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Radiative Heat Conduction and the Magnetorotational Instability

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ABSTRACT
A photon or neutrino gas–semi-contained by a baryonic species through scattering–comprises a rather peculiar MHD fluid where the magnetic field is truly frozen only to the co-moving volume associated with the mass density. Although radiative diffusion precludes an adiabatic treatment of compressive perturbations, we show that the energy equation may be cast in "quasi-adiabatic" form for exponentially growing, non-propagating wave modes. Defining a generalized quasi-adiabatic index leads to a relatively straightforward dispersion relation for non-axisymmetric magnetorotational modes in the horizontal regime when an accretion the disk has comparable stress contributions from diffusive and non-diffusive particle species. This analysis is generally applicable to optically thick, neutrino-cooled disks since the pressure contributions from photons, pairs and neutrinos, all have the same temperature dependence whereas only the neutrino component has radiative heat conduction properties on the time and length scales of the instability. We discuss the energy deposition process and the temporal and spatial properties of the ensuing turbulent disk structure on the basis of the derived dispersion relation.

Key words: accretion disks–MHD—instabilities—black hole physics

1 PRELIMINARIES

Understanding the magnetorotational instability is key to the development of realistic models of accretion. As a physical process, the MRI justifies two *sine qua non* ingredients of accretion disk theory: entropy generation from the differential shear flow and turbulent angular momentum transport. Consequently, clarifying the dynamics that leads to MRI initiated turbulence in a few relevant astrophysical regimes is paramount in uncovering how the accretion process takes place.

In the context of accretion onto compact objects, the stress associated with a radiative particle species often dominates the dynamics at high temperatures but our understanding of its effect on MHD processes is incomplete at best. Agol & Krolik (1998) pointed out that compressive radiation-MHD wave modes will be strongly suppressed by diffusive loss of pressure support in a range of wave numbers: $c_{\rm ph}/c < \tilde{k}^{-1} < 1$, where $\tilde{k} \equiv \ell_{\rm mfp}\, k$ represents the wave number normalized to the mean-free-path and $c_{\rm ph}$ is the phase speed of compressive waves in the single fluid view. Such analysis is applicable to accretion disks at large wave numbers where the effect of inertial forces and stratification are minor. For the MRI, however, both inertial and stratification effects play key roles in dictating the dynamics and polarization of the modes.

Blaes & Socrates (2001) have reported on a radiation-MHD dispersion relation for axisymmetric modes that comprises the MRI under rather general circumstances. Our analysis is different from theirs mainly in the consideration of a purely toroidal field where the compressibility of the non-axisymmetric modes leads to the largest impact from radiative diffusive damping. Furthermore, aiming to understand ultra-relativistic flows very near black holes, we emphasize toroidal mode dynamics because these modes may become predominant when shear effects begin to overwhelm other inertial accelerations leading to fastest growing modes at large azimuthal scales (Araya-Góchez 2002). The scale and degree of intermittency in the turbulent flow bear a direct impact on gravitational wave emission models from hyper-accreting black holes (Araya-Góchez 2003). For a relativistically hot accretion disk with comparable vertical pressure support from diffusive and non-diffusive species (see §2), we solve for an approximate dispersion relation involving Lagrangian displacements in the *horizontal regime* ($\xi_\theta/\xi_r \simeq k_r/k_\theta \to \emptyset$). This simplification affords a lowering of the rank in the dispersion relation to second order (in ω^2) which, in turn, enables a relatively straightforward approximate treatment of the effects of radiative heat conduction (in contrast with Blaes and Socrates' more complex but exact, relation for axisymmetric modes).

2 RADIATIVE VISCOSITY *VS* DIFFUSION

The half optical depth to scattering of a standard radiation-pressure–dominated α-disk is related to the vertical scale height, $\mathcal{H}$, and to the angular rotation profile, $\Omega(r)$, through

$$\tau_{\rm disk} = -\frac{c}{2\alpha\hat{\rm A}\Omega\mathcal{H}} \qquad (1)$$

where Oort's A parameter is normalized $\hat{\rm A} \equiv \frac{1}{2}{\rm d}_{\ln r}\ln\Omega$ ($\doteq -3/4$ for a non-relativistic, Keplerian flow). This relation ensues from balancing the local heating rate, $\propto \dot{M}\Omega^2\hat{\rm A}$, with the local cooling rate from radiative diffusion, $\propto \nabla_\tau p$; while eliminating the accretion rate in favor of the vertically integrated component of the viscous stress responsible for angular momentum transport $\propto \alpha\,\mathcal{H}p$. Eq 1 is thus explicitly sensitive only to the local nature of the cooling (although it is implicitly subject to a suitable vertical gradient of heat deposition; see, e.g., Krolik 1999).

In the neutron-rich, neutrino cooled gas associated with a hyper-accreting black hole (see, e.g. Popham *et al.* 1999), the pressure contributions from radiation, pairs and neutrinos, all have the same temperature dependence, $\propto a/6\;T^4$, with relative contributions varying only by internal degrees of freedom times particle statistics factors: 2×1, $4\times7/8$, and $6\times7/8$, respectively (with only one helicity state per neutrino). Thus, the neutrino pressure is never greater than $\simeq p_{\rm thermal}$ where $p_{\rm thermal} = 11/12aT^4$ includes the pressure from photons and pairs. Yet, only the former species has radiative diffusion properties on a dynamical time scale. Thus, in this setting the neutrinos are the lone diffusive species and their stress is identified with $p_{\rm rad}$ below (e.g., radiative = diffusive species throughout this paper).

