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# Hydromagnetic stability of a slim disc in a stationary geometry

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**ABSTRACT**

The magnetorotational instability originates from the elastic coupling of fluid elements in orbit around a gravitational well. Since inertial accelerations play a fundamental dynamical role in the process, one may expect substantial modifications by strong gravity in the case of accretion on to a black hole. In this paper, we develop a fully covariant, Lagrangian displacement vector field formalism with the aim of addressing these issues for a disc embedded in a stationary geometry with negligible radial flow. This construction enables a transparent connection between particle dynamics and the ensuing dispersion relation for magnetohydrodynamic wave modes. The magnetorotational instability (MRI) – in its incompressible variant – is found to operate virtually unabated down to the marginally stable orbit; the putative inner boundary of standard accretion disc theory. To obtain a qualitative feel for the dynamical evolution of the flow below $r_{\rm ms}$, we assume a mildly advective accretion flow such that the angular velocity profile departs slowly from circular geodesic flow. This exercise suggests that turbulent eddies will occur at spatial scales approaching the radial distance while tracking the surfaces of null angular velocity gradients. The implied field topology, namely large-scale horizontal field domains, should yield strong mass segregation at the displacement nodes of the non-linear modes when radiation stress dominates the local disc structure (an expectation supported by quasi-linear arguments and by the non-linear behaviour of the MRI in a non-relativistic setting). Under this circumstance, baryon-poor flux in horizontal field domains will be subject to radial buoyancy and to the Parker instability, thereby promoting the growth of poloidal field.

**Key words:** black hole physics – gravitational waves – instabilities – MHD.

## 1 INTRODUCTION

The process of accretion on to compact objects has long been recognized as the primary mechanism in powering the most luminous events in space. In the traditional picture of Shakura & Sunyaev (1973) and Novikov & Thorne (1973), entropy is generated and radiated locally from the free energy available in a shear flow with a Keplerian angular velocity profile. Two salient oversimplifications of this framework have been the focus of intense research and progress in the previous decade: energy advection by the flow and free energy tapping and angular momentum transport through magnetohydrodynamical (MHD) processes.

The magnetorotational instability or MRI (Velikhov 1959; Chandrasekhar 1961; Balbus & Hawley 1991), justifies the long-sought mechanism for efficient, turbulent transport of angular momentum that enables accretion discs to operate with astrophysically interesting mass accretion rates (Pringle 1981). The importance of this process cannot be overstated: by catalysing accretion into gravitational wells, the MRI enables a plethora of astrophysical phenomena to occur, from protostar formation inside molecular clouds to jet launching in quasars. The MRI also holds the key to understand the extraction of free energy from the differential shear flow of otherwise hydrodynamically stable discs (Balbus, Hawley & Winters 1999; Godon & Livio 1999).

On the observational front, the wealth of high-quality data from spectral and timing devices aboard space-borne high-energy observatories has turned out the most compelling evidence yet of accretion on to black holes. The discoveries of pairs of high-frequency quasi-periodic oscillations in *RXTE* X-ray timing data from microquasars *GRO* J1655-40 and GRO 1915+105 (Strohmayer 2001a,b) have brought the spotlight to hydrodynamical models of adiabatic global excitations of the inner disc, also known as diskoseismology models (Perez et al. 1997) or relativistic precession models (Stella, Vietri & Morsink 1999). Interestingly, the role of magnetic fields has been largely ignored in spite of clear evidence that quasi-periodic objects (QPOs), being non-thermal, hard X-ray phenomena, probably do not originate in the accretion disc proper but rather on a magnetically active accretion disc corona (Blandford, private communication). Likewise, the recent report

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(Wilms et al. 2001) of the detection of a very broad Fe K$\alpha$ feature on the *XMM-EPIC* spectrum of MCG-6-30-15, has made a strong case for the inadequacy of standard models of energy deposition in accretion discs. The proposed solutions to this paradox – extraction of black hole spin energy (Blandford & Znajek 1977), or non-zero torque at the marginally stable circular orbit (ISCO) radius, $r_{\rm ms}$ (Agol & Krolik 2000) – both rely on magnetic coupling between a standard disc and the flow inside $r_{\rm ms}$.

On a more exotic front, theoretical progress in our understanding of accretion processes at the most extreme imaginable conditions – stellar-mass black holes hyper-accreting at 12 orders of magnitude above the Eddington limit – requires attention to a detailed physical account of highly relativistic accretion flows. Aside from the potential to explain gamma-ray burst phenomenology, such studies are chiefly relevant to assessing the likelihood of 'failed' supernovae as gravitational wave sources (Fryer, Holz & Hughes 2002). Indeed, when neutrino trapping occurs at $\dot{M} \gtrsim 1\ {\rm M}_{\odot}\ {\rm s}^{-1}$ (Popham & Gammie 1998), the associated dynamical stress will mimic the effects of radiation stress in standard discs where clumpy accretion ensues (Turner, Stone & Sano 2001). If the mass fraction in the clumps is large, prolific gravitational waves will be emitted from the mass quadrupole moment associated with the bulk motion of large mass overdensities. Such a scenario will also lead to excitations of the geometry of the black hole, which at high values of the spin parameter $a$, can produce highly characteristic, monochromatic black hole ringing as the geometry settles towards a quiescent Kerr state. Remarkably, the expectation of a large mass fraction in the clumps is reasonable and justifiable by the physical picture of near-hole accretion presented herein.

An outstanding issue yet to be addressed in light of recent theoretical progress is our view of black hole accretion inside the marginally stable orbit, the putative inner boundary of standard accretion disc theory. In particular, very little is known concretely about the inertial effects of strong gravity on the relevant MHD processes. Previous work has either assumed pure hydrodynamical flow (and energetically negligible energy release) or, alternatively, laminar flow under ideal MHD conditions (Gammie 1999; Krolik 1999). Krolik (1999) has made an interesting point: under mere flux freezing conditions the assumption of ballistic orbits in the plunging region is *never* self-consistent; when the radial velocity component is significant, the magnetic field energy density becomes comparable to the rest mass energy density of the matter.

In this paper, we address the issue of stability of the magnetic field (comoving frame) in a stationary, axially symmetric background geometry. Curiously, the two key developments in accretion disc theory over the previous few years may have come of age to properly address the problem at hand: inside $r_{\rm ms}$ the accretion flow will be mildly advective, with a slightly sub-Keplerian angular velocity profile and possibly supported in part by the radial pressure gradient of a hot MHD fluid with significant relativistic enthalpy [see Popham & Gammie (1998)] solutions for moderate values of $\alpha$ and advected fraction $f$). In this spirit, we argue in Section 5 that the natural evolution of the MRI inside the marginally stable orbit is at least consistent with this view.

The (magnetohydro) dynamics of black hole accretion comprises two important aspects that have received relatively little attention: the effects of radiation pressure (see, however, Blaes & Socrates 2001; Turner et al. 2001), and the effects of strong gravity (see footnote 7 of Gammie & Popham 1998). We will address the former problem in a future paper (Araya-Góchez & Vishniac 2001), while concentrating on general relativity in this one. As a background, Section 2 looks at the Lagrangian displacement vector field formulation of the MRI concentrating on inertial and compressibility effects. In Section 3, we develop a fully covariant theory of the instability. The intention is to build a theoretical framework from first principles in order to avoid missing any subtleties associated to the full incorporation of gravitational effects (e.g. reference is made to the Cowling approximation and to the fixing of the gauge associated with the component of the Lagrangian displacement along the four-velocity of the fluid). The elastic response of the field is computed by noting that the surface of invariance of the Faraday tensor attributes mathematically identical variational properties to the two four-vectors that span it: the magnetic field four-vector and the four-velocity of the fluid. We then make the minimal modifications to the relativistic fluid equations that allow for the inclusion of a coherent magnetic field and undertake a local stability analysis of this field in the medium of a slim disc around a rotating black hole, while suppressing compression. The role of compressibility in a photon gas is then briefly assessed.

## 2 A LAGRANGIAN FORMULATION OF THE MRI IN COMPRESSIBLE MEDIA

The MRI is essentially a local instability. In the frame of the fluid, the interplay of inertial 'forces' with the elastic coupling of fluid elements creates an unstable situation for the redistribution of specific angular momentum, $\ell$. Without the elastic coupling provided by the *bending* of field lines, such inertial forces – namely, the shear (tide) and the coriolis terms – induce radial epicyclic motions while preserving specific angular momentum in collisionless fluids (e.g. stars in the Galaxy). This is related to the Rayleigh criterion for stability of a differentially rotating fluid: $r^{-3}{\rm d}_r \ell^2 = \kappa^2 \geqslant 0$, where $\kappa$ is the frequency of epicyclic motions.

In the weak-field limit, one may construct a dispersion relation quite independently of the specific magnetic field topology: highly subthermal fields, $v_{\rm Alf}/c_{\rm s} \ll 1$, guarantee that the instability is truly local,[1] occurring at large values of $k_{\parallel} \simeq \Omega/v_{\rm Alf} (\equiv \boldsymbol{k} \cdot \mathbf{1}_{\hat{B}})$. In this simplified approach, the global disc structure is ignored (no curvature nor radial structure) and the response of the field amounts to nothing more than providing a restoring force to displacements from equilibrium (Balbus & Hawley 1992). Indeed, in the *horizontal regime of Lagrangian displacement* two orthogonal field topologies yield nearly identical mathematical dispersion relations for wavemodes: axisymmetric perturbations of a meridional field and non-axisymmetric perturbations of a toroidal field. The former case corresponds to the 'classical' Balbus–Hawley instability and its physical relevance is free of controversy. The relative importance of the latter analyses is a more subtle issue.

[1] When the field is non-negligible, $k_{\parallel}^{-1}$ may approach the pressure scaleheight of the disc and in the case of supra-thermal toroidal fields, non-axisymmetric modes have fastest growing wavenumbers that may approach the inverse radial scalelength (Foglizzo & Tagger 1995).

