NOTE ON THE NATURE OF COSMIC RAYS

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1. The recent experiments of Bothe and Kolhoerster[1] have led to an important and interesting conclusion. The ionization phenomena, by means of which cosmic rays are measured, are produced by highly penetrating corpuscular rays. It remains, however, an open question whether this corpuscular radiation is identical with the cosmic rays themselves, or is secondary in its nature, produced by primary rays of an electromagnetic character. Bothe and Kolhoerster admit both possibilities, but they seem to lean toward the first, owing to the fact that their corpuscular rays show a penetration of the same order of magnitude as that of the cosmic rays themselves. Further studies on the nature of these corpuscular rays are due to Rossi,[2] Tuve[3] and Mott-Smith.[4] Rossi gives some preliminary results about the deflection of these rays in a magnetic field. If they are fast-moving electrons, he finds that their energy is a little lower than 10^9 volt. He admits, however, that his observations were not sufficiently numerous to make this figure conclusive.

The purpose of the present note is to find out whether the hypothesis of an electronic nature of the primary cosmic rays is consistent with their observed intensity distribution on the surface of the earth. For lack of a better figure, we shall assume their energy to be 10^9 volt. It has been pointed out to the author by Dr. R. A. Millikan that electrons of that speed are appreciably deflected in the magnetic field of the earth. In fact, the radius of curvature a of an electron moving normally to a homogeneous magnetic field H is given by the relation

$$mv^2/a\sqrt{1-\beta^2} = e\beta H, \qquad (1)$$

where e, m denote the charge and mass of the electron, v and $\beta = v/c$ its velocity in absolute measure and referred to the velocity of light c.

On the other hand, the energy is expressed in the following way by the voltage V

$$mc^2\left[\frac{1}{\sqrt{1-\beta^2}} - 1\right] = eV. \qquad (2)$$

In the case of high speed, β is very close to 1 and $\sqrt{1-\beta^2}$ a very small number. We may neglect 1 in the bracket of Eq. (2) and, dividing the two expressions, obtain as a good approximation

$$a = V/H. \qquad (3)$$

With $V = 10^9$ volt $= 0.33.10^7$ abs. and $H = 0.50$ Gauss this gives $a = 6.6.10^6$ cm. $=$ 66 km. Compared with terrestrial dimensions, this is a very small radius, indeed, and this fact makes us expect a big influence of the earth field on the path of our particles.

The equations of motion turn out to be so complicated that a solution cannot readily be given in an analytical form, either rigorously or by approximations. Fortunately, however, it is possible to find certain maximum and minimum relations which give an unambiguous answer to the question we are interested in: *The assumption under discussion is inconsistent with observed facts.* Electrons of 10^9 volt energy coming from outside can strike the earth only in two limited zones around its magnetic poles. Practically all the countries where cosmic rays have been measured are outside these zones. This leaves the following possibilities: (*a*) cosmic rays are electromagnetic waves, (*b*) they are corpuscular rays of a very much higher energy than 10^9 volt (at least 6.10^{10} volt), (*c*) they are not cosmic but have terrestrial origin.

2. We treat the earth as a simple magnetic dipole and express the field in polar coordinates r, ϑ, φ

$$eH_r = -2fc\cos\vartheta/r^3,\ eH_\vartheta = -fc\sin\vartheta/r^3,\ eH_\varphi = 0. \tag{4}$$

The force acting on the electron is $\overline{F} = \frac{e}{c}\,\bar{v} \times \overline{H}$, with the Lagrangean components

$$\begin{cases} Q_r = f\sin^2\vartheta\dot{\varphi}/r^2,\ Q_\vartheta = -\,2f\sin\vartheta\cos\vartheta\dot{\varphi}/r, \\ \qquad Q_\varphi = -f\sin\vartheta\left(\dfrac{\sin\vartheta\dot{r}}{r^2} - \dfrac{2\cos\vartheta\,\dot{\vartheta}}{r}\right) \end{cases} \tag{5}$$