Partial pressure support by the non-diffusive components (baryons, toroidal field and the thermal component) does not modify Eq 1 as long as the contribution to the disk's flare from this extra pressure is inversely proportional to the mass density decrease from the same, e.g., 1D expansion. Consequently, for a neutrino cooled α-disk, a replacement in the source of opacity by neutrino scattering in the non-advective accretion regime: $0.1\,{\rm M}_\odot\,{\rm sec}^{-1} \le \dot{\rm M} \le 1\,{\rm M}_\odot\,{\rm sec}^{-1}$ (Di Matteo *et al.* 2002), is all that is needed for Eq 1 to hold.

On the other hand, the "parallel" size the of fastest growing MRI eddies is $l_{\rm eddy} = k_{\rm MRI}^{-1} \simeq v_{\rm Alf}/\Omega$. For magnetic angular momentum transport such that $\alpha \simeq (v_{\rm Alf}/c_{\mathcal{H}})^2$, the *isotropic* diffusion time through these eddies, $t_{\rm diff} \sim \Omega^{-1}$, turns out to be similar to the timescale for fastest MRI modes to develop $t_{\rm MRI} \simeq -{\rm A}_{\rm Oort}^{-1}$ (see §5). Moreover, the optical depth through these eddies is (with $c_{\mathcal{H}}^2 \equiv p_{\rm tot}/\rho$)

$$\tau_{\rm eddy} \sim \frac{v_{\rm Alf}}{c_{\mathcal{H}}} \times \tau_{\rm disk} \sim -\frac{1}{2\hat{\rm A}\sqrt{\alpha}}\,\frac{c}{c_{\mathcal{H}}} \gg 1. \qquad (2)$$

Since Compton/neutrino drag go as $\mathbf{f}_{\rm drag} \propto \tilde{k}^2\,\delta\boldsymbol{\beta}$ and $\tilde{k}^2 \simeq \mathcal{O}(\alpha\,c_{\mathcal{H}}^2/c^2)$, these estimates indicate that the effect of photon/neutrino viscosity is negligible for the optically thick eddies of a radiatively cooled disk.

3 A LAGRANGIAN FORMULATION OF THE MRI IN SOFT MEDIA

We take a semi-local approach by computing instantaneous (co-moving) growth rates in the limit $k_r/k_\theta \to 0$ and referring to these as the fastest growing modes on the implicit understanding of their transient nature (since $k_r(t) = k_{0r} - [{\rm d}_{\ln r}\Omega]\,k_\varphi t$). In terms of a local, co-moving observer, azimuthal wave numbers are no longer discrete and consequently, neither are the co-moving frequencies (Ogilvie & Pringle 1996). Further, we discard field curvature and radial gradient terms. A simple meridional stratification profile sets the physical scale-length of the problem: ${\rm d}_z\ln\rho = 1/\mathcal{H}$, with gas, radiative and magnetic pressures tracking the unperturbed density profile.

The first order relation between Lagrangian, Δ, and Eulerian, δ, variations (see, e.g., Chandrasekhar & Lebovitz 1964, Lynden-Bell & Ostriker 1967) is

$$\Delta = \delta + \boldsymbol{\xi}\cdot\boldsymbol{\nabla}. \qquad (3)$$

The Euler velocity perturbation, $\mathbf{v} \equiv \delta\mathbf{V}$, is thus related to the Lagrangian displacement, $\boldsymbol{\xi}$, through

$$\mathbf{v} = \{\partial_t + \mathbf{V}\cdot\boldsymbol{\nabla}\}\,\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\mathbf{V} \to i\sigma\boldsymbol{\xi} - \xi^r\Omega_{,r}\mathbf{1}_\varphi. \qquad (4)$$

The algebraic relation follows from the assumption of differential rotation and from writing $\exp i(\omega t + m\varphi + k_z z)$ dependencies for $\boldsymbol{\xi}$, while $\sigma \doteq \omega + m\Omega$ denotes the co-moving frequency of the perturbations.

These geometrical equations have their more traditional equivalents in the so-called shearing sheet approximation where a co-moving, "locally Cartesian frame" $(\hat{r},\hat{\varphi},\hat{\theta}) \to (x,y,z)$, is used along with the linearized shear velocity field, $\mathbf{V}(x) = [{\rm d}_{\ln r}\Omega]\,x\mathbf{1}_y$, to treat the problem locally while introducing the coriolis terms by hand. (The reader is invited to examine a companion paper: Araya-Góchez (2002), where the inertial terms are induced in a fully covariant, geometric way).

