THE EXPANSION OF THE UNIVERSE AND THE INTENSITY OF COSMIC RAYS

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1. It was pointed out by Zwicky[1] that the distance which a light quantum can travel is, perhaps, limited. Its frequency ν and energy $h\nu$ are affected by the astronomical red shift and may be reduced to the vanishing point if the distance is sufficiently large. Experimentally the dependence of the red shift upon the travel time $t - t_0$ follows very nearly (within the limits of error) a linear law up to $t - t_0 = 10^8$ years and amounts to $k = -\Delta\nu/\nu = 0.56 \times 10^{-9}$ per year. If it is permissible to extrapolate the linear law to larger distances, the light quantum will vanish for $t - t_0 = 1.8 \times 10^9$ years. All the radiation received by a terrestrial observer must, in this case, be derived from a space limited by this radius and this involves serious difficulties for the explanation of the rather high intensity of cosmic rays.

The question arises, therefore, to what extent the linear extrapolation of the red shift is justifiable and its analysis forms the main content of the present paper. The answer is that the embarrassingly short range of the cosmic rays is only a new disguise for the old difficulty of the short time scale of the expanding universe. In fact, the first rough estimates of the age of the universe were also made by a linear extrapolation of the red shift (which is of the nature of a Doppler effect) and, therefore, led to precisely the same figure 1.8×10^9 years. It is desirable, however, to bring out this connection with more detail. The shortness of the time scale was particularly emphasized by Lemaitre who showed that it can be avoided by assigning a proper value to Einstein's cosmological constant λ. Curiously, therefore, the high intensity of cosmic rays which we actually measure is, to a certain degree, an additional argument in favor of retaining the cosmological constant which some authors wished to discard. In this connection it will be necessary to discuss the different possible types of expansion of the universe and the time scales belonging to them. It is possible to do this in an extremely simple and general and, at the same time, more exhaustive way than it has been done heretofore.

2. We restrict ourselves, in this paper, to the case of the homogeneous (quasi-static) universe and take as our starting point Friedmann's form[2] of the non-static line element

$$ds^2 = -R^2(t)[d\chi^2 + \sin^2 \chi(d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2)] + c^2dt^2, \qquad (1)$$

where χ, ϑ, φ are the space coördinates which remain unchanged by the expansion, $R(t)$ the variable radius of the world and c a constant (velocity of light), and we wish to determine the energy of free motion in a geodesic. We can simplify the problem by choosing the coördinates ϑ, φ in a proper way. If the origin lies on the geodesic, we have $\varphi = \text{const.}$, $d\varphi = 0$, moreover we can define ϑ so as to make the geodesic lie in the equatorial sphere: $\vartheta = \pi/2$, $d\vartheta = 0$.

$$ds^2 = -R^2(t)\, d\chi^2 + c^2\, dt^2. \tag{1'}$$

In the case of light, $ds = 0$ and we obtain

$$d\chi = c\, R^{-1}\, dt, \tag{2}$$

or for the angular distance corresponding to a travel time from t_0 to t

$$\chi = c \int_{t_0}^{t} R^{-1}\, dt. \tag{3}$$

If at one end of this distance two signals are sent out at the interval Δt_0 from each other, they will arrive at the other end with the interval Δt. The ratio $\Delta t/\Delta t_0$ is obtained by differentiating with respect to t_0 the equation (3), where χ is independent of time,

$$\Delta t : \Delta t_0 = R : R_0, \tag{4}$$

if we write for short $R = R(t)$ and $R_0 = R(t_0)$, a formula which has been given in an equivalent form by Tolman[3] and others. In particular, we could consider the time intervals between two planes of equal phase of the light wave: In this case, the eq. (4) will give us the ratio between the periods which the light has in the points $\chi = \chi$ and $\chi = 0$. As the frequencies are reciprocal to the periods, the energies ($E = h\nu$) of a light quantum at the time of its arrival and at the time of its emission stand in the inverse ratio

$$E : E_0 = R_0 : R. \tag{5}$$

Almost equally simple are the conditions in the general case of any moving particle. The equations of its geodesic are obtained from (1′) by Lagrangian differentiation

$$\frac{d}{ds}\left(R^2 \frac{d\chi}{ds}\right) = 0, \qquad c^2 \frac{d^2 t}{ds^2} + R \frac{dR}{ds}\left(\frac{d\chi}{ds}\right)^2 = 0.$$

