that there are more possible electron positions available than there are valency electrons to fill them, and the question also arises as to which of the radii are to be selected in those cases when two types of lattice are possible. If we knew definitely just how a single electron scatters X-rays these two questions could be easily settled by a quantitative consideration of the intensities of the observed X-ray diffraction patterns. In the present state, however, of our knowledge of scattering, the carrying through of such computations does not seem worthwhile.

In conclusion I wish to express my thanks to Professor Henry G. Gale for his interest in this work.

[1] This investigation was commenced by the author under a NATIONAL RESEARCH FELLOWSHIP.

[2] W. L. Bragg, *Phil. Mag.*, **42,** 169–189, 1920.

[3] W. P. Davey, *Physic. Rev.*, **22,** 211–220, 1923.

[4] W. P. Davey, *Ibid.*, **23,** 218–231, 1924.

[5] T. D. Bernal, *Proc. Roy. Soc.*, **106A,** 749–773, 1924.

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# *THE MAGNETIC DIPOLE IN UNDULATORY MECHANICS*

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1. In the following lines we present a method for the computation of the characteristic values of the parameter contained in linear differential equations. This method is applicable in certain cases when the equation cannot be reduced to the hypergeometric type. As the special example with which to illustrate our procedure we select the motion of an electron in the combined fields of a nucleus and of a magnetic dipole attached to this nucleus. This problem has an interesting bearing on the theory of the spinning electron, as will be fully discussed in section 6. The wave equation for this case has been set up by Fock;[1] we prefer, however, to generalize the problem by taking in the effect of relativity neglected by Fock, and so to arrive at an equation which differs from equation (5) of our last paper[2] (to which we shall refer as loc. cit.) only by a term proportional to $\chi/r^3$:

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left[A + \frac{2B}{r} + \frac{p_0^2 - k(k-1)}{r^2} + \frac{f}{r^3}\right]\chi = 0. \quad (1)$$

The notations are the same as loc. cit., only the constant $f$ is introduced here for the first time and is connected with the moment $\mu$ of the dipole by the relation[3]

$$f = 2\mu p_0 j/e, \tag{2}$$

$j$ being a new quantic integer.

The term depending on $f$ is a small correction term. Higher powers than the first of $f$ were neglected in obtaining equation (1). Therefore, we shall restrict ourselves to solutions taking into account $f$ only to the first power. We believe that the method here used will prove of assistance in many other problems of wave dynamics.

2. We use again the substitutions loc. cit. (7), and obtain the equation for $M$:

$$\frac{d^2M}{dr^2} + 2\left(\alpha + \frac{k}{r}\right)\frac{dM}{dr} + \left(2\,\frac{\alpha k + B}{r} + \frac{p_0^2}{r^2} + \frac{f}{r^3}\right) M = 0. \tag{3}$$

The singular points of this equation are at $r = 0$ and $r = \infty$ and both are indefinite, representing essential singularities. However, in the vicinity of the point $r = \infty$ there exists one integral that can be represented by a descending power series. We can, therefore, start our discussion from this integral, putting

$$M = \sum A_p r^{-p}, \tag{4}$$

and we obtain from (5) the formula connecting the coefficients $A_p$:

$$fA_{p-2} + (p - k - B/\alpha - \sigma_1 - 1)\,(p - k - B/\alpha - \sigma_2 - 1)A_{p-1} - 2\alpha(p - k - B/\alpha)A_p = 0, \tag{5}$$

if we denote

$$\sigma_{1,2} = \mp \sqrt{(k - {}^1/_2)^2 - p_0^2} - B/\alpha - {}^1/_2. \tag{6}$$