A somewhat technical point – well discussed in the review by Balbus & Hawley (1998) – is the non-locality induced by shear on wave-modes with Eulerian coordinate phase-dependences. For $k_\varphi \neq 0$ modes, shear evolves the radial component of wavenumbers according to $k_r(t) = k_{0r} - [\mathrm{d}_{\ln r}\Omega]\, k_\varphi t$, which means that modes that could be 'unstable' are only so, transiently. The maximum instantaneous growth rate occurs when $k_r \to 0$ and matches that of the local axisymmetric modes. In the end, this issue turns out to be more academic than practical but it stresses the importance of treating the instability locally, in comoving coordinates. The down side is that this greatly complicates global approaches that rely on eigenmode solutions in Eulerian coordinates extrinsic to the fluid. On the other hand, in a local approach azimuthal wavenumbers are no longer discrete (Ogilvie & Pringle 1996) and consequently, neither are the comoving frequencies (see below).

A related issue concerns the relevance of non-axisymmetric mode analyses when the magnetic field is not purely toroidal. Balbus & Hawley (1998) argue that the strict ordering of wavenumber components (and narrow phase space) necessitated to achieve fastest growth: $k_\varphi \ll k_r \ll k_\theta$, ensue in violent poloidal Alfvénic couplings that promptly take over the dynamics. Non-axisymmetric modes, however, are important for at least two key reasons: (i) the ordering is not so restrictive when the fields are not weak (as needed to explain $\alpha$ values of a few tenths), and (ii) *compressive*, non-axisymmetric modes are fundamental to examine energy deposition when radiation stress becomes significant (Araya-Góchez & Vishniac, in preparation). Moreover, because the dispersion relations relate simply (at least in the horizontal regime of fastest growth) it is rather useful to examine both cases at once.

Aiming to formulate a fully covariant relativistic theory of the MRI in Section 3, this section conducts the same task in three dimensions. The linear stability analysis is carried out in terms of the Lagrangian displacement vector field, $\boldsymbol{\xi}$. Foglizzo (1995) has stressed the usefulness of this approach in accounting for the polarization of compressive MHD modes. A simple meridional stratification profile sets the physical scalelength of the problem: $\mathrm{d}_z \ln \rho = \mathcal{H}^{-1}$, with gas, radiation (and possibly magnetic) pressures tracking the unperturbed density profile $\rho\mathcal{H}\Omega = p_{r+g} + p_{\boldsymbol{B}}$. The problem naturally splits into two parts: computation of the inertial–geometric terms Section 2.1, and computation of the body forces from gas, radiation and electromagnetic stresses in Section 2.2. We avoid going into the rotating frame from the onset in order to preserve a transparent connection to a 'universal' standard of rest frame (to be associated with Boyer–Lindquist coordinates).

## 2.1 Inertial terms

Inertial accelerations are geometrically imprinted in the connection terms for the covariant derivatives of the Eulerian velocity components. For spherical *coordinate motion* $(\dot{r}, \dot{\varphi}, \dot{\theta}) \longrightarrow (V^r, V^\varphi, V^\theta)$, the only non-trivial connections are $\Gamma^r_{\varphi\varphi} \wedge \Gamma^\varphi_{r\varphi}$. Denoting the Lagrangian time derivative by $\mathrm{d}_t \equiv \partial_t + \boldsymbol{V}\cdot\nabla$, the three components of Euler's equation read

$$\begin{aligned}
&\mathrm{d}_t V^r = \partial_t V^r + V^j V^r_{,j} + (-r)V^\varphi V^\varphi = g^{rj}\boldsymbol{f}_j\\
&\mathrm{d}_t V^\varphi = \partial_t V^\varphi + V^j V^\varphi_{,j} + (2/r)V^r V^\varphi = g^{\varphi j}\boldsymbol{f}_j\\
&\mathrm{d}_t V^\theta = \partial_t V^\theta + V^j V^\theta_{,j} = g^{\theta j}\boldsymbol{f}_j\\
&\text{where}\quad \boldsymbol{f} \equiv -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}\boldsymbol{J}\times\boldsymbol{B},
\end{aligned} \tag{1}$$

$g^{ij}$ is the flat-space metric for spherical coordinates and $\nabla$ denotes the covariant derivative hereon.

Assuming an equilibrium from purely azimuthal (but differential) bulk motion $\boldsymbol{V} = \Omega\mathbf{1}_\varphi$, an Eulerian perturbation of such a state $(V^r, V^\varphi, V^z) \longrightarrow (v^r, \Omega + v^\varphi, v^z)$, leads to the usual equations associated with a rotating frame and its coriolis and centrifugal terms. For coordinate motion, the Euler equations for the perturbations of the fluid read

$$\begin{aligned}
&(\partial_t + \Omega\partial_\varphi)v^r - 2r\Omega v^\varphi = g^{rj}\delta\boldsymbol{f}_j\\
&(\partial_t + \Omega\partial_\varphi)v^\varphi + \left(2\frac{\Omega}{r} + \Omega_{,r}\right)v^r = g^{\varphi j}\delta\boldsymbol{f}_j\\
&(\partial_t + \Omega\partial_\varphi)v^\theta = g^{\theta j}\delta\boldsymbol{f}_j
\end{aligned} \tag{2}$$

where $\delta\boldsymbol{f}$ denotes the Eulerian perturbation of the sum of specific body forces. The standard form of these equations, e.g. for *non-coordinate motion* (see Chandrasekhar 1961), may be obtained from equations (1) above by 'dimensionalizing' $V^\varphi$ (i.e. in the second equation, multiplying by $r$ and completing the differential while recalling that the covariant derivative and the metric commute $[\nabla, g_{ij}] = \emptyset$).

Next, one switches dynamical variables from the Euler velocity perturbation, $\boldsymbol{v}$, to the Lagrangian displacement, $\boldsymbol{\xi}$, using the first-order relation between Lagrangian and Eulerian variations, $\tilde{\Delta} = \delta + \boldsymbol{\xi}\cdot\nabla$, whilst denoting[2] $\tilde{\Delta}\boldsymbol{V} \equiv \mathrm{d}_t\boldsymbol{\xi}$ and $\delta\boldsymbol{V} \equiv \boldsymbol{v}$ (see, e.g., Chandrasekhar & Lebovitz 1964; Lynden-Bell & Ostriker 1967)

$$\boldsymbol{v} = \{\partial_t + \boldsymbol{V}\cdot\nabla\} \sim -(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{V} \to \mathrm{i}\sigma \sim -\xi^r\Omega_{,r}\mathbf{1}_\varphi. \tag{3}$$

The algebraic relation follows from the assumption of differential rotation and from writing $\exp \mathrm{i}(\omega t + m\varphi + k_z z)$ dependences for $\boldsymbol{\xi}$. Note that the connection coefficients in equation (3) cancel one another and that $\sigma \doteq \omega + m\Omega$ denotes the comoving frequency of the perturbations.

These geometrical equations have their more traditional equivalents in the so-called shearing sheet approximation where a comoving, 'locally Cartesian frame' $(\hat{r}, \hat{\varphi}, \hat{\theta}) \to (x, y, z)$, is used along with the linearized shear velocity field, $\boldsymbol{V}(x) = [\mathrm{d}_{\ln r}\Omega]\, x\mathbf{1}_y$, to treat the problem

[2] The tilde indicates that this form of Lagrangian displacement – which is generally non-unique – has had its gauge 'fixed' in accordance to the non-relativistic regime. Mathematically, this amounts to a choice of Universal time direction, $\mathbf{1}_t$ (e.g. unaffected by the motion of the fluid), while adopting the gauge-fixing condition $\xi\cdot\mathbf{1}_t \doteq \emptyset$.

locally while introducing the coriolis terms by hand. Defining the Cartesian derivative operator $\boldsymbol{\partial}$, then the equivalent to equation (3) is $(\partial_t + \boldsymbol{V} \cdot \boldsymbol{\partial}) \sim = \boldsymbol{v} + \boldsymbol{\xi} \cdot \boldsymbol{\partial V}$, which has Galilean invariance in the sense that Lagrangian time derivatives produce comoving frequency factors in the dispersion relation: $(\partial_t + \boldsymbol{V} \cdot \boldsymbol{\partial})\boldsymbol{\xi} \equiv \mathrm{d}_t \boldsymbol{\xi} \rightarrow \mathrm{i}\sigma\boldsymbol{\xi}$.

The equations of motion (EoM) for the (coordinate) Lagrangian displacement are

$$\begin{aligned} &\left(-\sigma^2 + 2\Omega r \Omega_{,r}\right)\xi^r - 2r\Omega \mathrm{i}\sigma\xi^\varphi = \delta \boldsymbol{f}^r \\ &-\sigma^2\xi^\varphi + 2\frac{\Omega}{r}\mathrm{i}\sigma\xi^r = \delta \boldsymbol{f}^\varphi \\ &-\sigma^2\xi^\theta = \delta \boldsymbol{f}^\theta \end{aligned} \tag{4}$$

Note that the Eulerian shear term, $\propto \mathbf{1}_\varphi$, becomes the tide term, $\propto \mathbf{1}_r$, in terms of $\boldsymbol{\xi}$.

Let us re-cast these equations in a more compact form

$$\ddot{\xi}^i + 2\Gamma^i_{jk} V^j \dot{\xi}^k - 2\Gamma^i_{jk} V^j (v - \dot{\xi})^k = \delta \boldsymbol{f}^i, \tag{5}$$

where each overdot denotes a factor of $\mathrm{i}\sigma$ (from a Lagrangian time derivative). In the shearing sheet approximation, these equations correspond to the Hill equations for *non-coordinate* motion Chandrasekhar 1961; Balbus & Hawley 1992) $\ddot{\boldsymbol{\xi}} + 2\Omega \times \dot{\boldsymbol{\xi}} + 2r\Omega\Omega_{,r}\xi_x \mathbf{1}_x = \delta \boldsymbol{f}$.