Eliminating ds between them, we find an easily integrable equation which leads to

$$v = R\frac{d\chi}{dt} = c(1 + \alpha^2 R^2)^{-1/2}, \tag{6}$$

α being the constant of integration. If we take as the expression of the energy the usual $E = mc^2 dt/ds = mc^2(1 - v^2/c^2)^{-1/2}$, we obtain

$$\frac{E}{E_0} = \frac{R_0}{R}\left(\frac{1 + \alpha^2R^2}{1 + \alpha^2R_0^2}\right)^{1/2}. \tag{7}$$

We see from (6) that for very fast particles of very high energy with velocities v practically equal to the velocity of light c, the constant α must be so small as to make α^2R^2 negligible compared with 1. In this case (7) becomes identical with (5). If material particles form a part or the whole of the cosmic radiation, they unquestionably have velocities of this order. We can, therefore, apply the formula (5) to corpuscles in general without bothering about their physical nature (light quanta or matter). In the other limiting case, when α is very large, we find the approximation $E \propto (1 + 1/2\alpha^2R^2)$. If we denote the constant of proportionality by mc^2,

$$E = mc^2(1 + 1/2\alpha^2R^2), \tag{5'}$$

we recognize that the first term can be interpreted as the intrinsic, the second as the kinetic energy of the particle or body moving with a comparatively small velocity.

3. Although the law (5) of energy decrease of photons is not new, it seems that its main implications have never been discussed. It shows us that the energy of the corpuscles we observe at the present time is proportional to the radius of the universe at the time they were emitted. If the history of our universe is such that at some time in the past its radius was zero or nearly zero, then the case surmised by Zwicky is realized. The corpuscles which arrive with their energies completely exhausted are those which were sent out "in the beginning of the world" and their travel time $t - t_0$ is equal to the age of the universe. There are no signals possible from earlier states: The beginning of the world is at the same time the beginning of time.

Lemaitre expressed the hypothesis that the cosmic rays were produced just in these incipient stages of the universe when the density was high and matter was, perhaps, in a state quite different from what it is today. Radioactive materials with enormous molecules might have existed then whose breaking down produced correspondingly energetic α-, β- and γ-rays. These rays may have been circulating round the universe ever since and they manifest themselves even today in the form of cosmic rays. Attractive as this picture is, the relations just stated present grave difficulties to it: The longer the hyper-radioactive rays circulate the more they are whittled down by red shift. In order to have the properties of cosmic rays, they must have started with energies millions of times higher than the very large ones we find in them now. If the hypothesis is given a less extreme form, to the effect that the super-radioactive processes went on not only in the initial but in the early stages of the universe, it meets with less difficulty. Yet its advantages are somewhat decreased by the above considerations.

For short travel times Δt, we have only to substitute in (5) $E_0 = E - \Delta E$, $R_0 = R - \Delta R$, to obtain the well-known form of the red shift coefficient

$$k = -\frac{1}{E}\frac{\Delta E}{\Delta t} = \frac{1}{R}\frac{\Delta R}{\Delta t}. \tag{8}$$

We know the present value of this coefficient, and the question is how it was changing in former times. This will be answered if we come to know what function of time the radius R of the universe is. The linear extrapolation of Zwicky's and others' amounts to assuming that $kR = \Delta R/\Delta t =$ const. This means, if we plot R against t (Fig. 1), that the slope of the curve was at all times the same as it is at the present moment and the law is represented by a straight line, for instance, AB. The age of the universe would then be given by the segment AC. As this extrapolation is very dubious, we wish to investigate the problem somewhat closer. Much work has already been done on this subject: On the one hand, Friedmann[1] and Lemaitre discussed the possible types of expansion in the special case when the pressure can be neglected. On the other hand, Tolman[4] and Tolman and Ward[5] succeeded in excluding certain types of behavior as impossible. A general investigation, however, making no restrictions (but that the pressure remain positive) does not yet exist and shall be given in this and the next sections.[10]