The linearized equations of motion for the displacement vector correspond to the Hill equations (Chandrasekhar 1961, Balbus & Hawley 1992)

$$\ddot{\xi}_r - 2\Omega\,\dot{\xi}_\varphi = -4{\rm A}_{\rm Oort}\Omega\,\xi_r + \frac{f_r}{\rho}, \quad \ddot{\xi}_\varphi + 2\Omega\,\dot{\xi}_r = \frac{f_y}{\rho}, \quad \text{and} \quad \ddot{\xi}_\theta = \frac{f_\theta}{\rho}, \qquad (5)$$

where each over-dot stands for a Lagrangian time derivative and $\mathbf{f}$, for the sum of body forces.

The Lagrangian perturbation of mass density and the Eulerian perturbation of the field follow straightforwardly from Eq [3], and from mass and magnetic flux conservations (under ideal MHD conditions, notwithstanding the peculiar nature of the fluid under study)

$$\frac{\Delta\rho}{\rho} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}, \qquad \delta\,\mathbf{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}). \tag{6}$$

Using the latter relation, the Lorentz force in the co-moving frame is readily laid out

$$\begin{aligned}\delta\frac{1}{4\pi\rho}\mathbf{J}\times\mathbf{B} \;&=\; v^2_{\rm Alf} \;\times\; \left\{\, \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \nabla^2_{\hat{\mathbf{B}}}\boldsymbol{\xi} \;-\nabla_{\hat{\mathbf{B}}}\boldsymbol{\nabla}(\mathbf{1}_{\hat{\mathbf{B}}}\cdot\boldsymbol{\xi}) - \mathbf{1}_{\hat{\mathbf{B}}}\nabla_{\hat{\mathbf{B}}}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\right\} \\ &\overset{\nabla\to i\mathbf{k}}{\longrightarrow} -v^2_{\rm Alf}\times\{(k_i\xi_i - k_{\hat{\mathbf{B}}}\xi_{\hat{\mathbf{B}}})\mathbf{k} + k^2_{\hat{\mathbf{B}}}\boldsymbol{\xi} - \mathbf{1}_{\hat{\mathbf{B}}}k_{\hat{\mathbf{B}}}k_i\xi_i.\}\end{aligned} \tag{7}$$

where $\mathbf{1}_{\hat{\mathbf{B}}}$ is a unit vector in the direction of the unperturbed field and where the scalar operator $\nabla_{\hat{\mathbf{B}}} \equiv \mathbf{1}_{\hat{\mathbf{B}}}\cdot\boldsymbol{\nabla}$. Note that the term $(k_i\xi_i - k_{\hat{\mathbf{B}}}\xi_{\hat{\mathbf{B}}}) \doteq k_\perp\xi_\perp$ may be interpreted as a restoring force due to the compression of field lines (Foglizzo & Tagger 1995).

Next, we look in detail at the non-magnetic stress terms.

In the single fluid view, the Lagrangian variation of the specific pressure gradient, $\rho^{-1}\boldsymbol{\nabla}p_{\rm r+g}$, contains two terms (see, e.g., Lynden-Bell & Ostriker 1967): one $\propto \Delta\rho^{-1}$, and another one $\propto \Delta\boldsymbol{\nabla}p_{\rm r+g}$. In terms of the displacement vector, the first term is proportional to the equilibrium value of $\boldsymbol{\nabla}p_{\rm r+g}$ which is negligible in the local treatment ($\propto$ radial gradient when $\xi_\theta/\xi_r \to \emptyset$). For the same reason, the Eulerian and Lagrangian variations of the specific pressure gradient are nearly identical.

The "thermodynamic" pressure term is thus given by

$$-\delta\left(\frac{1}{\rho}\boldsymbol{\nabla}p_{\rm r+g}\right) = \Gamma\,\frac{p_{\rm r+g}}{\rho}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) \overset{\nabla\to i\mathbf{k}}{\longrightarrow} c_s^2\,\mathbf{k}\,k_i\xi_i, \tag{8}$$

where, for a heterogeneous fluid, $\Gamma(\equiv d_{[\ln\rho]}\ln p)$ represents a generalized adiabatic index (see, e.g., Chandrasekhar 1939, Mihalas & Mihalas 1984).

Assembling $\mathbf{f}$ from Eqs [7 & 8], and plugging this form into Eqs [5] yields

$$\ddot{\boldsymbol{\xi}} + 2\,\boldsymbol{\Omega}\times\dot{\boldsymbol{\xi}} = \{c_s^2\,\mathbf{k} + v^2_{\rm Alf}(\mathbf{k} - k_{\hat{\mathbf{B}}}\,\mathbf{1}_{\hat{\mathbf{B}}})\}(\mathbf{k}\cdot\boldsymbol{\xi}) + v^2_{\rm Alf}(k^2_{\hat{\mathbf{B}}}\,\boldsymbol{\xi} - k_{\hat{\mathbf{B}}}\xi_{\hat{\mathbf{B}}}\,\mathbf{k}) - 4\mathrm{A}\Omega\,\xi_r\,\mathbf{1}_r. \tag{9}$$

This equation agrees with the matrix de-composition of Foglizzo & Tagger (1995) for $\Gamma = 1$. Moreover, these authors also pointed out that because of vertical stratification and rotation, the polarization of the slow MHD (e.g., MRI) modes obey

$$\frac{\xi_\theta}{\xi_r} \simeq \mathcal{O}\left(\frac{k_r}{k_\theta}\right) \tag{10}$$

when $k_r/k_\theta \to \emptyset$. Eqs [9 & 10] impose an anisotropy constraint on the components of the Lagrangian displacement

$$k_\perp\xi_\perp = -\frac{\Gamma}{\Gamma + 2\Theta}\;k_\parallel\xi_\parallel \quad \equiv -\Lambda\,k_\parallel\xi_\parallel, \tag{11}$$

where $\Theta \equiv p_{\mathbf{B}}/p_{\rm r+g}$.