### 2.2 Compressibility

The Lagrangian perturbation of mass density and the Eulerian perturbation of the field follow from mass and magnetic flux conservations (recall the non-relativistic relation $\tilde{\Delta} = \delta + \boldsymbol{\xi} \cdot \nabla$)

$$\frac{\Delta\rho}{\rho} = -\nabla \cdot \boldsymbol{\xi}, \quad \delta \boldsymbol{B} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}). \tag{6}$$

The latter equation includes possible gradients of the background field $\boldsymbol{\partial B} \neq \emptyset$; however, in the spirit of examining the instability as a local phenomenon the global structure of the field is ignored herein. The Lorentz force variation (comoving frame) may then be written as

$$\delta\left(\frac{1}{4\pi\rho}\boldsymbol{J} \times \boldsymbol{B}\right) = v^2_{\mathrm{Alf}} \times \left[\nabla(\nabla \cdot \boldsymbol{\xi}) + \nabla^2_{\hat{\boldsymbol{B}}}\boldsymbol{\xi} - \nabla_{\hat{\boldsymbol{B}}}\nabla(\mathbf{1}_{\hat{\boldsymbol{B}}} \cdot \boldsymbol{\xi}) - \mathbf{1}_{\hat{\boldsymbol{B}}}\nabla_{\hat{\boldsymbol{B}}}(\nabla \cdot \boldsymbol{\xi})\right] \stackrel{\nabla \rightarrow \mathrm{i}\boldsymbol{k}}{\longrightarrow} -v^2_{\mathrm{Alf}} \times \left[(k_i\xi_i - k_{\hat{\boldsymbol{B}}}\xi_{\hat{\boldsymbol{B}}})\boldsymbol{k} + k^2_{\hat{\boldsymbol{B}}} \sim -\mathbf{1}_{\hat{\boldsymbol{B}}}k_{\hat{\boldsymbol{B}}}k_i\xi_i\right] \tag{7}$$

where the scalar operator $\nabla_{\hat{\boldsymbol{B}}} \equiv \mathbf{1}_{\hat{\boldsymbol{B}}} \cdot \nabla$, and $\mathbf{1}_{\hat{\boldsymbol{B}}}$ is a unit vector in the direction of the unperturbed field. Note that the term $(k_i\xi_i - k_{\hat{\boldsymbol{B}}}\xi_{\hat{\boldsymbol{B}}}) \doteq k_\perp\xi_\perp$ may be interpreted as a restoring force arising from the *compression* of field lines (distinct from line bending, Foglizzo & Tagger 1995).

The Lagrangian variation of the specific pressure gradient contains two terms (Lynden-Bell & Ostriker 1967): one $\propto \Delta\rho^{-1}$ and another $\propto \Delta\nabla p_{\mathrm{r+g}}$. In terms of the displacement vector, the first term is proportional to the equilibrium value of $\nabla p_{\mathrm{r+g}}$ which is negligible[3] in the local treatment (proportional to a radial gradient). For the same reason, the Eulerian and Lagrangian variations of the pressure 'force' are identical.

The thermodynamic pressure term is then given by

$$-\delta\left(\frac{1}{\rho}\nabla p_{\mathrm{r+g}}\right) = \Gamma\frac{p_{\mathrm{r+g}}}{\rho}\nabla(\nabla \cdot \boldsymbol{\xi}) \stackrel{\nabla \rightarrow \mathrm{i}\boldsymbol{k}}{\longrightarrow} c^2_{\mathrm{s}}\boldsymbol{k}\ k_i\xi_i, \tag{8}$$

where, for heterogeneous media, $\Gamma \equiv \mathrm{d}_{[\ln\rho]} \ln p$ represents a generalized adiabatic index (see, e.g., Chandrasekhar 1939; Mihalas & Mihalas 1984).

Putting the above equations together, one obtains the EoM in Fourier-space

$$\ddot{\xi}^i + 2\Gamma^i_{jk} V^j \dot{\xi}^k - 2\Gamma^i_{jk} V^j (v - \dot{\xi})^k = g^{ij}\left\{\left[c^2_{\mathrm{s}}\boldsymbol{k} + v^2_{\mathrm{Alf}}(\boldsymbol{k} - k_{\hat{\boldsymbol{B}}}\mathbf{1}_{\hat{\boldsymbol{B}}})\right](\boldsymbol{k} \cdot \boldsymbol{\xi}) + v^2_{\mathrm{Alf}}\left(k^2_{\hat{\boldsymbol{B}}}\boldsymbol{\xi} - k_{\hat{\boldsymbol{B}}}\xi_{\hat{\boldsymbol{B}}}\boldsymbol{k}\right)\right\}_j, \tag{9}$$

which agrees with the matrix decomposition of Foglizzo & Tagger (1995) in the case of a purely toroidal field embedded in a gas with adiabatic index $\Gamma = 1$.

Reckoning of fluid compressibility has complicated somewhat the equations of motion. Yet, these generally unwieldy equations simplify greatly in the regime of fastest growth (also known as the horizontal regime) and for two ideal field topologies of interest. When the field is meridional, the fastest growth modes have $\xi_{\hat{\boldsymbol{B}}} \doteq \emptyset$, and $\boldsymbol{k} \simeq k_{\hat{\boldsymbol{B}}}\mathbf{1}_{\hat{\boldsymbol{B}}}$, thus yielding a simple isotropic elastic response $\propto -v^2_{\mathrm{Alf}}(\mathrm{i}k_{\hat{\boldsymbol{B}}})^2\boldsymbol{\xi}$.

Alternatively, when the field is purely toroidal, the meridional component of equation (9) yields an anisotropy constraint: $v^2_{\mathrm{Alf}}(k_\perp\xi_\perp) = -c^2_{\mathrm{s}}(\boldsymbol{k} \cdot \boldsymbol{\xi})$ (Foglizzo & Tagger 1995), which allows for a straightforward solution in this regime. Defining $\Lambda$ through

$$1 - \Lambda = -\frac{\nabla \cdot \boldsymbol{\xi}}{\mathbf{i}k\|\xi\|} = \frac{2\Theta}{\Gamma + 2\Theta}, \tag{10}$$

where $\Theta \equiv p_{\boldsymbol{B}\varphi}/p_{\mathrm{r+g}}$, the dispersion relation out of equation (9) reads

$$\hat{\sigma}^4 - \left[(\Lambda + 1)\hat{q}^2_{\hat{\boldsymbol{B}}} + \hat{\chi}^2\right]\hat{\sigma}^2 + \Lambda\hat{q}^2_{\hat{\boldsymbol{B}}}\left(\hat{q}^2_{\hat{\boldsymbol{B}}} + 4\hat{A}\right) = \emptyset, \tag{11}$$

where all frequencies are normalized to the rotation rate, $\hat{A} \equiv \frac{1}{2}\mathrm{d}_{\ln r} \ln \Omega$ is the Oort $A$ 'constant', $\hat{\chi}^2 \equiv 4(1 + \hat{A})$ is the squared of the epicyclic frequency, and $\hat{q}_{\hat{\boldsymbol{B}}} \equiv (\boldsymbol{k} \cdot \boldsymbol{v}_{\mathrm{Alf}})/\Omega$ is a frequency related to the component of the wave vector along the field (in velocity units).

[3] The variation of the mass density is also negligible when the focus is on the effects of radiative heat conduction: loss of pressure support out of compressive modes involves only the pressure term (Araya-Góchez & Vishniac, in preparation).

The non-axisymmetric modes of fastest growth conform with (Araya-Góchez & Vishniac, in preparation)

$$\hat{q}_{\hat{B}}^2 = -2\hat{A} + \left(\frac{1+\Lambda}{2\Lambda}\right) \times \left\{-\frac{2\Lambda\hat{A}^2}{\mathcal{D}}\right\}, \quad \text{where} \quad \mathcal{D} \equiv 1 + \left(\frac{1-\Lambda}{2}\right)\hat{A} + \sqrt{1+(1-\Lambda)\hat{A}}, \tag{12}$$

where the expression in curly brackets corresponds to the negative root of the dispersion relation.

Note that the compressibility of non-axisymmetric modes is imprinted on the deviations of $\Lambda$ from unity. From equation (10) one reads that the degree of compression of these modes becomes stronger with the (toroidal) field strength and, naturally, with a softer equation of state. In the companion paper (Araya-Góchez & Vishniac, in preparation), we find that when the radiation pressure begins to dominate the disc dynamics, an 'ultra-soft' effective index accentuates the effects of mode compressibility. On the other hand, setting $\Lambda \doteq 1$ and re-orienting the field vertically produces the standard (incompressible) dispersion form for the Balbus–Hawley instability of a meridional field in the horizontal regime.

## 3 GENERAL RELATIVISTIC EFFECTS IN THE COWLING LIMIT

In contrast with the Newtonian case, the formulation of a covariant theory of accretion disc oscillations requires more than mere application of the Lagrangian rate of change operator $\mathrm{d}_t$ (or its relativistic counterpart $\mathrm{d}_\tau$) to the Eulerian velocity perturbation. This is insufficient to carry out a normal-mode analysis because of the freedom associated with the choice of coordinates. It is much more useful and proper to free the eigenmodes from the coordinate representation, treating them rather as being intrinsic to the physical system. It is here that a Lagrangian construction comes in handy.

A working covariant definition of the Lagrangian displacement is that of a vector field that moves the world line of a fluid element from its unperturbed position in space–time to its perturbed one. The fundamental relation between the Lagrangian (following the world line of the fluid) and Eulerian (taken at a fixed coordinate point) variational operators is

$$\Delta = \delta + \mathcal{L}_\xi, \tag{13}$$

where $\mathcal{L}$ denotes the Lie derivative.