The exponent of our series is, therefore, $p = k + B/\alpha$, and we arrive at the infinite system of equations:

$$\begin{aligned} -1.2\alpha A_1 \ldots\ldots\ldots\ldots &= -\sigma_1\sigma_2 A_0 \\ (\sigma_1 - 1)\,(\sigma_2 - 1)A_1 - 2.2\alpha A_2 \ldots\ldots &= -fA_0 \\ fA_1 + (\sigma_1 - 2)\,(\sigma_2 - 2)A_2 - 3.2\alpha A_3 \ldots &= 0 \end{aligned}$$

generally

$$fA_{\tau-2} + (\sigma_1 - \tau + 1)\,(\sigma_2 - \tau + 1)A_{\tau-1} - \tau.2\alpha A_\tau = 0. \tag{7}$$

3. The choice of $A_0$ is arbitrary while the rest of the coefficients must

be determined in terms of $A_0$. If we take the first $\tau$ equations of this system, we can determine $A_\tau$ as a quotient of two determinants

$$A_\tau = \Delta_\tau / \Delta. \tag{8}$$

The denominator $\Delta$ is the determinant of the system and simply reduces to

$$\Delta = \tau!(-2\alpha)^\tau. \tag{9}$$

The numerator $\Delta_\tau$ is obtained by substituting into $\Delta$ the right side of our equations instead of the last column. So we can write

$$\Delta_\tau/\tau!\tau! = (-1)^\tau/\tau!\tau! \cdot \begin{vmatrix} \sigma_1.\sigma_2, & -1.2\alpha, & 0, & \ldots & 0 \\ f, & (\sigma_1-1)(\sigma_2-1), & -2.2\alpha, & \ldots & 0 \\ 0, & f, & (\sigma_1-2)(\sigma_2-2), & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & \ldots & (\sigma_1-\tau+1)(\sigma_2-\tau+1) \end{vmatrix} \tag{10}$$

We have added the factor $1/\tau!\tau!$ in order to secure the convergence of this expression if we should go to the limit $\tau = \infty$.

The integral series (4) is a solution of equation (3) which vanishes for $r = \infty$. Because of this it satisfies one of the physical requirements of finiteness. The second requirement that it be finite for $r = 0$ has still to be satisfied by a suitable choice of $\alpha$ and $\sigma$. It is evident that our series (4) will not be convergent unless the coefficients $A_p$ decrease with increasing index, $p$, so that

$$\lim_{p=\infty} A_p = 0. \tag{11}$$

This is a necessary condition for the finiteness of $M$ in the point $r = 0$. It will be shown in a later publication that this condition is also sufficient. In this note we only wish to point out the numerical possibilities of this method. Since the denominator $\Delta/\tau!\tau!$ does not become infinite in the limit $\tau = \infty$, our condition (11) is equivalent to

$$\lim_{\tau=\infty}(\Delta_\tau/\tau!\tau!) = 0. \tag{12}$$

We have to choose our parameter $E$ on which both $\alpha$ and $\sigma$ depend, so as to satisfy this condition. In other words $E$ must be a root of the infinite determinant (12).

4. In the present case we wish to find the roots only to the terms of the first order in $f$. Instead of discussing the infinite determinant we can, therefore, follow a different line of reasoning. We regard quantities of the order $f^2$ as negligible; if we can determine $E$ in such a way that all the coefficients $A_p$ following a certain one of them $A_\tau$ be proportional to $f^2$, we will have solved our problem, since we then obtain what corresponds to a finite series. Let us first consider the case $f = 0$, the determinant $\Delta_\tau$ then reduces to the product.

$$(\Pi = \sigma_1(\sigma_1 - 1)\dots(\sigma_1 - \tau + 1)\sigma_2(\sigma_2 - 1)\dots(\sigma_2 - \tau + 1), \quad (13)$$

and the roots of this expression are simply the integral values $\sigma_1 = l$. This is exactly condition loc. cit. (12) since $\sigma_1 = -(\sigma_0 + k + B/\alpha)$, we find, therefore, the same values for our parameters as loc. cit.