The kinetic potential is, therefore,

$$L = -mc^2\sqrt{1-\beta^2} + f\sin^2\vartheta\dot{\varphi}/r. \tag{6}$$

In fact, it can be easily verified that Eq. (5) are obtained by Lagrangean differentiation of the second term of this expression. The momenta, on the other hand, result by differentiation with respect to $\dot{r}$, $\dot{\vartheta}$, $\dot{\varphi}$

$$p_r = \frac{m\dot{r}}{\sqrt{1-\beta^2}},\ p_\vartheta = \frac{mr^2\dot{\vartheta}}{\sqrt{1-\beta^2}},$$
$$p_\varphi = \frac{mr^2\sin^2\vartheta\dot{\varphi}}{\sqrt{1-\beta^2}} + \frac{f\sin^2\vartheta}{r}. \tag{7}$$

and the Hamiltonian $H = -L + \dot{r}p_r + \dot{\vartheta}p_\vartheta + \dot{\varphi}p_\varphi$ assumes the form

$$H = mc^2/\sqrt{1-\beta^2}$$
$$= mc^2\left\{1 + \frac{1}{m^2c^2}\left[p_\vartheta{}^2 + \frac{1}{r^2}p_\vartheta^2 + \left(\frac{p_\varphi}{r\sin\vartheta} - \frac{f\sin\vartheta}{r^2}\right)^2\right]\right\}^{1/2}. \quad (8)$$

From this form of the Hamiltonian, there follow two facts which are obvious but which must be emphasized as they are of great importance for the following discussion. In the first place, H does not contain the time t; this leads to the existence of an equation of conservation of energy $H = mc^2 + \alpha$, where α denotes the kinetic energy.

$$mc^2/\sqrt{1-\beta^2} = mc^2 + \alpha, \quad (9)$$

or

$$p_r^2 + \frac{1}{r^2}p_\vartheta^2 + \left(\frac{p_\vartheta}{r\sin\vartheta} - \frac{f\sin\vartheta}{r^2}\right)^2 = A^2,\ A^2 = 2m\alpha + \alpha^2/c^2. \quad (10)$$

Equation (9) *shows that the velocity remains strictly constant throughout.* This could be anticipated as the force is always normal to the path and does not produce any work.

In the second place, H does not contain the variable φ, so that $p_\varphi = p =$ const. We can rewrite the third of equations (7) in the form

$$\frac{m\dot{\varphi}}{\sqrt{1-\beta^2}} = \frac{p}{r^2\sin^2\vartheta} - \frac{f}{r^3}. \quad (11)$$

When the distance of the electron from the dipole is extremely large, the term f/r^3 is negligible compared with $p/r^2\sin^2\vartheta$, and p has the meaning of an ordinary angular momentum. As the electron approaches the last term gains in importance: The angular velocity is altered by an amount proportional to r^{-3}. This is a generalization of Larmor's theorem: a homogeneous magnetic field produces the constant Larmor precession, a dipole produces a variable precession (proportional to r^{-3}) around the magnetic axis.

It follows from the two facts, just outlined, that the electron cannot come infinitely close to the dipole, except in the directions $\vartheta = 0$, or $\vartheta = \pi$. In fact, as the electron approaches the last term begins to dominate for any finite ϑ, $\dot{\varphi}$ increases as r^{-3}, and the absolute value of the azimutal component of the velocity $v_\varphi = r\sin\vartheta\dot{\varphi}$ increases as $\sin\vartheta r^{-2}$. However, we have found that the total velocity v is constant, so that $v_\varphi \leq v$ cannot increase indefinitely. This shows that, for any given ϑ, there are points which cannot be reached, or in other words: unless the electron ultimately approaches the dipole in one of the directions $\vartheta = 0$ or $\vartheta = \pi$, there is a position of nearest approach, for which $\dot{r} = p_r = 0$. Our objective will be the finding of a minimum value for this position of nearest approach.