An elemental use of this relation involves particle number conservation (Schutz & Sorkin 1977): with the use of a number flux density $\mathcal{N}^\nu \equiv n\sqrt{-g}U^\nu$, such a law reads $\Delta\mathcal{N} \doteq \emptyset$, where $g = \det|g^{\mu\nu}|$ and $U^\nu$ is the four-velocity of the fluid. In the Cowling approximation, $\delta g \doteq \emptyset$, and in the absence of comoving sources (or sinks) of particles, one obtains for the variations of the four-velocity of an ideal fluid:

$$\Delta U \doteq 0 = \delta U + \mathcal{L}_\xi U, \tag{14}$$

which demonstrates that Eulerian perturbations of the four-velocity, $\delta U \equiv u$, obey $u^\nu = -\mathcal{L}_\xi U^\nu$.

The connection to the Newtonian limit is recuperated upon identifying $cU \cdot \nabla$ with the convective (or material) rate of change $cU \cdot \nabla \xrightarrow{c\to\infty} (\partial_t + \boldsymbol{V}\cdot\nabla)$ so that, with $\tilde{\Delta}\boldsymbol{V} \equiv \mathrm{d}_t\boldsymbol{\xi}$, one has

$$\tilde{\Delta} = \delta + \boldsymbol{\xi}\cdot\nabla \tag{15}$$

[see, e.g., Chandrasekhar & Lebovitz 1964; Lynden-Bell & Ostriker 1967, and compare $\delta\boldsymbol{V} \equiv \boldsymbol{v}$ with equation (3)].

The fluid particles that constitute a thin accretion disc (with negligible radial inflow) embedded in a Kerr space–time geometry have unperturbed four-velocity $U^\nu = \gamma(\mathbf{1}_t + \Omega\mathbf{1}_\varphi)$ where $\mathbf{1}_t = (1, 0, 0, 0)$ and $\mathbf{1}_\varphi = (0, 0, 1, 0)$ are the Killing vector fields of the stationary, axisymmetric geometry and where $\gamma$ is the 'redshift' factor of the fluid elements at fixed radius $\Gamma = U^t = \mathrm{d}_\tau t$. In terms of the Lagrangian displacement vector field, each Lie derivative with respect to one of the Killing vector fields of the geometry 'brings down' a wavenumber co-factor in the dispersion relation (modulo spatial gradients of the four-velocity). This leads to an algebraic relation between $\xi$ and $u \equiv \delta U$ in the case of a differentially rotating fluid:

$$\begin{aligned} u &= \mathcal{L}_{\gamma(\mathbf{1}_t+\Omega\mathbf{1}_\varphi)}\xi \\ &= \dot{\xi} - \gamma\xi^r\Omega_{,r}\mathbf{1}_\varphi - U\xi\cdot\nabla\ln\gamma. \end{aligned} \tag{16}$$

Here $\dot{\xi} \equiv \mathrm{i}\sigma\gamma\xi^\nu$, with $\sigma = \omega + m\Omega$ the comoving frequency of the perturbation as measured at asymptotic infinity (see Ipser & Lindblom 1992).

The relativistic generalization of equation (3) is found upon projecting $\xi$ on the three-surface perpendicular to the (unperturbed) four-velocity. With $h^{\alpha\beta} \equiv U^\alpha U^\beta + g^{\alpha\beta}$ the projection operator, one has

$$h^\alpha{}_\beta u^\beta \equiv \hat{u}^\alpha = \mathrm{i}\sigma\gamma\hat{\xi}^\alpha - \gamma\xi^r\Omega_{,r}\hat{\mathbf{1}}^\alpha_\varphi, \tag{17}$$

while fixing the gauge freedom associated with the component of the Lagrangian displacement perpendicular to space-like hypersurfaces (Schutz & Sorkin 1977), i.e. along the local 'time' direction. Note that the requirement of unit normal for the *perturbed* velocity, $U + u$, under the Cowling approximation fixes the gauge accordingly: $2U^\alpha u_\alpha = -U^\alpha U^\beta\delta g_{\alpha\beta} \doteq 0$.

Dynamical conservation laws for an ideal fluid in the presence of a large-scale electromagnetic field are written succinctly through the Einstein–Maxwell equation (Cowling approximation)

$$T^{\mu\nu}{}_{;\nu} - F^{\mu\nu}J_\nu = 0,$$

where the first term denotes the matter stress and the second term equals the Maxwell stress. The notation is standard fare: $F$ is the electromagnetic field tensor and $J = neU$ is the four-current. Ideal MHD makes things easy by stating that the electric field in the comoving frame vanishes everywhere. Since the latter is the contraction of the field tensor with the four-velocity, it follows that $F_{\mu\nu} J^\nu = 0$, and the four-acceleration from the Maxwell stress vanishes as well (but not its perturbation).

We shall concern ourselves with the material stress first. Denoting the relativistic enthalpy of the fluid by $\varrho \equiv \rho + \varepsilon + p$, it is straightforward to show that for a non-dissipative, ideal fluid such that $T^{\mu\nu} = \varrho U^\mu U^\nu + p g^{\mu\nu}$,

$$T^{\mu\nu}{}_{;\nu} = \varrho \mathrm{d}_\tau U^\mu + g^{\mu\nu} p_{,\nu} \quad \text{where} \quad \mathrm{d}_\tau \equiv U \cdot \nabla \tag{18}$$

denotes the generalization of the convective rate of change, i.e. the Lagrangian proper-time derivative. The projection of this equation along the four-velocity states energy conservation while the perpendicular components express conservation of momentum.

The specific Eulerian perturbation of the four-acceleration (normalized to the enthalpy) looks like $u \cdot \nabla U + U \cdot \nabla u$, and one can use equation (17) to switch the dynamical variable in favour of the projected $\xi$:[4]

$$\hat{u} \cdot \nabla U + h(U \cdot \nabla \hat{u}^\alpha) = h(U \cdot \nabla \dot{\xi} - \dot{\xi} \cdot \nabla U) + 2\dot{\xi} \cdot \nabla U - \gamma \xi^r \Omega_{,r} \hat{\mathbf{1}}_\varphi \cdot \nabla U - hU \cdot \nabla \left( \gamma \xi^r \Omega_{,r} \hat{\mathbf{1}}_\varphi \right) \tag{19}$$

where the hats on the $\xi$ (signifying projected components) have been dropped.

The first term on the right-hand side of equation (19) may be readily identified with (the projection of) the Lie derivative of $\dot{\xi}$ along $U$. Defining $q \equiv \mathbf{1}_t + \Omega \mathbf{1}_\varphi$ (so that $U = \gamma q$), one computes

$$\begin{aligned} \mathcal{L}_U \dot{\xi} &= \gamma \mathcal{L}_q \dot{\xi} - q \dot{\xi} \cdot \nabla \gamma \\ &= \ddot{\xi} - \gamma \dot{\xi}^r \Omega_{,r} \hat{\mathbf{1}}_\varphi + \dot{\xi} U \cdot \nabla \ln \gamma - U \dot{\xi} \cdot \nabla \ln \gamma, \end{aligned} \tag{20}$$

where $\ddot{\xi} \equiv (\mathrm{i}\sigma\gamma)^2 \xi$. Note that the final term disappears upon (re)projection on to proper space-like hypersurfaces.

The second term on the right-hand side of equation (19) is easily evaluated, $\dot{\xi} \cdot \nabla U^\alpha = \gamma \dot{\xi}^r \Omega_{,r} \hat{\mathbf{1}}^\alpha_\varphi + \Gamma^\alpha_{\mu\nu} U^\mu \dot{\xi}^\nu$, and the third simply involves a projected affine connection. Evaluation of the (non-projected) last term yields four parts:

$$U \cdot \nabla \left( \gamma \xi^r \Omega_{,r} \hat{\mathbf{1}}_\varphi \right) = \xi^r \Omega_{,r} \hat{\mathbf{1}}_\varphi U \cdot \nabla \gamma + \gamma \Omega_{,r} \hat{\mathbf{1}}_\varphi U \cdot \nabla \xi^r + \gamma \xi^r \hat{\mathbf{1}}_\varphi U \cdot \nabla \Omega_{,r} + \gamma \xi^r \Omega_{,r} U \cdot \nabla \hat{\mathbf{1}}_\varphi.$$

Under the premise of negligible radial motion, in the second part above $U \cdot \nabla \xi^r \simeq \dot{\xi}^r + \mathcal{O}(U^r)$, while the third is $\mathcal{O}(U^r)$. Likewise, pairing of all terms proportional to the logarithmic gradient of the redshift factor yields the same order of (negligible) corrections $(\dot{\xi} - \gamma \xi^r \Omega_{,r} \hat{\mathbf{1}}_\varphi) U \cdot \nabla \ln \gamma \simeq \mathcal{O}(U^r)$. Moreover, the final term above involves the same connection coefficient as the third term on the right-hand side of equation (19).

When all this is said and done, one gets for the specific Eulerian perturbation of the four-acceleration:

$$\hat{u} \cdot \nabla U + h(U \cdot \nabla \hat{u}) \xrightarrow{\mathcal{O}(U^1)} \ddot{\xi}^\nu + 2\Gamma^\nu_{\alpha\beta} U^\alpha \dot{\xi}^\beta - 2\Gamma^\nu_{\alpha\beta} U^\alpha (\hat{u} - \dot{\xi})^\beta. \tag{21}$$

The resemblance with equation (5) is remarkable but not accidental.

The shear (tidal) term is embodied by the third term on the right-hand side: $\hat{u} - \dot{\xi} = -\gamma \xi^r \Omega_{,r} \hat{\mathbf{1}}_\varphi$. Note the non-trivially hatted unit vector $\hat{\mathbf{1}}^\nu_\varphi = h^\nu_\mu \mathbf{1}^\mu_\varphi = \mathbf{1}^\nu_\varphi + U^\nu U_\varphi$. We evaluate this term first using the standard form of the Kerr metric in the equatorial plane (Boyer–Lindquist coordinates):

$$\mathrm{d}s^2 = -\frac{\mathcal{D}}{\mathcal{A}} \mathrm{d}t^2 + r^2 \mathcal{A} (\mathrm{d}\varphi - \omega \, \mathrm{d}t)^2 + \frac{1}{\mathcal{D}} \mathrm{d}r^2, \tag{22}$$

with $\omega \equiv 2a/\mathcal{A}r^3$ the rate of frame dragging by the hole and where the metric functions of the radial Boyer–Lindquist force (BLF) coordinate are written as relativistic corrections (e.g. Novikov & Thorne 1973):

$$\mathcal{A} \equiv 1 + a^2/r^2 + 2a^2/r^3 \quad \text{and} \quad \mathcal{D} \equiv 1 - 2/r + a^2/r^2,$$

in normalized geometrical units ($c = G = M_{\mathrm{bh}} = 1$).