In this horizontal regime, with $k_z\xi_z$ finite, the dispersion relation derived from Eq [9] and from the above constraint reads

$$\hat{\omega}^4 - \{(\Lambda+1)\hat{q}^2_{\hat{\mathbf{B}}} + \hat{\chi}^2\}\,\hat{\omega}^2 + \Lambda\,\hat{q}^2_{\hat{\mathbf{B}}}\{\hat{q}^2_{\hat{\mathbf{B}}} + 4\hat{\mathrm{A}}\} = \emptyset, \tag{12}$$

where all frequencies are normalized to the local rotation rate, $\hat{\chi}^2 \equiv 4(1+\hat{\mathrm{A}})$ is the squared of the epicyclic frequency, and $\hat{q}_{\hat{\mathbf{B}}} \equiv (\mathbf{k}\cdot\mathbf{v}_{\rm Alf})/\Omega$ is a frequency related to the component of the wave vector along the field (in velocity units). This expression matches the form given by T. Foglizzo (1995).

In terms of the function, $\Lambda(\Gamma,\Theta)$, we find the fastest growing wave numbers to be given by

$$\hat{q}^2_{\hat{\mathbf{B}}} = -2\hat{\mathrm{A}} + (\tfrac{1+\Lambda}{2\Lambda})\times\left\{-\frac{2\Lambda\hat{\mathrm{A}}^2}{\mathcal{D}}\right\}, \quad \text{where} \quad \mathcal{D} \equiv 1 + (\tfrac{1-\Lambda}{2})\hat{\mathrm{A}} + \sqrt{1 + (1-\Lambda)\hat{\mathrm{A}}}, \tag{13}$$

and where the expression in the curly brackets corresponds to the growth rate of the modes.

Notably, the compressibility of non-axisymmetric, MRI modes is imprint on deviations of Λ from unity

$$\frac{\Delta\rho}{\rho} = (1-\Lambda)(ik_\parallel)\;\xi_\parallel, \tag{14}$$

e.g., the degree of compression gets stronger with field strength and, naturally, with a softer equation of state.

4 A QUASI-ADIABATIC INDEX FOR WAVE PHENOMENA

In a heterogeneous fluid composed of a neutral baryonic plasma (with ideal gas properties) plus a neutrino and/or photon component, ideal MHD dictates that the field lines remain frozen to the charged massive component while pressure perturbations associated with such material gas must track density perturbations on all scales. On the other hand, collisional coupling between the gas and the diffusive species translates into a scale dependent containment of the pressure perturbation associated with the radiative species. Consequently, acoustic wave modes partially supported by radiative stress will be damped from a non-adiabatic loss of pressure support when the diffusive and wave time scales are comparable (Agol & Krolik 1998). Such phenomenology can be quantitatively implemented by considering the mathematical equivalent to the equation of state for the diffusive component of the fluid.

We start with the co-moving frame, frequency-integrated, radiative transfer equation–corrected to first order in the fluid's motion, $\boldsymbol{\beta}$, and with isotropic, elastic scattering as the lone source of opacity:

$$\tilde{\partial}_t I(\mathbf{n}) + \mathbf{n}\cdot\tilde{\boldsymbol{\nabla}} I(\mathbf{n}) = (1+3\boldsymbol{\beta}\cdot\mathbf{n})J - 2\boldsymbol{\beta}\cdot\mathbf{H} - (1-\mathbf{n}\cdot\boldsymbol{\beta})I(\mathbf{n}). \tag{15}$$

A tilde means normalization to the mean free path, $\ell_{\rm mfp}$, or to the mean crossing time, $\ell_{\rm mfp}/c$, while $I_\nu(\mathbf{n})$ is the radiative intensity in the direction of $\mathbf{n}$, and J and $\mathbf{H}$ are the first two moments of the intensity.

Agol and Krolik (1998) computed the first three radiative moment equations of Eq [15] and, by assuming that multipoles of $I(\mathbf{n})$ higher than quadrupole vanish, found an otherwise exact form for the mean intensity perturbation, $\delta J \propto \frac{1}{\tilde{\omega}}\tilde{\mathbf{k}}\cdot\delta\boldsymbol{\beta}$, in a background with $J = I$, $\mathbf{H} = \emptyset$ (e.g., such that $\delta = \Delta$). Tacitly conforming with the premise of isotropy, we re-write Eq [23] of Agol & Krolik (1998) by taking each of the perturbed diagonal components of the radiative field stress tensor, $\delta\underline{K} = (1/4\pi)\delta\int d\nu\, d\Omega\, \mathbf{nn}\, I_\nu(\mathbf{n})$, to be identical and proportional to the perturbed radiative pressure $\delta\, p_{\rm rad}$. Thus $\delta K_{ii} = \frac{1}{3}\delta J$ and, upon insertion of $-\delta\ln V$ in lieu of $\frac{1}{\tilde{\omega}}\tilde{\mathbf{k}}\cdot\delta\boldsymbol{\beta}$, this yields the following relation for quasi-adiabatic perturbations of the radiative component of the fluid

$$\frac{\delta p_{\rm rad}}{p_{\rm rad}} = -\frac{4}{3}\left\{(1-i\tilde{\omega}) + \frac{i}{15}\,\frac{\tilde{k}^2}{\tilde{\omega}}\,\frac{5-9i\tilde{\omega}}{1-i\tilde{\omega}}\right\}^{-1}\frac{\delta V}{V}. \tag{16}$$

