity among diffusive and non-diffusive sources of stress. Moreover, the net ef-

fect is to soften the effective index of the fluid toward isothermality,  $\Gamma_1 \rightarrow 1$  (cf. equations 25 and 26), along with more modest dynamical effects. Furthermore, if thermalization processes are ‘slow’ compared to the instability time-scale, the energy deposition process may be associated primarily with conductive losses out of MRI eddies, and, secondarily, with thermalization interactions between the diffusive and the non-diffusive particle species. Since the shortest dimension of the fastest-growing modes is meridional, conductive losses move preferentially along the local meridian and this may favour an efficient cooling regime.

These conclusions are broadly supported by the numerical simulations of Turner, Stone & Salo (2002) and Turner et al. (2003). Their report of a standard radiation-pressure-dominated  $\alpha$ -disc, with  $p_{\text{rad}} \gtrsim p_{\text{gas}} \simeq p_B$  shows that the non-linear outcome of the MRI is a porous medium with drastic density contrasts. Under nearly constant total pressure and temperature, the non-linear regime shows that density enhancements anticorrelate with azimuthal field domains – just as expected from the linear theory – and that turbulent eddies live for about a dynamical time-scale while matter clumps are destroyed through collisions or by running through localized regions of shear on a similar time-scale. Notably, the non-linear density contrasts may be quite large ( $\rho_{\text{max}}/\rho_{\text{min}} \gtrsim 10$  when  $p_{\text{rad}} \gg p_{\text{gas}}$ ).

At a very fundamental level, the formation of these clumps may be connected to the effects of radiative heat conduction out of compressive perturbations. This can be understood by computing the polarization properties of the fastest-growing modes in the horizontal regime:  $\xi_r = -\sqrt{\Lambda}\xi_{\varphi}$  and  $|\xi_{\theta}| \ll |\xi_r|$ , and reading from equation (14) that at the linear stage of the instability there exists a converging flow toward every other Lagrangian displacement node of the modes (see fig. 2 of Foglizzo & Tagger 1995). In a fluid with entirely elastic (adiabatic) properties, the pressure perturbation associated with such compression will act as a restoring force to decompress the fluid in the non-linear stage. On the other hand, when the fluid has radiative heat conduction properties on the scale of the density perturbations, no such restoring force persists on time-scales longer than the inverse of the growth rate (which, following equation 17, is of the same order as the diffusion time). In the non-linear stage, material clumps thus formed will maintain their integrity and decompress only in proportion to the pressure contribution from the non-diffusive particle species.

Blaes & Socrates (2001) predicted qualitatively similar results for axisymmetric modes. Our analytical solutions agree with their result,  $|\hat{\omega}_{\text{MRI}}| \sim c_g/v_{\text{AIF}}$ , in the limit  $k_r = c_r \rightarrow 0$  and  $c_g \rightarrow 0$ . The present paper complements their findings and the non-linear numerical explorations of Turner et al. (2002, 2003) by presenting a simplified approximate dispersion relation for non-axisymmetric modes while interpreting the effects of radiative heat conduction in terms of an ‘ultrasoft’ quasi-adiabatic index (i.e.  $\tilde{\Gamma} < 1$ ) for the contribution of the radiative species to the energy equation. Thermalization processes are excluded by the radiative conduction approximation. Thermalization effects are discussed by Blaes & Socrates (2003, Section C) in terms of an effective sound speed for the combined fluid. Our results are applicable to optically thick, neutrino-cooled discs when the electron chemical potential rises significantly above the disc temperature such that non-conductive, thermalizing processes (i.e. neutrino annihilation into electron–positron pairs and absorption on to nuclei) are effectively blocked by high occupation of the Fermi levels. Our linear analysis is suggestive of a highly clumpy flow structure where neutrinos escape from neutron-rich mass overdensities with moderate electron degeneracy ( $\mu > kT$ ). Notably, an intermittent, porous disc structure is conducive to

efficient cooling as the radiative species escape easily once they diffuse out of the mass enhancements (Ruszkowski & Begelman 2002).

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