In expanded form, the projection of the Killing vector associated with the azimuthal symmetry is

$$\hat{\mathbf{1}}_\varphi = [1 + \tilde{\gamma}^2 \tilde{r} v^{\tilde{\varphi}} \Omega] \mathbf{1}_\varphi + \tilde{\gamma}^2 \tilde{r} v^{\tilde{\varphi}} \mathbf{1}_t,$$

where $\tilde{\gamma} = \gamma \sqrt{\mathcal{D}/\mathcal{A}}$ is the redshift factor relative to 'locally non-rotating observers' (Bardeen, Press & Teukolsky 1972) and $\tilde{r} \equiv r\mathcal{A}/\sqrt{\mathcal{D}}$ is the radius of gyration for the physical velocity in that frame, $v^{\tilde{\varphi}} = \tilde{r}(\Omega - \omega)$.

Evaluation of the tidal term is a bit lengthy but straightforward,

$$-2\Gamma^r_{\alpha\beta} U^\alpha (\hat{u} - \dot{\xi})^\beta = -\frac{4}{r^3} \left\{ \frac{1}{2} \gamma^2 \mathcal{D} \, \mathrm{d}_{\ln r} \Omega \right\} \left[ \left( \frac{\Omega}{\Omega_+ \Omega_-} - a \right) + \tilde{\gamma}^2 \tilde{r} v^{\tilde{\varphi}} \left( 1 - \frac{\Omega}{\Omega_+} \right) \left( 1 - \frac{\Omega}{\Omega_-} \right) \right] \xi^r \tag{23}$$

where $\Omega_\pm = \pm (r^{3/2} \pm a)^{-1}$ refer to prograde and retrograde circular orbits and where the expression in the curly brackets equals (minus) the shear of the congruence of circular, equatorial geodesics (Novikov & Thorne 1973).

[4] Note that with this form of the stress–energy tensor, $h^\alpha{}_\beta \mathrm{d}_\tau U^\beta \doteq \mathrm{d}_\tau U^\alpha$, i.e. the four-acceleration automatically lies in proper space-like hypersurfaces.

Next, to evaluate the coriolis terms one finds $\xi^t$ from the gauge-fixing condition $\xi \cdot U \doteq 0$. Accordingly, one finds

$$2\Gamma^r_{\alpha\beta}U^\alpha\dot{\xi}^\beta = 2\gamma\frac{\mathcal{D}}{r^2}\left[\frac{\Omega}{\Omega_+\Omega_-} - a + (1-a\Omega)\frac{r^2\mathcal{A}(\Omega-\omega)}{1-(2/r)(1-a\Omega)}\right]\dot{\xi}^\varphi,$$

$$2\Gamma^\varphi_{\alpha\beta}U^\alpha\dot{\xi}^\beta = -2\gamma\frac{1}{r^4\mathcal{D}}\left(\frac{\Omega}{\Omega_+\Omega_-} - a + 2r^2\Omega\right)\dot{\xi}^r \tag{24}$$

and

$$2\Gamma^t_{\alpha\beta}U^\alpha\dot{\xi}^\beta = 2\gamma\frac{1}{r^4\mathcal{D}}[r^2 + a^2 - a\Omega(3r^2+a^2)]\dot{\xi}^r.$$

Equation (21) for the (Eulerian) perturbation of the four-acceleration, $a^\mu = \hat{u}\cdot\nabla U^\mu + h(U\cdot\nabla\hat{u}^\mu)$, was derived for the components of the Lagrangian displacement in a coordinate frame that is fixed with respect to distant stars, i.e. in the Boyer–Lindquist 'frame'. However, because the instability is local (at least for weak fields in thin discs) one needs to transform the components of equation (21) for manipulation in terms of the local tetrad carried by comoving observers. This simply involves (matrix) multiplication by the basis vectors of such a tetrad (e.g. Novikov & Thorne 1973). In our notation, the relevant basis vectors are $e^{\hat{r}}_\alpha = 1/\sqrt{\mathcal{D}}(0,1,0,0)$ and $e^{\hat{\varphi}}_\alpha = \gamma r\sqrt{\mathcal{D}}(-\Omega,0,0,1)$ (note that transformation to the local tetrad yields equations for non-coordinate motion, i.e. equivalent to motion in a local Cartesian basis).

Transformation of the $r$-component is trivial (since one needs to transform both the acceleration and the displacement vector in the basic EoM below, the radial scale, $1/\sqrt{\mathcal{D}}$, has no net effect):

$$\begin{aligned}\sqrt{\mathcal{D}}\,a^{\hat{r}} &= \ddot{\xi}^r + 2\gamma\frac{\mathcal{D}}{r^2}\left[\frac{\Omega}{\Omega_+\Omega_-} - a + (1-a\Omega)\frac{r^2\mathcal{A}(\Omega-\omega)}{1-(2/r)(1-a\Omega)}\right]\dot{\xi}^\varphi \\ &\quad -\frac{4}{r^3}\left\{\frac{1}{2}\gamma^2\mathcal{D}\mathrm{d}_{\ln r}\Omega\right\}\left[\left(\frac{\Omega}{\Omega_+\Omega_-} - a\right) + \tilde{\gamma}^2\tilde{r}v^{\tilde{\varphi}}\left(1-\frac{\Omega}{\Omega_+}\right)\left(1-\frac{\Omega}{\Omega_-}\right)\right]\xi^r.\end{aligned} \tag{25}$$

On the other hand, using the local azimuthal base vector, equations (24) and the gauge-fixing condition $\xi\cdot U = 0$, computation of the local $\varphi$-component, $\propto -\Omega a^t + a^\varphi$, is a bit more involved (again, the radial scale factors out of the EoM and does not affect the dispersion relation)

$$\frac{1}{\gamma r\sqrt{\mathcal{D}}}a^{\hat{\varphi}} = \left[\frac{1-(2/r)(1-2a\Omega) - r^2\mathcal{A}\Omega^2}{1-(2/r)(1-a\Omega)}\right]\ddot{\xi}^\varphi - 2\gamma\frac{1}{r^4\mathcal{D}}\left[\frac{\Omega}{\Omega_+\Omega_-} - a + (1-a\Omega)\Omega(3r^2+a^2)\right]\dot{\xi}^r. \tag{26}$$

Let us take a look at the Maxwell stress next.

The fundamental premise of ideal MHD may be stated rather succinctly: in the rest frame of the fluid currents will flow uninhibited to (instantaneously) cancel any hint of an electric field. A relativistic generalization of ideal MHD may be achieved by a similar covariant (albeit imperfect) postulate: $E_{\mathrm{rf}} = F\cdot U \doteq \emptyset$, e.g. the Faraday field tensor is 'purely magnetic' in the fluid frame.

Such postulate brings a few mathematical consequences (Phinney 1983):

(i) the second electromagnetic invariant vanishes everywhere,

$$\frac{1}{4}F_{\mu\nu}\mathcal{F}^{\mu\nu} \doteq \emptyset;$$

(ii) the Faraday field tensor is Lie transported along the worldlines of the fluid,

$$\mathcal{L}_U F \doteq \emptyset;$$

(iii) the other zero eigenvector of the field tensor is the (space-like) magnetic field $B \equiv \mathcal{F}\cdot U$,

$$F\cdot(\mathcal{F}\cdot U) \doteq \emptyset;$$

with $\mathcal{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ the dual to the field tensor, and

(iv) the field tensor is also invariant when transported along the magnetic field four-vector:

$$\mathcal{L}_{\mathcal{F}\cdot U}F \doteq \emptyset. \tag{27}$$

Note further that, since the four-velocity and the four-magnetic field are orthogonal, $U\cdot B = 0$, properties (ii) and (iv) above define a two-surface of invariance for the Faraday tensor

$$\mathcal{L}_{aU+bB}F \doteq \emptyset, \tag{28}$$

where $a$ and $b$ are arbitrary real numbers.

Aside from the intrinsic (physical) difference in their space–time orientation, the mathematical similarities between $U$ and $B$ are uncanny.

Let us go back to the Newtonian case for a moment. From our definition of the Lagrangian variation of the three-velocity: $\mathrm{d}_t\boldsymbol{\xi} \equiv \tilde{\Delta}\boldsymbol{V}$, one finds the equation governing the Lagrangian change of three-velocity: $\tilde{\Delta}\boldsymbol{V} = (\boldsymbol{V}\cdot\nabla)\boldsymbol{\xi}$ (e.g. equation 15). As noted in the footnote, the difference between $\tilde{\Delta}$ and $\Delta$ is related to the choice of gauge for $\xi^\alpha$. The induction equation of non-relativistic MHD yields a virtually identical relation for the magnetic field variation (in a frame where the fluid was originally at rest), which is spoiled by fluid compressibility: $\tilde{\Delta}\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{\xi} - \boldsymbol{B}\nabla\cdot\boldsymbol{\xi}$. Nevertheless, making use of the continuity equation and weighting the field by the inverse of the mass density of the fluid $\tilde{\boldsymbol{B}} \equiv \boldsymbol{B}/\rho$ cleans up its connection to the displacement vector field $\tilde{\Delta}\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}\cdot\nabla\boldsymbol{\xi}$. If only conservative forces, $U\cdot f \doteq 0$, act on the fluid, it can be shown that use of energy conservation in lieu of mass conservation simply swaps the rest mass density of the fluid by the relativistic enthalpy (a world scalar), above. Thus, we choose to work below with a specific measure of the magnetic four-vector weighted by the inverse of the relativistic enthalpy of the fluid $\tilde{B} \equiv \frac{1}{\varrho}(\mathcal{F} \cdot U)$. Such combination of observables (and its perturbation) occurs naturally in the problem at hand.