The basis of this investigation is formed by the expressions of the energy-momentum tensor corresponding to Friedmann's line element (1) which in our notations have the form:

$$\left.\begin{aligned} -8\pi p &= (c^2 + \dot{R}^2 + 2R\ddot{R})R^{-2}c^{-2} - \lambda, \\ 8\pi\rho/3 &= (c^2 + \dot{R}^2)R^{-2}c^{-2} - \lambda/3, \end{aligned}\right\} \tag{9}$$

where p is the pressure, ρ the density and λ the cosmological constant. The second of these eqs. gives us $\dot{R}$, while we obtain $\ddot{R}$ by taking the difference of the two

$$\dot{R} = c[(8\pi\rho + \lambda)R^2/3 - 1]^{1/2}, \tag{10}$$

$$\ddot{R} = c^2[\lambda - 4\pi(\rho + 3p)]R/3. \tag{11}$$

It will be useful to remark that, if we exclude negative pressures ($p \geqslant 0$), both ρ and p are necessarily decreasing functions of R. In fact, from (9) and (10) follows the identity

$$d(\rho R^3) + p dR^3 = 0, \tag{12}$$

(eq. of conservation) which is formally identical with the law of compression of an adiabatic fluid. When the pressure is so low as to be negligible ($p = 0$), this gives $\rho = C/R^3$; when p is finite ($p > 0$), the rate of decrease of ρ is even faster. The pressure p decreases, of course, with ρ for reasons of stability.

4. The case of $\lambda \leqslant 0$ is now immediately disposed of: $\ddot{R}$ is, then, under all circumstances negative so that $\dot{R}$ decreases monotonically. Let us go backward in time from the point B on curve I (Fig. 1) which represents the present state of the universe. The slope of the curve becomes steeper and steeper as is shown in the branch BD. Since ρ is infinite for $R = 0$, the slope is vertical in the point D.[6] The age of the universe is represented by the segment DC which is shorter than the segment AC obtained by linear extrapolation. On the other hand, if we proceed from A forward, the slope will decrease, becoming horizontal when $\dot{R}$ vanishes. This will occur when

$$Q(R) = (8\pi\rho + \lambda)R^2/3 - 1 = 0 \tag{13}$$

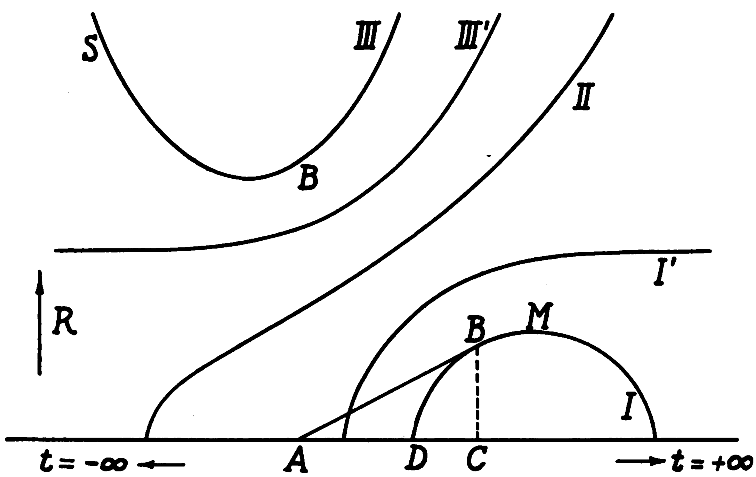

FIGURE 1

and, since the first term is a monotonic function of R, this eq. has always one and only one simple root. It corresponds to the maximum M of the curve which is reached, according to well-known theorems,[7] in a finite time. Beyond it the slope becomes negative and the second half of the curve is symmetrical to the first.

In the case $\lambda > 0$, the character of the curve depends on the roots of both bracket expressions of the eqs. (10) and (11). Since $\rho + 3p$ is a monotonic function of R, the eq. $\lambda - 4\pi(\rho + 3p)$ has always one and only one root which we shall denote by R_i. For $R = R_i$ we have $\ddot{R} = 0$, so that it corresponds to a *point of inflexion* of our curve. On the other hand, in the expression $Q(R)$ of eq. (13) the term $8\pi\rho R^2/3$ is monotonically decreasing and λR^2 is monotonically increasing. $Q(R)$ is infinite both for $R = 0$ and $R = \infty$ and has a minimum in between. We should expect that the position of the minimum is $R = R_i$ (the root of the other bracket expression). This is, in fact, the case as is, easily, checked with the help of eq. (12). We have now to distinguish two cases:

(1) $Q(R_i) > 0$, the minimum value of $Q(R)$ is positive. Then $Q(R)$ and $\dot{R}$ are positive throughout and R increases monotonically as shown in curve II of figure 1. The slope is vertical for $R = 0$ and decreases until the point of inflexion $R = R_i$ is reached. From then on, it increases and becomes again vertical for $R = \infty$.