Expression (10) differs from $\Pi$ by terms of the order $f$; if we substitute into it for $E$ the last root of $\Pi$, our purpose will not be accomplished, since the coefficients beginning with $A_\tau$ will be of the order $f$ and not smaller. The situation will not be changed, if we take for $E$ the last root of $A_\tau$. This is apparent from the relation

$$\tau.2\alpha A_{\tau+1} = (\sigma_1 - \tau)(\sigma_2 - \tau)A_\tau + fA_{\tau-1}.$$

However, if we take the last but one root of $A\tau$, it is different. This root differs only by a quantity of the order $f$ from the last root of $A_{\tau-1}$, and will make this coefficient itself a small quantity. Therefore, $A_{\tau+1}$ will become quadratic in $f$, as well as the next coefficient $A_{\tau+2}$. But if two coefficients in succession are of the order $f^2$, all the following will be small of the same order. It remains to evaluate the last but one root of every coefficient $A_\tau$, or of the corresponding determinant $\Delta_\tau$.

5. We take as the index $\tau + 2$ instead of $\tau$ and expand $\Delta_{\tau+2}$ with respect to the elements of the first column. If we introduce the notation $\Delta^{(m)}_{\tau+2}$ for the determinant resulting from $\Delta_{\tau+2}$ by crossing off the first $m$ lines and the first $m$ columns, this expansion gives

$$\Delta_{\tau+2} = \sigma_1\sigma_2\Delta^{(1)}_{\tau+2} + 1.2\alpha f\Delta^{(2)}_{\tau+2}.$$

$\Delta^{(2)}_{\tau+2}$ is multiplied by the factor $f$. Since we wish to determine the roots only to terms of the first order in $f$, we have to retain only terms of the same order in our equation, and we may substitute 0 for $f$ in the determinant $\Delta^{(2)}_{\tau+2}$

$$\Delta_{\tau+2} = \sigma_1\sigma_2\Delta^{(1)}_{\tau+2} + 2\alpha f\Pi/\sigma_1(\sigma_1 - 1)\sigma_2(\sigma_2 - 1).$$

On the other hand, $\Delta^{(1)}_{\tau+2}$ has the same structure as $\Delta_{\tau+2}$, and we can subject it to the same expansion. Repeating this procedure $\tau + 1$ times, we obtain

$$\Delta_{\tau+2} = \Pi + 2\alpha f \Pi \cdot \cdot \sum_{1}^{\tau+1}{}^{m}\, m/(\sigma_1 - m + 1)(\sigma_1 - m)(\sigma_2 - m + 1)(\sigma_2 - m).$$

We have to find the last but one root of this expression. The first approximation is obtained by omitting the sum altogether and putting $\Pi = 0$. This case was discussed in the preceding section and the last but one root of $\Pi_{\tau+2}$ is $\sigma_1 = \tau$. The second approximation will differ from this value only by a small correction of the order $f$. Therefore, $\Pi$ will contain a small factor and the product $f\Pi$ will be of the second order. This will make all the terms of the sum negligible except the last two containing the same factor in the denominator. Cancelling out all the factors of $\Pi$, except the small one, we can write

$$\sigma_1 - \tau + 2\alpha f\left[\frac{\tau}{(\sigma_1 - \tau + 1)(\sigma_2 - \tau)(\sigma_2 - \tau + 1)} + \frac{\tau + 1}{(\sigma_1 - \tau - 1)(\sigma_2 - \tau)(\sigma_2 - \tau - 1)}\right] = 0.$$

Into the correction term we may substitute the approximate value $\sigma_1 = \tau$. At the same time we may neglect in this term $p_0^2$ which is of the same order of magnitude as $f$ and write[4] $\alpha = \alpha_0 = -me^2/K^2 l$ and $\sigma_2 = \sigma_1 + 2k - 1 = \tau + 2k - 1$, where $l$ is an abbreviation: $l = k + \tau$. We so obtain the second and final approximation[5]