Let us inspect the natural parameterization of this expression. When the non-diffusive pressure is negligible, this index represents a truly adiabatic processes for *wave* modes only if $\tilde{\omega}$, $\tilde{k}^2$ & $\tilde{k}^2/\tilde{\omega} \ll 1$. Individually, both $\tilde{\omega} = \omega(\ell_{\rm mfp}/c) \simeq |\hat{\rm A}|(c_{\mathcal{H}}/c)\tau_{\rm disk}^{-1} \simeq 2\alpha\hat{\rm A}^2(c_{\mathcal{H}}/c)^2$, and $\tilde{k} \simeq k\mathcal{H}\,\tau_{\rm disk}^{-1} \simeq 2\sqrt{\alpha}|\hat{\rm A}|(c_{\mathcal{H}}/c)$, are expected to be small for the scales of interest in the magnetorotational instability problem. However, the ratio $\tilde{k}^2/\tilde{\omega}$ is not small and this has a non-trivial interpretation:

$$\left\{\begin{matrix} k_\parallel^2 \\ \mathbf{k}^2 \end{matrix}\right\}\frac{\tilde{k}^2}{\tilde{\omega}} \doteq -i\,\frac{\ell_{\rm mfp}^2/l_{\rm eddy}^2}{\ell_{\rm mfp}\,|\omega_{\rm MRI}|/c} \simeq -\frac{i}{|{\rm A}|}\,\frac{(c/\tau_{\rm eddy})}{l_{\rm eddy}} \sim \mathcal{O}\,(2i) \tag{17}$$

which identifies with the ratio of eddy diffusion rate to MRI growth rate. $\alpha_4\varphi\upsilon$

To zeroth order, from Eq [16] one thus has

$$\tilde{\Gamma} \simeq \tfrac{4}{3}(1+\tfrac{i}{3}\tilde{k}^2/\tilde{\omega})^{-1} \tag{18}$$

which indicates that acoustic wave perturbations (i.e., with ω real) on the scale of the MRI are strongly damped if radiative pressure is the predominant source of stress. Indeed, Eq [18] may be interpreted as a form of energy equation characterizing radiative "heat conduction" out of compressive wave modes (Agol & Krolik 1998, compare this with Eq[51.12] of Mihalas & Mihalas 1984).

If there is more parity between the diffusive and non-diffusive sources of stress, we find a generalized quasi-adiabatic index by defining a thermodynamic "quasi-volume", $\tilde{V} \equiv V^{\tilde{\eta}}$, through Eq [16] as follows

$$\frac{dp_{\rm rad}}{p_{\rm rad}} = -\frac{4}{3}\,\tilde{\eta}\,\frac{dV}{V} \equiv -\frac{4}{3}\,\frac{d\tilde{V}}{\tilde{V}}, \tag{19}$$

while computing the diffusive stress contribution to thermodynamic processes in terms of the logarithmic differential of such quasi-volume: $d\ln\tilde{V} = \tilde{\eta}\, d\ln V$. (This yields the effective volume filled by the radiative, "leaky" component of the fluid; the scale-dependent function $\tilde{\eta}$ is nothing but the logarithmic differential ratio between the effective and actual volumes occupied by the gas.)

We consider two cases in turn: 1- an admixture of photon radiation plus an ideal gas (with $p_{\rm gas} \simeq p_{\rm rad}$) and 2- an admixture of neutrinos ($p_{\rm rad} = 7/8aT^4$) plus gas/radiation/pairs ($p_{\rm gas} = 11/12aT^4$) and where the latter species are collisionally coupled at all scales. Furthermore, for neutrino cooled flows we make the simplification that the relatively cool neucleons ($T \lesssim 20$ MeV) make a negligible contribution to the total pressure (see, e.g., Di Matteo *et al.* 2002)