3. With condition $p_r = 0$, Eq. (10) becomes

$$\left(\frac{p}{r\sin\vartheta} - \frac{f\sin\vartheta}{r^2}\right)^2 = B^2, \tag{12}$$

with the abbreviation

$$B^2 = A^2 - p_\vartheta^2/r^2,\ 0 \leqq B^2 \leqq A^2. \tag{13}$$

Solving (12) with respect to r, we find

$$\frac{1}{r} = \frac{p \pm \sqrt{p^2 \pm 4Bf\sin^3\vartheta}}{2f\sin^2\vartheta}. \tag{14}$$

This expression represents four different solutions, which we may denote by the symbols r_1 corresponding to $(+, +)$, r_2 to $(-, +)$, r_3 to $(+, -)$, r_4 to $(-, -)$. Solution r_2 must be ruled out as it is negative and has no physical significance. If we plot these solutions, for constant p, f, A, in the polar r, ϑ-plane, the solutions r_3 and r_4 represent parts of the same curve and join together in the vertex $\sin^3\vartheta = p^2/4Bf$. If we replace B by A (always $\geqq B$)

$$\frac{1}{\rho} = \frac{p \pm \sqrt{p^2 \pm 4Af\sin^3\vartheta},}{2f\sin^2\vartheta} \tag{15}$$

we have $\rho_1 \leqq r_1$, $\rho_3 \geqq r_3$, $\rho_4 \leqq r_4$, provided that the solutions are real. This shows us, among other things, that the vertex of ρ_3, ρ_4 coincides with the vertex r_3, r_4 so that here we have $B = A$.

Let us now put the question as follows: We consider a point $r = R$, $\vartheta = \theta$ and we ask whether an electron, coming from outside, can reach it. We have to distinguish the following two cases

(I). Constant p negative, or positive and small

$$p \leqq 2(Bf\sin^3\theta)^{1/2},$$

and *a fortiori*

$$p \leqq 2(Af\sin^3\theta)^{1/2}.$$

This rules out solutions r_3, r_4 and ρ_3, ρ_4. The motion of the electron can take place only outside the curve ρ_1. Comparing electrons with different constants p, we see that ρ_1 is smallest for those where $p = 2(Af\sin^3\theta)^{1/2}$, namely,

$$(1 + \sqrt{2})\rho_1 = (f\sin\theta/A)^{1/2}.$$

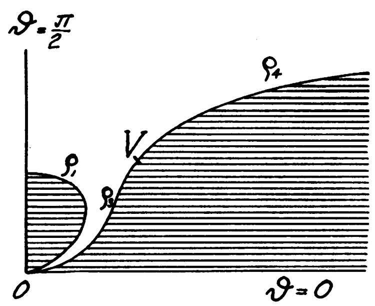
Consequently the point R, θ *cannot* be reached if $R < \rho_1$, or

$$2.41R < (f\sin\theta/A)^{1/2}. \tag{16}$$

(II). The constant p is positive and

$$p \geqslant 2(Bf \sin^3 \theta)^{1/2}.$$

In this case all three barriers r_1, r_3, r_4 of solution (14) have a physical existence, as well as the corresponding ρ_1, ρ_3, ρ_4 of (15). We give in figure 1 a graphical representation of these curves for a quadrant of the r, ϑ-plane.



The electrons, under no circumstances, can get into the shaded region. Again we have to consider two sub-cases. (II*a*). R is larger than the distance of the vertex V from the origine O. This can only happen when p is small, namely, $p < 2\ A^2R^3/f$. With the numerical data of the next section, this turns out to be such a small number that it is permissible to neglect it altogether in the expression of the curve ρ_1 which limits the motion of our electron. R cannot become smaller than ρ_1 and this leads to the following condition.