(2) $Q(R_i) < 0$. In this case, $Q(R)$ has two roots R_1 and R_2 so that $R_1 < R_i$ and $R_2 > R_i$. The radius R cannot assume values between R_1 and R_2 because this would make the rate of expansion $\dot{R}$ imaginary. We have, therefore, two alternatives: (Case $2a$, when $R < R_1$ and, *a fortiori*, $R < R_i$). The acceleration $\ddot{R}$ is, then, permanently negative so that the conditions are quite the same as in the case $\lambda \leqslant 0$ and we are led again to a curve of the type I. Its maximum M corresponds to $R = R_1$. (Case $2b$, when $R > R_2$ and, *a fortiori*, $R > R_i$). The second derivative R is then permanently positive and the curvature opposite to that of curve I. We find then a curve of the type III which goes into infinity for $t = \pm\infty$ and touches a minimum for $R = R_2$. Its slope tends toward the vertical at both extremities.

In the intermediate case $Q(R_i) = 0$, R_i is a double root of $Q(R)$, so that the time of approach of this value becomes logarithmically infinite. The horizontal $R = R_i$ is, then, an asymptote which can be approached either from below, as in curve I′, or from above, as in curve III′.

5. By the types of figure 1 all the possibilities are exhausted. We have said that this applied to the general case, in the sense that it will be true for any law of change of pressure applicable to fluids (i.e., p must be positive and increase with ρ). If the law is such that ρ becomes infinite already for a finite volume, smaller volumes have no physical reality and the axis of abscissae of figure 1 corresponds not to $R = 0$ but to a finite minimum value of the radius without any change in the type of the curves.

It must not be forgotten, however, that the Einstein theory of continuous matter which leads to Friedmann's eq. (1) is based on the consideration of a system of material points. It is, so to speak, the theory of a gravitating perfect fluid, in which the size of the particles and non-gravitational forces between them are omitted. It would be futile to try to improve the accuracy by discussing laws of pressure which take the interaction into consideration because they are outside the validity of the fundamental equation. But this does not impair the theory as a good approximation to the real conditions. Because of the present very low density of matter even the big agglomerations are sufficiently wide apart. On the other hand, if the volume of the universe should become (or have been) considerably smaller, these large units will dissolve and so the process will go on by successive stages. We can, therefore, expect important deviations from the law of perfect fluids only for extremely small radii R.

The equation of state for our perfect fluid follows directly from our energy expressions (5) and (5′). In the stellar space we have, on one hand, bodies and particles moving with astronomical velocities which are small compared with the velocity c of light; on the other hand photons and other corpuscles having quite or almost the velocity c. For the first group applies the energy expression (5′), for the second (5). If we take the sum of the energies of all these bodies, the total energy of the universe U will consist of three terms: the intrinsic energy of the slow moving elements U_i = const., their kinetic energy $U_k \propto 1/R^2$, the radiant energy (including the total energy of the α- and β-rays) $U_r \propto 1/R$. Differentiating the logarithms of U_k, U_r, we find the expressions $dU_k/dR = -2U_k/R$, $dU_r/dR = -U_r/R$. If V is the volume of the universe, $U = \rho V \propto \rho R^3$. We can, therefore, write

$$\frac{d(\rho_k R^3)}{dR} = -2\rho_k R^2, \qquad \frac{d(\rho_r R^3)}{dR} = -\rho_r R^2. \tag{14}$$

Because of the linear character of eq. (12), the two types of particles produce their separate partial pressures: $p = p_k + p_r$, and this eq. separates into two with the indices k and r. Comparing them with (14), we obtain

$$p_k = \frac{2}{3}\rho_k, \qquad p_r = \frac{1}{3}\rho_r \tag{15}$$

precisely the same relations as in the non-relativistic theory.

It follows from these considerations that when the universe expands into infinity ($R \rightarrow \infty$), both U_k and U_r vanish and only U_i can remain finite. Conversely, when it contracts to nothing ($R \rightarrow 0$), not only the energies U_k, U_r but even the densities ρ_k, ρ_r become infinite.

6. We return now to the question about the maximum travel time of photons and other fast corpuscles. As was shown in section 3, it coincides with the age of the universe represented in figure 1 by the projection of the expansion curve upon the time axis from the intercept to a point B giving the present state. It must be compared to the subtangent at the point B which is known from the red shift measurements to be equal to 1.8×10^9 years. There follows from the analysis of section 4 that, in the case of negative or vanishing cosmological constant, the range of all corpuscles is necessarily smaller than 1.8×10^9 years. If λ is positive, the range depends on the value of this constant and of $R(t)$ and can assume all possible values up to infinity.