The standard lore (see, e.g., Mihalas & Mihalas 1984, Chandrasekhar 1939) computation of the generalized adiabatic exponent, Γ_1, for an ideal gas plus a radiative component involves setting $dQ = dU + dW \doteq \emptyset$ while working out expressions for dU & dW in terms of two logarithmic differentials, say, $d\ln T$ and $d\ln V$. Using $dQ \doteq \emptyset$ is clearly artificial but this is an adequate artifact when the main concern is with the non-elastic properties of the fluid and not with the amount of heat loss. Thus, we set $dQ_{\rm tot} = dQ_{\rm gas} + d\tilde{Q}_{\rm rad} = \emptyset$ with both $dQ_{\rm gas} = \emptyset$, and $d\tilde{Q}_{\rm rad} = \emptyset$ (recall that the interactions between the two species are assumed to be entirely elastic). Computation of the specific heat contribution from the gas component is straightforward $dQ_{\rm gas}/V = p_{\rm gas}/(\Gamma_{\rm gas}-1)d\ln T + p_{\rm gas}\, d\ln V$; while use of the quasi-volume for the radiative component yields

$$d\tilde{Q}_{\rm rad}/\tilde{V} = 12\,p_{\rm rad}\,d\ln T + 4\,p_{\rm rad}\,d\ln\tilde{V} = 12\,p_{\rm rad}\,d\ln T + 4\tilde{\eta}\,p_{\rm rad}\,d\ln V. \tag{20}$$

Elimination of $d\ln T$ in favor of $d\ln p$ is achieved by computing $dp\,(d\ln T,\, d\ln V)$ for the combined gas. Furthermore, utilizing $\Gamma_{\rm gas} = 5/3$ and the standard definition of $\Gamma_1 \equiv -d_{[\ln V]}\ln p\ (= d_{[\ln\rho]}\ln p)$ yields

$$\Gamma_1 = \frac{(4\tilde{\eta}\beta + 1)(4\beta + 1) + 12\beta + 3/2}{(1+\beta)(12\beta + 3/2)}, \tag{21}$$

where $\beta \equiv p_{\rm rad}/p_{\rm gas}$. Naturally, the $\Re(\Gamma_1)$ agrees with the standard result (Chandrasekhar 1939, Mihalas & Mihalas 1984). Moreover, it is relatively straightforward to use the $\Im(\Gamma_1)$ to find the decay rate of acoustic waves partially supported by radiative stress (aside from viscosity).

For neutrino cooled accretion flows, Eq [20] is still valid under the understanding that $p_{\rm rad} = 7/8aT^4$ corresponds to the neutrino pressure, while $p_{\rm gas} = 11/12aT^4$ corresponds to photons and pairs and $dQ_{\rm rad}/V = 12\, p_{\rm gas}\, d\ln T + 4\, p_{\rm gas}\, d\ln V$. With these substitutions, an identical procedure to the one above yields

$$\Gamma_1 = \frac{4}{3}\frac{(1+\tilde{\eta}\beta)}{(1+\beta)}. \tag{22}$$

5 IMPACT OF RADIATIVE DIFFUSION AT MRI SCALES

Thus far, we have aimed to maintain the mathematical simplicity afforded by analytical solutions in order to gain physical insight on the phenomenology of radiative heat conduction and on its impact on the MRI.

We have thus worked out solutions of a simplified version of the dispersion relation for non-axisymmetric MRI modes in the limit of purely horizontal fluid displacements. Since horizontal displacements maximize the efficiency of free-energy tapping from a differential shear flow, this regime generally encompasses the unstable modes of fastest growth. The equation of motion, Eq [9], is applicable to standard heterogeneous fluids with arbitrary generalized index Γ_1. The ensuing dispersion relation, Eq [12], shows that the MRI is sensitive to a combination of two fluid properties–the "softness" of the fluid and the strength of the field–through a single parameter: Λ. Such parametrization, in turn, bears a direct connection to the compressibility of the modes through Eq [14]. Some physical insight may be gained by re-writing the compressibility parameter, $1-\Lambda$ ($\propto \Delta\rho/\rho$) as follows (c.f. Eq [14])

$$1 - \Lambda = \left(1 + \frac{\Gamma_1}{2}\frac{p_{\rm r+g}}{p_{\mathbf{B}}}\right)^{-1},$$

while recapitulating that "2" represents the effective index of a magnetic field to perpendicular compression (see, e.g., Shu 1992). MRI modes thus involve a large degree of compression when the effective sound speed of the material medium is much smaller than the Alfvén speed.

On the other hand, in §4 (c.f. Eq [17]) we have demonstrated that when the disk fluid is very hot such that a radiative, diffusive species (photons or neutrinos) constitutes a significant source of stress, radiative heat conduction precludes a formal treatment of the fluid as a single fluid at the scales of relevance to the MRI; that is, compression at these scales becomes non-adiabatic. A key conceptual point of this paper is that when the diffusive species dominates the stress tensor (e.g., radiation-pressure–dominated disks), the horizontal regime becomes inaccessible because of the brisk loss of pressure support from meridional diffusion[1]. Nevertheless, we are looking to read the gross properties of the fastest growth modes from the 2D dispersion relation Eq [12] by requiring parity among diffusive and non-diffusive sources of stress while finding an expression for the *generalized quasi-adiabatic index* of the combined fluid, Eq [21 & 22]. It is noteworthy that such parity among sources of stress is to be expected in neutrino cooled accretion flows such as those invoked in gamma-ray burst progenitor models.