$$\sigma_1 = \tau - me^2 f/2K^2(k - {}^1/_2)k(k - 1). \qquad (14)$$

6. It remains to express $E$ in terms of the integers $k,\tau,l$. Going back to equation (6) and comparing it with loc. cit. (12), we see that the only difference is that the integer $n$ is replaced by our expression $\sigma_1$. We have, therefore, to substitute $\sigma_1$, for $n$ into the energy expression loc. cit. (13). Expanding with respect to $p_0^2$ and neglecting $f$ in the term of order $p_0^4$, we obtain

$$E = -\frac{mc^2}{2}\left\{\frac{p_0^2}{(\sigma_1 + k)^2} + \frac{p_0^4}{(k - {}^1/_2)l^3} - \frac{3}{4}\frac{p_0^4}{l^4}\right\}. \qquad (15)$$

Substituting for $\sigma_1$ its value (14)

$$E = -\frac{mc^2}{2}\left\{\frac{p_0^2}{l^2} + \frac{p_0^4}{(k - {}^1/_2)l^3} - \frac{3}{4}\frac{p_0^4}{l^4} + \frac{me^2 p_0^2 f}{K^2} \cdot \frac{1}{l^3(k - {}^1/_2)k(k-1)}\right\}. \qquad (16)$$

It is known that in the old theory the problem of the magnetic dipole has a close relation to that of the spinning electron: One has only to replace the moment $\mu$ by one-half of the spin moment taken with the negative sign

$$\mu = -Ke/4mc, \tag{17}$$

in order to obtain the identical spin formula (including the Thomas correction). It is interesting to see what we obtain by the same procedure in the new theory. Of interest are only the second and fourth terms of our braces, since they are the only ones dependent on $k$ and giving the fine structure. Their sum now becomes

$$p_0^4[1 + j/2k(k-1)]/l^3(k-{}^1/_2). \tag{18}$$

The geometrical significance of $j$ in our problem is the projection of $k$ on the dipole axis. It will be noticed that for the two values $j = k$ and $j = -(k - 1)$ our expression reduces to $p_0^4/l^3(k - 1)$ and $p_0^4/l^3k$, which are identical with the levels of the old Sommerfeld theory. *This shows that our formulae (16) and (18) in fact completely represent the effect of the spinning electron as it was suggested by Goudsmit and Uhlenbeck.*

This result must have its cause in a mathematical parallelism of the two problems. When the wave equation for the spinning electron will be set up, it, probably, will prove of the same or a similar type as our equation (1). Taking our formula (18) to be identical with that for the spinning electron, it is easy to see that in this case the absolute value of $j$ must be either $k$ or $k-1$. The procedure of compounding the total momentum vectorially has been worked out by the spectroscopists. According to their normalization the orbital momentum is an integral multiple of $K$ and proportional to $k-1$. The spin momentum of a single electron is $K/2$ and, since this momentum is half integral, the total momentum must be also half integral and proportional to $j-{}^1/_2$. It follows that $k-1+{}^1/_2 \geqslant j-{}^1/_2 \geqslant k-1-{}^1/_2$, whence $j$ is, in absolute value, either $k$ or $k-1$. I.e., the spin axis is either parallel or antiparallel to the orbital momentum $k-{}^1/_2$. This leads to the negative sign of $k-1$ since in the second case spin and orbital momentum have opposite directions.

[1] V. Fock, *Zs. Physik*, **38**, p. 242, 1926.

[2] P. S. Epstein, these PROCEEDINGS, **13**, p. 94, 1927.

[3] For working out this relation I am indebted to Mr. C. F. Richter of the Norman Bridge Laboratory. The magnetic dipole is taken to have a fixed direction in space. However, in the case of a free dipole the reaction of the electron on it would not change the type of equation (1) but only the constant of equation (2).

[4] P. S. Epstein, these PROCEEDINGS, **12**, p. 629, 1926, formula (4).

[5] The case $k = 1$ requires a separate treatment. We have thus to take the last root of every determinant $\Delta_\tau$ and obtain $\sigma_1 = \tau - me^2f/K^2$.