Applying the Lagrangian variational operator, cf. equation (15), on $\tilde{B}$ under the constraints from ideal MHD noted above (equation 28), yields (contrast this with equation 14)

$$\Delta \tilde{B} \doteq \emptyset = \delta \tilde{B} + \mathcal{L}_\xi \tilde{B}. \tag{29}$$

This equation states a manifestly covariant expression for the Eulerian perturbation of the (enthalpy-weighted) Faraday tensor under ideal MHD constraints: $\tilde{b} = \mathcal{L}_{\tilde{B}} \xi$. Imposing the constraint that the total magnetic field four-vector be orthogonal to the (unperturbed) four-velocity is equivalent to projecting its Eulerian perturbation into proper space-like hypersurfaces: $\tilde{b}^\mu \to \hat{b}^\mu = h^\mu_\nu \mathcal{L}_{\tilde{B}} \xi^\nu$. Again, we suppress the hats below while tacitly imposing the condition $\xi \cdot U \doteq \emptyset$ throughout.

The Eulerian perturbation of the specific measure of the Lorentz force, $\delta(\tilde{T}_{\rm em})^{\mu\nu}_{;\nu} = \delta F^{\mu\nu} \tilde{J}_\nu + F^{\mu\nu} \delta \tilde{J}_\nu$, may now be written in terms of the Lagrangian displacement but the general expressions are not particularly illuminating. Evaluated in the frame where the fluid was originally at rest, the Eulerian perturbations of the field tensor and of the four-current depend linearly on the components of $b$: $\delta F = F(b)$ and $4\pi \delta \tilde{J} = \mathrm{d} \cdot \delta \tilde{F}(b)$.

We proceed by assuming negligible gradients of the background-specific field ($\nabla \tilde{B} \doteq \emptyset$):

$$\tilde{b} = \tilde{B} \cdot \nabla \xi - \xi \cdot \nabla \tilde{B} \longrightarrow \tilde{B} \cdot (\mathrm{i}k)\xi, \tag{30}$$

and, consistent with this assumption, we also ignore the $\delta F \cdot \tilde{J}$ term in the perturbation of the Maxwell stress (i.e. gradients of the background field tensor ($\propto J$) gentler than those of the perturbations).

The simple 'linear poking' of the field tensor may now be written in a manifestly covariant manner

$$F^{\mu\nu} \delta \tilde{J}_\nu = \frac{1}{\varrho}(B \cdot \mathrm{i}k)^2 \xi^\mu. \tag{31}$$

Naturally, evaluation of the elastic response of the field is straightforward in the rest frame of the fluid where one has $F^{\mu\nu}\delta\tilde{J}_\nu \doteq -(v_{\rm Alf} k_{\hat{\boldsymbol{B}}})^2 \xi^\mu$. The only difference with the non-relativistic analogue is that the Alfvén speed is now weighted by the relativistic enthalpy of the fluid $\varrho v^2_{\rm Alf} \equiv \frac{1}{2} F^{\mu\nu} F_{\mu\nu}$.

We are now all geared up to put together the pieces of the puzzle. In terms of the Lagrangian displacement vector field, the right-hand sides of equations (25) and (26) are to be balanced by the elastic response of the field tensor to the poking by $\xi$, cf. equation (31) (note that the radial scales of the transformation into the rest frame of the fluid cancel one another). This balance is locally equivalent to $a^\mu = -q^2_{\hat{\boldsymbol{B}}} \xi^\mu$, i.e. the covariant components of the acceleration of the fluid respond to a force proportional to the displacement vector [with the unnormalized 'spring constant' $q_{\hat{\boldsymbol{B}}} = (v_{\rm Alf} k_{\hat{\boldsymbol{B}}})$ provided by the field]. By construction, both of these vectors are orthogonal to $U$ and collinear. Furthermore, since $\ddot{\xi}^\mu \equiv (\mathrm{i}\gamma\sigma)^2 \xi^\mu$ and $\gamma\sigma$ is a world scalar to be identified with the true comoving frequency (as measured by an observer riding along with the fluid), it follows that $\ddot{\xi}^{\hat{\varphi}} \equiv (\mathrm{i}\gamma\sigma)^2 \xi^{\hat{\varphi}}$. With these relations and the aforementioned equations for the tidal and coriolis terms, one arrives at lengthy component equations for $\xi^r$ and $\xi^\varphi$, for general $\Omega \equiv U^\varphi / U^t$ and negligible radial flow.

In the case of circular geodesic flow, the equations simplify nicely (horizontal regime)

$$\begin{aligned} &\ddot{\xi}^r - 2\gamma \frac{\mathcal{D}}{r^{1/2}} \Omega_\pm \left( \frac{r^3 - 3r^2 \pm 2ar^{3/2}}{r^{3/2} \pm a - 2r^{1/2}} \right) \dot{\xi}^\varphi - \frac{4}{r^{3/2}} \left\{ \frac{3}{4} \gamma^2 \mathcal{D} r^{3/2} \Omega^2_\pm \right\} \xi^r = -q^2_{\hat{\boldsymbol{B}}} \xi^r \\ &\ddot{\xi}^\varphi + 2\gamma \frac{1}{r^{5/2}\mathcal{D}} \Omega_\pm (r^{3/2} \pm a - 2r^{1/2}) \dot{\xi}^r = -q^2_{\hat{\boldsymbol{B}}} \xi^\varphi. \end{aligned} \tag{32}$$

These immediately yield the sought-after dispersion relation near a rotating hole

$$(\gamma\sigma)^4 - \left[ 4\gamma^2 \Omega^2_\pm \left( \mathcal{C}_\pm - \frac{3}{4}\mathcal{D} \right) + 2q^2_{\hat{\boldsymbol{B}}} \right] (\gamma\sigma)^2 + q^2_{\hat{\boldsymbol{B}}} \left[ q^2_{\hat{\boldsymbol{B}}} - 4 \left\{ \frac{3}{4} \gamma^2 \mathcal{D} \Omega^2_\pm \right\} \right] = \emptyset, \tag{33}$$

where $\mathcal{C}_\pm \equiv 1 - 3/r \pm 2a/r^{3/2}$ corresponds to the $\mathcal{C}$ function of Novikov & Thorne (1973) for prograde orbits.

Factoring out the extrinsic[5] dynamical frequency, $\Omega_\pm$, one arrives to the normalized dispersion relation (with $\gamma\sigma \equiv \Omega_\pm \hat{\sigma}$)

$$\hat{\sigma}^4 - \left( \hat{q}^2_{\hat{\boldsymbol{B}}} + \hat{\chi}^2_\pm \right) \hat{\sigma}^2 + \hat{q}^2_{\hat{\boldsymbol{B}}} \left( \hat{q}^2_{\hat{\boldsymbol{B}}} + 4\hat{A} \right) = \emptyset, \tag{34}$$

where

$$\hat{A} \equiv -\left\{ \frac{3}{4} \gamma^2 \mathcal{D} \right\} \quad \text{and} \quad \hat{\chi}^2_\pm = 4\gamma^2 \left( \mathcal{C}_\pm - \frac{3}{4}\mathcal{D} \right)$$

denote the normalized shear parameter and (comoving) epicycle frequency (note that $\frac{1}{\gamma}\hat{\chi}$ corresponds to the well-known result of epicycle frequency as measured at asymptotic infinity). One thus sees that with the proper generalizations of the epicycle frequency and shear parameter, the local dispersion relation is identical to the Newtonian case in the limit of no fluid compression and $\mathbf{1}_{\hat{\boldsymbol{B}}} \cdot \boldsymbol{\xi} \doteq \emptyset$ (i.e. the 'classical' Balbus–Hawley instability, equation 11).

[5] As defined, $\Omega \equiv U^\varphi / U^t$ reflects motion as observed in the Boyer–Lindquist frame, i.e. in a frame extrinsic to the fluid. It follows that the time-scale associated with $\Omega^{-1}$ does not reflect a proper dynamical time-scale.

Using the relation $\gamma^2 = (1 \pm a/r^{3/2})^2 \mathcal{C}_\pm^{-1}$ for cold, circular, *geodesic* flow (Novikov & Thorne 1973), one finds the fastest growing modes to conform with

$$\hat{q}_{\hat{B}}^2 = 1 - \frac{1}{16}\hat{\chi}^4 = 1 - \left(1 \pm \frac{a}{r^{3/2}}\right)^4 \left\{1 - \frac{3}{4}\frac{\mathcal{D}}{\mathcal{C}_\pm}\right\}^2 \tag{35}$$

which remains finite and close to the Newtonian value of $\frac{15}{16}$ for all radii *outside* the ISCO (and for any value of the rotation parameter).

To attach meaning to the polynomial functions that appear naturally in the dispersion relation for the magnetorotational instability, recall the range of radii that define particle dynamics in the Kerr geometry (Bardeen et al. 1972):

(i) The ISCO, $r_{\rm ms}$, corresponds to the root of $\hat{\chi}_\pm = 0$.
(ii) The radius of the circular photon orbit, $r_{\rm ph}$, is where $\mathcal{C}_\pm = 0$.
(iii) The event horizon, $r_+$, happens at the outer root of $\mathcal{D} = 0$.

One therefore has the following ordering of radii for any value of the rotation parameter $a$: $r_{\rm ms} > r_{\rm ph} > r_+$. As remarked by Bardeen et al. (1972), when $a = 1$, the proper radial distance between these radii is non-zero in spite of 'coinciding with the horizon', i.e. in spite of laying at the same Boyer–Lindquist radial coordinate.