To place a zeroth order (recall $|\tilde{\omega}| \simeq \tilde{k}^2 \ll 1,\ \ \tilde{\omega}/\tilde{k}^2 \sim \mathcal{O}[2i]$) upper limit on the quasi-adiabatic index, recall that for wave phase $\propto i(\mathbf{k}\cdot\mathbf{x} - \omega t)$, the MRI growth rate corresponds to a purely imaginary wave frequency (i.e., since the wave phase is stationary at the co-rotation radius): $\omega_{\rm MRI} = +i|\omega_{\rm MRI}|$. Thus, combining Eqs [17 & 18] yields

$$\tilde{\Gamma} = \frac{4}{3}\frac{1}{1+\frac{1}{3}\tilde{k}^2/|\tilde{\omega}_{\rm MRI}|} \longrightarrow \frac{4}{3}\left(1 + \frac{2}{3}\left\{\frac{\mathbf{k}^2}{k_\parallel^2}\right\} \times \frac{\hat{q}^2_{\hat{\mathbf{B}}}}{|\hat{\omega}_{\rm MRI}|}\right)^{-1}. \tag{23}$$

An accurate estimate of the ratio $\hat{q}^2_{\hat{\mathbf{B}}}/|\hat{\omega}_{\rm MRI}|$ for arbitrary values of $\Lambda(\Gamma_1,\,\Theta)$ may be found through the iterative use of Eq [13] (since $\tilde{\eta}$ and thus Γ_1 itself depends on $\hat{q}^2_{\hat{\mathbf{B}}}/|\hat{\omega}_{\rm MRI}|$), but the value of the *generalized* index is rather insensitive to small changes in this ratio. For Keplerian rotation and in the weak field (i.e., incompressible) limit, we have

$$\hat{q}^2_{\hat{\mathbf{B}}}/|\hat{\omega}_{\rm MRI}| \xrightarrow{\Theta\to 0} 5/4.$$

On the other hand, the ratio $\mathbf{k}^2/k_\parallel^2 \gg 1$ for 2D motions so it should be clear that by setting $\mathbf{k}^2 \geq 2k_\parallel^2$ we are merely imposing a rough upper limit to the quasi-adiabatic index:

$$\tilde{\Gamma} \leq \frac{4}{3}\,\tilde{\eta}_{\rm max} \xrightarrow{\Theta\to 0} \frac{1}{2} \tag{24}$$

[1] Note that the ordering of wavenumber components in the horizontal regime, $k_\varphi, k_r \ll k_\theta$, implies that the smallest length scale of the eddies is meridional; i.e., that the eddies are very nearly flat.

(again, the result we seek is rather insensitive to the precise value of $\tilde{\eta}_{\rm max}$).

Using $\tilde{\eta}_{\rm max} = 3/8$, and $p_{\rm rad} = p_{\rm gas}$, in Eq [21], the generalized quasi-adiabatic index for an ideal gas plus photon radiation works out to be

$$\Gamma_1 \longrightarrow 1^+. \tag{25}$$

On the other hand, using $\tilde{\eta}_{\rm max} = 3/8$, $p_{\rm rad} = 7/8aT^4$ and $p_{\rm gas} = 11/12aT^4$ in Eq [22], yields

$$\Gamma_1 = \frac{956}{1032} \longrightarrow 1^- \tag{26}$$

for a neutrtino cooled accretion flow. In either case, the fluid behavior is close to isothermal (in a quasi-adiabatic sense) when the quasi-adiabatic index associated with the diffusive species drops below the isothermal value $\tilde{\Gamma} \leq 1$.

6 DISCUSSION

Our phenomenological discussion is focused on two broad dynamical properties of the instability on the presumption that these help clarify the physical picture in the non-linear stage. Thus, the growth rate is identified in order of magnitude with the inverse of the correlation time of the turbulence and the geometrical regime of fastest growth, with the shape of the turbulent eddies. We also pay particular attention to the energy deposition process at the outer scale since a major portion of this paper is centered on the issue of entropy generation through non-adiabatic compression when a diffusive species contributes non-trivially to the stress tensor.

On the basis of the discussion in the previous section, we find that the root of a sluggish growth rate for the MRI (discovered by Blaes & Socrates 2001 for axisymmetric modes) when radiative pressure dominates the non-diffusive component is the adiabatic inaccessibility of the horizontal regime associated with fastest growth in standard fluids. For non-axisymmetric modes, this regime becomes intrinsically non-adiabatic because of the brisk rate of heat transport out of anisotropic compressive perturbations. Because we begin with a toroidal field configuration where the non-axisymmetric modes are strongly compressive from the onset, the impact of radiative heat conduction is most dramatic and evidenced first hand from the dispersion relation.

While in an ideal fluid the dominant eddies tend to have k_θ several times larger than k_r and k_φ, adding a radiative diffusive species will induce conductive losses that move mostly along $\mathbf{1}_\theta$. This, in turn, means that the modes with high k_θ, i.e., horizontal modes, are preferentially damped. The magnitude of k_θ for the fastest growing modes is then be set by a marginal damping condition. Since the length scale for marginal damping is $\approx k_\parallel$ (c.f. Eq 17), it follows that MRI eddies should be more nearly isotropic in such an environment, which goes along with the reduction in growth rate. For the same reason, energy deposition on to the thermal bath (directly associated with the radiative species, and, secondarily, to the non-radiative species through scattering) occurs on the instability time scale. On the other hand, when the source of stress is equally divided among diffusive and non-diffusive particle species, the net effect is to soften the effective index of the fluid toward isothermality $\Gamma_1 \to 1$ (c.f. Eq [25 & 26]) along with similar but more modest dynamical effects.