No point can be reached for which

$$R < (f \sin \theta/A)^{1/2}. \qquad (17)$$

(II*b*). R is smaller than the distance of the vertex V from the origin O. In this case the electron must strike or have passed to the left side of the vertex V. We have seen that the point V is characterized by $\sin^3\vartheta = p^2/4Af$. There are points in the path of our electron for which the angle is larger than this, so that there we have $\sin^3 \vartheta \leqslant p^2/4Af$ or $p \leqslant 2(Af \sin^3 \vartheta)^{1/2}$. We can assert, *a fortiori*, $p \leqslant 2(Af)^{1/2}$. Eq. (15) with $(+, +)$ shows us that ρ has its minimum when p reaches its largest value, i.e., $2(Af)^{1/2}$. The smallest possible ρ is then

$$\rho_1 = \left(\frac{f}{A}\right)^{1/2} \frac{\sin^2 \theta}{1 + \sqrt{1 + \sin^2 \theta}}.$$

Points for which

$$R < \left(\frac{f}{A}\right)^{1/2} \frac{\sin^2 \theta}{1 + \sqrt{1 + \sin^2 \theta}} \qquad (18)$$

cannot be reached.

We may leave the question open whether the curves corresponding to the different solutions (14) are simple or consist of several branches, owing to the analytical character of B. We have only made use of the fact that they lie outside the curves of Eq. (15), i.e., outside the shaded region of figure 1. Since this is assured, our results will always apply.

4. We are now ready for numerical applications to the field of the earth, which is roughly equivalent to a dipole of the strength $f = 1.24 \cdot 10^6$. For electrons of energy 10^9 volt we get $\alpha = 1.6 \cdot 10^{-3}$, $A = 5.3 \cdot 10^{-14}$. There-

fore, $\sqrt{f/A} = 4.8 \cdot 10^9$ cm. Since the radius of the earth is $R = 6.35 \cdot 10^8$, we can say $\sqrt{f/A} = 7.6\ R$.

If we replace in expressions (16), (17), (18) the sign of inequality by that of equality, they will give us directly the maximum angular distance θ_m from the magnetic pole, in which an electron can strike the surface of the earth. We obtain for the three special cases,

$$\text{(I)} \qquad 7.6 \sin^{1/2}\theta_m = 2.41, \qquad \theta_m^{(1)} = 6°. \tag{19}$$

$$\text{(II}a\text{)} \qquad 7.6 \sin^{1/2}\theta_m = 1, \qquad \theta_m^{(2)} = 1°. \tag{20}$$

$$\text{(II}b\text{)} \qquad 7.6 \sin^2 \theta_m = 1 + \sqrt{1 + \sin^2 \theta_m}, \qquad \theta_m^{(3)} = 32°. \tag{21}$$

The net result of our considerations is, therefore, that electrons of 10^9 volt energy cannot hit the earth outside of two limited zones around the magnetic poles.[5] Our analysis does not permit us to say whether the maximum distance from the pole of 32° is actually reached.

[1] Bothe and Kolhoerster, *Zs. Physik*, **54**, 686 (1929).

[2] B. Rossi, *Rend. Acc. dei Lincei*, **2**, 478 (1930).

[3] M. A. Tuve, *Phys. Rev.*, **35**, 651 (1930).

[4] L. M. Mott-Smith, *Ibid.*, **35**, 1125 (1930).

[5] For electrons of energy 10^8 volt the maximum distance would be 17°, for 2.10^9 volt it would be 40°.

PERIODICITY IN SEQUENCES DEFINED BY LINEAR RECURRENCE RELATIONS

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A sequence of rational integers

$$u_0, u_1, u_2, \ldots \tag{1}$$

is defined in terms of an initial set $u_0, u_1, \ldots, u_{k-1}$ by the recurrence relation

$$u_{n+k} + a_1 u_{n+k-1} + \ldots + a_k u_n = a, \; n \geqq 0, \tag{2}$$

where $a_1, a_2, \ldots, a_k$ are given rational integers. The author examines (1) for periodicity with respect to a rational integral modulus m. Carmichael[1] has shown that (1) is periodic for $(a_k, p) = 1$ and has given periods (mod