Inspection of equation (35) now shows that $\hat{q}_{\hat{B}} \to 0^+$ as $r \to r_{\rm ph}^+$ so the most unstable MRI modes go to large scale just outside the photon orbit. Moreover, utilizing that expression for $\hat{q}_{\hat{B}}$ in the unstable root of the dispersion relation, one finds the growth rate (or frequency!) to be given by

$$-\hat{\sigma}^2 = \left\{\frac{3}{4}\frac{\mathcal{D}}{\mathcal{C}_\pm}\right\}^2 \left[\left(1 \pm \frac{a}{r^{3/2}}\right)^4 - \frac{8}{3}\frac{\mathcal{C}_\pm}{\mathcal{D}}\left(\pm 2\frac{a}{r^{3/2}} + 5\frac{a^2}{r^3} \pm 4\frac{a^3}{r^{9/2}} + \frac{a^4}{r^6}\right) + \left(\frac{4}{3}\frac{\mathcal{C}_\pm}{\mathcal{D}}\right)^2 \left(4\frac{a^2}{r^3} \pm 4\frac{a^3}{r^{9/2}} + \frac{a^4}{r^6}\right)\right] \tag{36}$$

For a non-rotating hole,

$$\hat{q}_{\hat{B}} \to 0 \quad \text{at}\, r = r_{\rm ph}\left(1 + \frac{1}{5}\right),$$

and the local growth rate

$$\hat{\sigma} = \frac{3}{4}\frac{\mathcal{D}}{\mathcal{C}_\pm} \to 2,$$

while for a rotating hole, the MRI quenching radii (for fastest growing modes) also occur just outside the circular photon orbit and may be readily extracted from the above relations. In Figs 1 and 2, we plot the general relativistic modifications to the fastest growing linear wavemodes, wavenumbers and growth rates, respectively, as functions of radius and for different values of the spin parameter $a$.

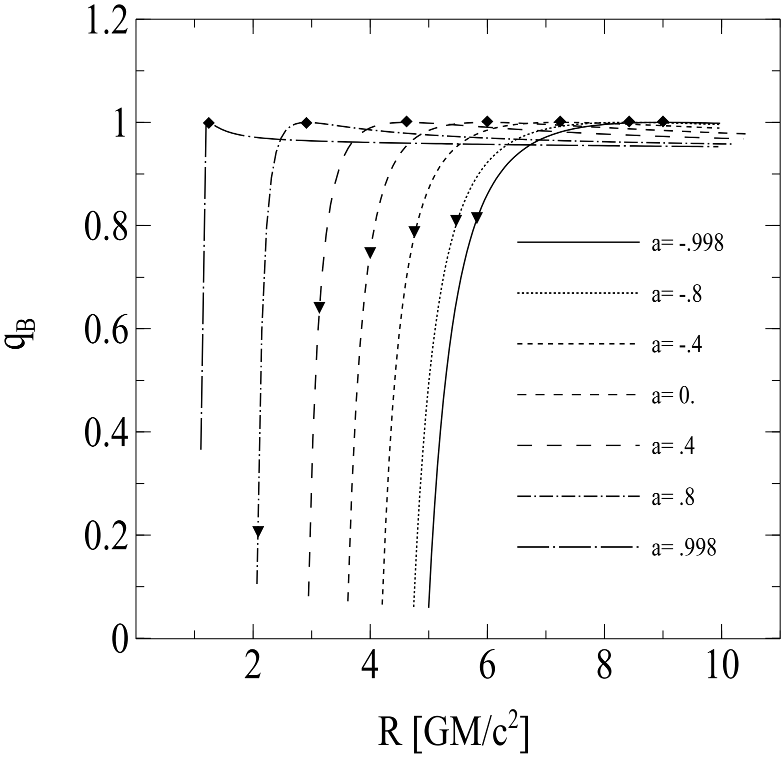


**Figure 1.** Normalized wavenumber, $\hat{q}_{\hat{B}}$, as a function of the radius (in gravitational radii) for several values of the spin parameter $a$. Diamonds indicate the location of the marginally stable orbit, $\hat{\chi} \doteq \emptyset$, and triangles, the location of the marginally bound orbit.

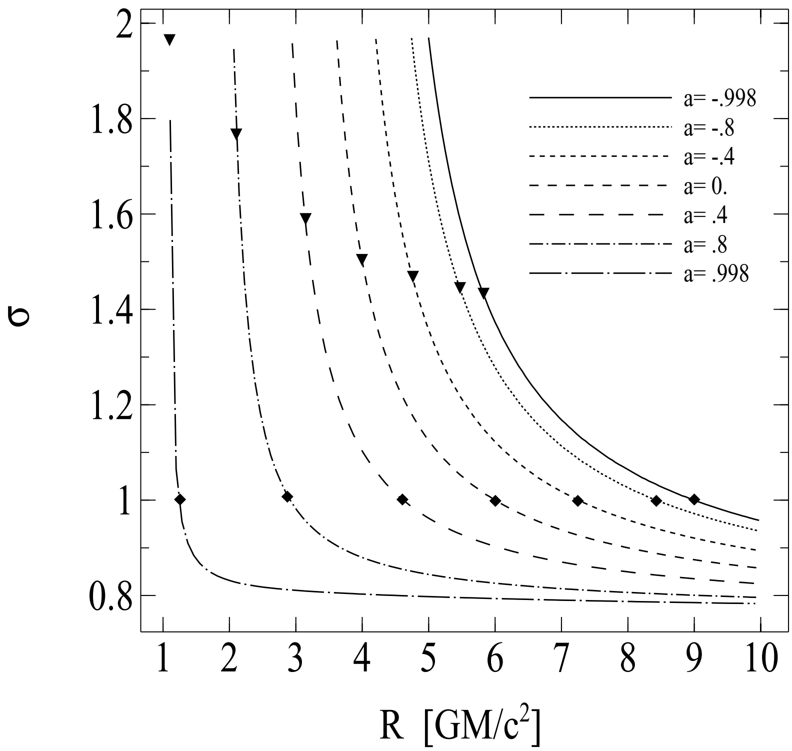


**Figure 2.** Normalized growth rate, $\hat{\sigma}$, as a function of radius for several values of the spin parameter $a$.

To go beyond this point, one would need to address global effects arising, for instance, from field curvature terms (see, e.g., Curry & Pudritz 1995; Ogilvie & Pringle 1996) and from the non-negligible radial velocity profile. Further investigation of the nature of the global instability is beyond the scope of this paper.

## 4 DISCUSSION

The MRI – in its simplest, local, incompressible variant – is found to operate virtually unabated down to the marginally stable orbit for massive particles. This radius is nearly coincident with the putative inner boundary of standard, thin accretion discs in the Kerr geometry. A vanishing epicycle frequency at $r_{\rm ms}$ means that the fastest growing wavenumbers tend to be of a bit smaller scale, $\hat{q}^2_{\hat{B}} : \frac{15}{16} \to 1 - \mathcal{O}(ar^{-3/2})$, while growing faster than classically, $\mathrm{i}\hat{\sigma} : \frac{3}{4} \to 1 + \mathcal{O}(ar^{-3/2})$. The effects of strong gravity become truly significant only in a regime where circular, cold, geodesic flow is unstable (i.e. where $\hat{\chi}^2_\pm < 0$).

Recall that particle trajectories with $U^\varphi / U^t = \Omega_\pm$ exist inside $r_{\rm ms}$ and all the way down to $r_{\rm ph}$ but, in the presence of turbulent velocity fluctuations, body forces such as a radial pressure gradient would be required to confine the flow to such circular orbits. Although very little is concretely known about the accretion flow inside $r_{\rm ms}$, two rather robust remarks may be ascertained: the flow inside $r_{\rm ms}$ cannot be supported centrifugally and it must therefore deviate from a standard thin disc. In addition, depending on the time-scale for infall, the flow may not have time to cool significantly and advection of entropy will become progressively more important as $r_+$ is approached. A *robust prediction of this paper is the expectation that free energy tapping from the differential shear flow goes on in the region immediately below* $r_{\rm ms}$.

One may envisage the situation inside the ISCO to evolve from a mildly advective accretion flow (MAAF) to a fully advection-dominated accretion flow (ADAF) as the photon orbit is approached. In fully or partly advective accretion flows, such as those modelled by Popham & Gammie (1998), the angular velocity profile 'peaks' precisely at $r_{\rm ph}$ and quickly drops therein to match the angular velocity at $r_+$. More importantly, when cooling by advection of entropy is moderately important – say, for advection fractions $f \simeq$ a few per cent – the angular velocity profile departs very slowly from circular geodesic flow, $U^\varphi / U^t \simeq \Omega_\pm$ down to a region below the marginally bound orbit. The transition from nearly Keplerian to plunging orbits can be clearly seen in one of the very few global slim disc models where the cooling fraction is calculated explicitly: the one-dimensional models of Popham, Woosley & Fryer (1999, albeit in the exotic scenario of a hyper-accreting black hole). In these models the radial velocity component is non-negligible when compared with the local speed of sound (the sonic point generally occurs below $r_{\rm ms}$, even near $r_{\rm mb}$ for low values of $\alpha$), but $v^{\hat{r}}$ is generally smaller than $v^{\hat{\varphi}}$ down to the region below $r_{\rm mb}$. [Note that the radial speed in the corotating frame (e.g. Gammie & Popham 1998) $V$, is related to the speed in the locally non-rotating frame by $v^{\hat{r}} = \gamma_\varphi^{-1} V$.]

The major limitation of the work presented herein is the presumption of negligible radial flow, which greatly simplifies matters from the outset (see equation 16). At this point, it is unclear how much the results will change when full consideration is made for the radial inflow. Since the changes could be qualitatively significant – recent reports negate the reversal of the centrifugal force when the radial speed overwhelms the azimuthal component (Mukhopadhyay & Prasanna 2001; Prasanna 2001) – this point should be a subject of close scrutiny in a future paper. Meanwhile, the adoption of an angular velocity profile corresponding to circular equatorial geodesic orbits seems a reasonable rough approximation in view of the above observations of advective flows. In this spirit, we argue below that the natural evolution of the MRI inside the marginally stable orbit is at least consistent with this assumption.