Lemaitre has already pointed out that the types of motion giving less than 1.8×10^9 years must be excluded because the geological time scale of the earth is of this order of magnitude. The difficulty of explaining the large intensity of cosmic rays, if their range is limited, point in the same direction. To what extent do astronomical data permit us to decide the question what type of expansion our universe is actually undergoing? The observational evidence is very scant: On the one hand, we have the Hubble-Humason red shift measurements which give $k/c = 5.6 \times 10^{-10}$ (years)$^{-1}$ = 5.8×10^{-28} cm.$^{-1}$ On the other, Hubble's determination of the minimum density $8\pi\rho = 8\pi 10^{-30}$ g. cm.$^{-3}$ = 1.86×10^{-57} cm.$^{-2}$ This value is derived for the luminous part of matter and does not include the dark part. The actual density may, therefore, be considerably above this minimum. Eliminating $\dot{R}$ from the eqs. (8) and (10) we find

$$(\lambda + 8\pi\rho - 3k^2c^{-2})R^2 = 3, \tag{16}$$

so that the parenthesis must be positive and we have the inequality

$$\lambda \geqslant 3k^2c^{-2} - 8\pi\rho. \tag{17}$$

Now $3k^2c^{-2} = 1.0 \times 10^{-54}$ and Hubble's minimum for $8\pi\rho$ is more than five hundred times smaller. If it is possible to assume that the true density is not higher than about 100 times this minimum, λ will be determined, essentially, by the first term on the right side and will be at least 10^{-54}. This value of λ coincides with that arrived at by Lemaitre on the basis of somewhat different considerations. Under the conditions assumed above we find $\lambda - 4\pi(\rho + 3p) > 0$ (since the pressure p is negligible compared with ρ) and this means, according to section 4, that in the present state of the universe the curvature of the expansion curve is convex toward the time axis. This leaves us the choice between the types II and III. With the help of our formulas and neglecting p we obtain the criterion $R \gtrless (4\pi\rho)^{-1/3}\lambda^{-1/6}$ or $R \gtrless 1.0 \times 10^{28}$ cm. (to 2×10^{27} cm.) the upper sign referring to the type II and the lower to III. Both possibilities are compatible with all known facts and a decision between the two types cannot be made. Even worse than that, it is possible that the density ρ is thousands of times higher than the lower limit given by Hubble, in which case even the type I would be permissible if there were not the difficulty of the short time scale.

Yet it must be said that the main argument which leads to a very high density ρ is to be taken with some caution. It is based on the fact that the irregular velocities of stars and nebulae are unusually high. Their explanation as the result of a Newtonian attraction by nebulae would require very large mutual forces and correspondingly large masses. However, the analogy between a classical system and the expanding universe is very imperfect. The observable kinetic energy in the Einstein-Friedmann universe is not the analogue to the kinetic energy of a point system subject to Newtonian forces because there is still the expansion of the radius. As we have seen in sections 2 and 5, it increases according to an entirely different law. We wish to illustrate this for the case of an expansion of the type given by curve III (Fig. 1) as this type has some remarkable properties which are well worth discussing. It was pointed out that in the initial state of this kind of a universe ($R = \infty$) both the total and the individual kinetic and radiant energies are infinitely small. A photon of the energy E_S could have been emitted only when the radius was already finite as represented, say, by the point S on curve III. As the universe continues to shrink the energy of the photon increases, according to our formula (5), in reciprocal proportion to the radius. After the minimum is passed, it decreases again and assumes the value E_B in the point B representing the present state of the universe. If E_S is the energy of a radioactive γ-ray, $E_B = (R_S/R_B)E_S$ may be well that of a cosmic ray, provided the emission took place a very long time back. In a similar way the kinetic energy of a slowly moving celestial body will first increase and then decrease according to formula (5′), i.e., reciprocally to the square of the radius, so that the velocity will again follow the law $v_B = (R_S/R_B)v_S$. Comparatively small velocities v_S due to accidental local causes will, therefore, be greatly increased.

The type of expansion III seems little attractive and we do not strongly advocate it, but it is an amusing fact that it could explain two great puzzles: The enormous energy of the cosmic rays and the high irregular velocities of stars. It is true that we did not take into account the collisions between celestial bodies: however, such collisions would only partially reduce the kinetic energy. In the case of the cosmic rays, there arises the question why we do not observe visible light coming from equally large distances. This might be explained by a difference in absorption: We shall see in the next section that it is possible to assume that light is completely absorbed in a comparatively short travel path, while cosmic rays could have a much longer range. On the other hand, apart from other considerations, it is against the type III that the necessary assumption of vanishing kinetic and radiant energies in the initial state seems rather artificial.