These conclusions are broadly supported by the numerical simulations of Turner *et al.* (2001, 2003). Their report of a standard radiation-pressure–dominated α-disk, with $p_{\rm rad} \gtrsim p_{\rm gas} \simeq p_{\mathbf{B}}$ shows that the non-linear outcome of the MRI is a porous medium with drastic density contrasts. Under nearly constant total pressure and temperature, the non-linear regime shows that density enhancements anti-correlate with azimuthal field domains–just as expected from the linear theory–and that turbulent eddies live for about a dynamical time scale while matter clumps are destroyed through collisions or by running through localized regions of shear on a similar time scale. Notably, the non-linear density contrasts may be quite large $< \rho_{\rm max}/\rho_{\rm min} > \gtrsim 10$ when $p_{\rm rad} \gg p_{\rm gas}$.

At a very fundamental level, clump formation is intimately connected to the effects of radiative heat conduction out of compressive perturbations. This can be understood by computing the polarization properties of the fastest growing modes in the horizontal regime: $\xi_r = -\sqrt{\Lambda}\ \xi_\varphi$ & $|\xi_\theta| \ll |\xi_r|$, and reading from Eq [14] that at the linear stage of the instability there exists a converging flow toward the Lagrangian displacement node of the modes (see Fig. 2 of Foglizzo & Tagger 1995). In a fluid with entirely elastic (adiabatic) properties, the pressure perturbation associated with such compression will act as a restoring force to de-compress the fluid in the non-linear stage. On the other hand, when the fluid has radiative heat conduction properties on the scale of the density perturbations, no such restoring force persists on time scales longer than the inverse of the growth rate (which, following Eq [17], is of the same order as the diffusion time). Material clumps thus formed will maintain their integrity and survive for longer times. When the diffusive and non-diffusive pressures are comparable, these arguments still hold but only in part. In this case, compressive perturbations in the non-linear stage will decompress only in proportion to the remaining pressure support from the non-diffusive species.

Blaes & Socrates (2001) predicted qualitatively similar results for axisymmetric modes. Our analytical solutions agree with their result, $|\hat{\omega}_{\rm MRI}| \sim c_g/v_{\rm Alf}$, in the limit $k_r = c_r \to 0$ and $c_g \to 0$. This paper complements these findings and the non-linear numerical explorations of Turner *et al.* (2001, 2003) by presenting a simplified approximate dispersion relation for non-axisymmetric modes while interpreting the effects of radiative heat conduction in terms of an "ultra-soft" quasi-adiabatic index (i.e., $\tilde{\Gamma} < 1$) for the radiative contribution to the stress tensor. The quasi-adiabatic index and all derived thermodynamic quantities (including "quasi-specific" quantities that ensue from division by the quasi-volume, §4) are mere mathematical artifacts to manipulate thermodynamical relations in the usual manner. Furthermore, we have demonstrated that when there is parity among the sources of stress, the combined gas behaves isothermally. Our results are generally applicable to optically thick, neutrino-cooled disks since the pressure contributions from photons, pairs and neutrinos, all have the same temperature dependence whereas only the neutrino component has radiative heat conduction properties on the time and length scales of the instability.

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REFERENCES

Agol, E. & Krolik, J. 1998, Ap.J., 507, 304.
Araya-Góchez , R.A. 2002, M.N.R.A.S. 337, 795
Araya-Góchez , R.A. 2003, M.N.R.A.S. submitted (astro-ph/0311001)
Balbus, S.A. & Hawley, J.F. 1992a, Ap.J. 392, 662
Blaes, O. & Socrates, A. 2001, Ap.J., 553, 987
Chandrasekhar, S. & Lebovitz, N. 1964, Ap.J., 140, 1517
Chandrasekhar, S. 1939, Stellar Structure and Evolution (University of Chicago Press: Chicago)
Chandrasekhar, S. 1961, Hydrodynamic and Hydromagnetic Stability (Oxford University Press: London)
Di Matteo, T., Perna, R. & Narayan, R. 2002, Ap.J., 579, 706
Foglizzo, T. 1995, Ph.D. Thesis, University of Paris VII
Foglizzo, T. & Tagger M. 1995, A. & A., 301, 293
Krolik, J.H. 1999, Active Galactic Nucleii. Princeton Series in Astrophysics, Eds. Jeremiah P. Ostriker & David N. Spergel (Princeton: Princeton University Press)
Lynden-Bell, D. & Ostriker, J. 1967, M.N.R.A.S., 136, 293
Mihalas, D. & Mihalas, B.W. 1984, Foundations of Radiation Hydrodynamics (Oxford University Press: New York)
Ogilvie, G.I. & Pringle, J.E. 1996, M.N.R.A.S., 279, 152
Popham, R., Woosley, S., & Fryer, C. 1999, Ap.J., 518, 356
Shu, F.H. 1992, The Physics of Astrophysics Vol. II: Gas Dynamics, University Science Books: Mill Valley, California
Turner, N., Stone, J., & Sano, T 2002, Ap.J., 566, 148
Turner, N., Stone, J., Krolik, J. & Sano, T 2003, Ap.J., 593, 992