Assume, in quasi-linear fashion, that the time- and length-scales provided by the linear dispersion relation reflect the growth and size of the dominant turbulent eddies to within factors of the order of unity to a few. Provided that $v^{\hat{r}} \lesssim v^{\hat{\varphi}}$, simple linear growth–non-linear decay arguments (e.g. Araya-Góchez 1999a,b) can be used to predict a predominantly toroidal field topology: the MRI constantly promotes radial–azimuthal field growth from 'horizontal' velocity fluctuations, $\xi^{\hat{r}} \simeq -\xi^{\hat{\varphi}}$, while the coherent, background azimuthal shear flow converts this field into a toroidal field at twice the rate of radial field generation. In the rest frame of the fluid, the tapping of free energy associated with the shear flow becomes very rapid as the flow turns relativistic. Indeed, a comoving observer measures the shear parameter, $2A/\Omega_\pm$ to be $\frac{3}{2}\gamma^2\mathcal{D} \simeq \frac{3}{2}(1 \pm a/r^{3/2})^2\mathcal{D}/\mathcal{C}_\pm$, higher than the 'Keplerian' frequency associated with the global dynamical time-scale as seen at large distances (the redshift factor comes in because we chose to measure the angular frequency in terms of Boyer–Lindquist coordinates).

The ratio $\mathcal{D}/\mathcal{C}_\pm$ represents a gauge of the relative strength of two inertial terms, shear and coriolis. Setting aside the issue of radial flow for a moment, our dispersion relation suggests that as material approaches the region just outside of the photon orbit where $\mathcal{C}_\pm$ vanishes, the slow branch of the dispersion relation (i.e. the MRI) is stabilized by the predominance of shear over the coriolis terms. Recall that the location of the circular photon orbit is the place where the centrifugal force reverses its direction: inside $r_{\rm ph}$, increasing the velocity of a test particle pulls it in further (see, e.g., Abramowicz & Prasanna 1990 and references therein). The limit of $\hat{q}_{\hat{\boldsymbol{B}}} \to 0^+$ means that what was essentially a local instability becomes a global phenomenon. Although such a regime is formally beyond the scope of the local analysis, one can anticipate a few rather interesting qualitative consequences.

At first glance, the dispersion relation equation (34) shows the appearance of an interchange, radially buoyant mode (T. Foglizzo, private communication; Araya-Góchez 1999a). More likely, this would simply imply the need for a steep radial stratification profile. Indeed, if the coherence length-scale of the field were to reach the comoving length associated with the radial scale, $\mathcal{A}r/\sqrt{\mathcal{D}}$, the disc could make a transition from centrifugally driven to magnetically driven: *MRI-modulated dynamics guarantee that the Alfvén speed associated with the toroidal field at this large scale would be comparable to the orbital speed.* Moreover, the field generated at large scales is less susceptible to decay through reconnection and also more buoyant. This has very important consequences for the energy fraction going into – and persisting in – electromagnetic channels.

The radial velocity profile will very likely change the expected outcome once the radial velocity becomes supersonic or super-Alfvénic, but some of the qualitative features of the this analysis may carry over when the full problem is solved, analytically or otherwise. If so, in this part of the so-called 'plunging region' of the flow, the turbulent eddies will tend to grow larger while the field direction will tend to track the surfaces of null angular velocity gradients (no longer purely toroidal). The implied field topology is that of large-scale horizontal field domains.

### 4.1 Effects of radiation stress and neutrino trapping

A precise assessment of the dynamical role of radiation in the general relativistic regime is hampered by the breakdown of one key assumption made to simplify the 'linear poking' on the Faraday field tensor: use of the enthalpy-weighted specific four magnetic field in equation (31). On the other hand, one expects a photon gas – semicontained by a neutral plasma through Compton scattering – to comprise a rather funny MHD fluid where the magnetic field is truly frozen only to the comoving volume associated with the mass density but for which pressure perturbations do not behave adiabatically. It follows that when the fluid becomes radiation-pressure dominated, compressive modes (e.g. toroidal field, non-axisymmetric modes) may lose pressure support in an unfavourable range of wavenumber phase-space (Agol & Krolik 1998). One can prove that the MRI falls squarely into such radiative heat conduction damping regime (Blaes & Socrates 2001). Araya-Góchez & Vishniac (in preparation) show that the behaviour of the energy equation is in some (algebraic) sense 'quasi-adiabatic' for exponentially growing, non-propagating modes. Mathematically, this means that a real, analytical, slowly varying function of the scale of the perturbations, $\tilde{\Gamma}(\mathrm{i}\tilde{\boldsymbol{k}}^2/\tilde{k}^0)$, can be used to treat the energy equation in quasi-adiabatic fashion. Radiative heat conduction isotropizes the modes and, to zeroth order, one can use such a quasi-adiabatic index in equation (12) to anticipate that the effects of radiative heat conduction out of compressive toroidal modes is to increment the threshold of the shear parameter where, $\hat{q}_{\hat{\boldsymbol{B}}} \to 0^+$ from $-2$ to $-\hat{A} \to 1 + 2\mathcal{D}/(1+\Lambda)$. Nevertheless, since $\hat{A} \propto \mathcal{D}/\mathcal{C}_\pm$ and $\mathcal{C}_\pm \to 0$ at $r_{\rm ph}$, the increase in shear threshold in this setting is rather inconsequential.

Note further that the qualitative nature of energy deposition in radiation-pressure-dominated fluids is insensitive to the details of the (global) cooling but it is explicitly sensitive to the optical thickness of the relevant eddies. Thus, upon the onset of neutrino trapping in the neutrino cooling regime of hyper-accreting black holes, one may reasonably expect MRI-modulated dynamics at $p_\nu \gtrsim p_{\rm r+g}$ (gas and radiation are tightly coupled) to resemble the standard disc case when $p_{\rm rad} \gtrsim p_{\rm gas}$. Turner et al. (2001) report that the non-linear outcome of the MRI in this setting is a porous medium with drastic density contrasts so as to cheat the Eddington limit at high accretion rates. Under nearly constant total pressure and temperature, the non-linear regime shows that density enhancements anticorrelate with azimuthal field domains (just as expected from the linear theory) and that turbulent eddies live for approximately a dynamical time-scale while mass clumps are destroyed through collisions or by running through a localized region of shear.

Since the turbulent eddies in the disc are largely instabilities of the toroidal field (at moderate values of the field), large-scale horizontal field domains near the marginally bound orbit would naturally force the baryonic component of the accretion flow into spatially segregated, massive clumps that occur near the nodes of non-axisymmetric (toroidal) MRI eddies (Araya-Góchez & Vishniac, in preparation). This expectation motivates the picture of massive clumpy accretion suggested in the introduction.

## 5 SUMMARY

In summary, this work shows that the MRI is virtually unaffected by strong gravity outside the innermost stable circular orbit. Secondly, it indicates that the instability becomes non-local inside this region. Indeed, the MRI may leave behind a large-scale ordered field as the fluid heads in towards the circular photon orbit (with an orientation that tracks surfaces of null angular velocity gradients). Assuming incompressibility and the angular velocity profile of circular geodesic flow, the fastest growing modes die off while going to large scales at a radius just inside the marginally bound orbit. Accountability of compressibility as required to address the effects of radiation stress will bring the critical MRI quenching radius in, slightly closer to the photon orbit. Radiation stress, when significant, will diminish the growth rate while increasing the threshold of the shear parameter to quench the MRI.

Radial inflow will affect the global field topology but the details depend on poorly understood fluid trajectories in a region where cold, circular geodesic flow is unstable. As was pointed out by Krolik (1999), the standard assumption of ballistic orbits is never self-consistent for ideal MHD accretion inside $r_{ms}$. Indeed, when magnetic turbulence is the culprit of angular momentum transport in the disc, the magnetic field energy density must become comparable to the rest-mass energy density of the fluid in the plunging region. Yet, unlike Krolik's suggestion, we do not believe that *linear* Alfvén waves could efficiently transport energy from inside $r_{ms}$; the magnetic field there is still highly unstable, and the range of stability of such waves is limited by inertial forces.

On the other hand, in this paper we demonstrate that energy deposition and angular momentum transport through the MRI go on virtually unscathed in the region just below $r_{ms}$. An important note is the prompt nature of this process at near Eddington rates since the MRI feeds the photon bath directly through compressive damping of the modes (Araya-Góchez & Vishniac, in preparation). Energy deposition into the radiation field thus occurs on the MRI time-scale! On the other hand, near $r_{ph}$ the flow will inevitably end up in the advective cooling regime. Assessing the magnetic field dynamics in the region $r_{ms} > r \gtrsim r_{mb}$ is essential to predicting the efficiency of accretion, and to addressing some large-scale effects such as jet launching and disc–hole coupling.

At highly super-Eddington accretion rates (such as those expected in the prompt stages of hyper-accreting black hole formation), the fluid may possess non-trivial amounts of internal energy per unit rest mass of baryons. For such a hot MHD fluid, $r_{ms}$ does not represent a significant boundary to the disc flow and this may occur rather closer to $r_{ph}$. This stresses the importance of addressing MHD processes in the region above the circular photon orbit. Along these lines, we have motivated the provocative conjecture that copious gravitational wave losses ensue through black hole ringing when a hyper-accreting black hole enters the accretion regime where neutrino trapping occurs. This argument, which combines linear regime phenomenology with the latest numerical results from accretion in radiation-stress-dominated environs, leads to a picture of near-hole accretion where large-scale horizontal field domains channel the flow into massive clumps that 'thump' the hole.

Finally, note that a strong, toroidal field topology is ripe ground for MHD instabilities that promote poloidal field generation such as the Parker and radial interchange instabilities (in the vertical and horizontal regime, respectively). These instabilities could provide a physical justification for desirable field topologies invoked in jet launching and the Blandford–Znajek processes.

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