Independently of any assumption about the type of expansion holds the following statement: As long as collisions and interactions are neglected the mean kinetic energy of celestial bodies is determined by a separate constant of integration and cannot be inferred from the density and radius of the universe. This fact is more striking in the relativistic theory than in the Newtonian: If in the former the kinetic energy is strictly zero at one time, it vanishes at all times.

7. In this section we shall discuss a few questions specifically relating to the motion of photons. We have mentioned the hypothesis that the cosmic rays were emitted in the early stages of the evolution of our universe. How many times did they circle around it during their long history? The angular distance χ which they cover follows from eqs. (2), (10) and (13) as

$$d\chi = R^{-1}Q^{-1/2}\,dR, \qquad \chi = \int_{R_0}^{R} R^{-1}Q^{-1/2}\,dR. \tag{18}$$

In the particular case considered by Einstein ($\lambda = 0$ and $p = 0$, $\rho = C/R^3$) this gives, for $R_0 = 0$, the eq. of a cylcoid $R = \frac{1}{2}R_1(1 - \cos\chi)$, $ct = \frac{1}{2}R_1(\chi - \sin\chi)$. This means that a pulse of light emitted in the "beginning of the world" ($R = 0$) comes back exactly at the "end of the world" ($R = 0$, again). If we take into consideration the finite values of the pressure p but leave λ unchanged, it is easy to see that the angular distance covered by the rays will be even smaller. This fact is very remarkable as it shows that the different parts of the universe act as one whole, with respect to the expansion, long before they could have received light signals from one another. Its closer investigation is contemplated by the writer since it should be very instructive. On the other hand, a positive cosmological constant increases the angular range of light but, for the ascending part of curve I, not very considerably. Even in the limiting case I', for a state in which the radius R has reached one half of its final value, the maximum angular range turns out to be only about 60°. We find similar conditions in the case of the curve III: If ρ can be neglected beside λ, the angular distance which a ray of light covers during the whole evolution (from $t = -\infty$ to $t = +\infty$) is π or one-half circle. As ρ becomes more important, this distance increases and becomes infinite in the limiting case III′. In the case of the type II, it is also short except when the curve approximates the limiting case III′. This shows that the life of the universe must be considered, in general, as extremely short. Only for laws of expansion in the neighborhood of the curve III′ could we expect some measure of equilibrium. Vice versa, the fact that we actually do not find equilibrium between matter and radiation, etc.,[8] is not at all surprising.

Let us now compute the total intensity of radiation received on the surface of the earth. Suppose that a star at the distance χ or ($t - t_0$ in travel time) from us emits in the time dt_0 the energy $du_0 = \epsilon_s dt_0$. By the time it reaches us it is spread out over a sphere of the surface $4\pi R^2 \sin^2\chi$ and, moreover, attenuated by the red shift according to our formula (5).

The energy received per unit area is, therefore, $du = \epsilon_s R_0 dt_0 / 4\pi R^3 \sin^2 \chi$. The intensity received from that star becomes $I_s = du/dt = \epsilon_s R_0^2 / 4\pi R^4 \sin^2 \chi$. If we wish to consider not a single star but all the radiating matter in a spherical layer between χ and $\chi + d\chi$, we must introduce a coefficient of emission $\epsilon(\chi)$ referred to unit mass. The total mass of the layer in question is $4\pi\rho_0 R_0^3 \sin^2 \chi d\chi$ and, to obtain the intensity of radiation due to it, we have to replace ϵ_s by the mass multiplied by $\epsilon(\chi)$. Integrating with respect to $d\chi$, we find the total intensity (with neglect of absorption)

$$I = \frac{1}{R^4} \int_0^{\chi} \epsilon(\chi) \rho_0 R_0^5 \, d\chi, \tag{19}$$

where the χ in the limit of the integral is the angular distance which a ray travels from the beinning of the universe to the present moment. If it is possible to neglect the pressure, the total mass remains constant: $\rho_0 R_0^3 = \rho R^3$, and the expression reduces to

$$\begin{aligned} I &= \frac{\rho}{R} \int_0^{\chi} \epsilon(\chi) R_0^2 d\chi = \frac{\rho}{R} \int_{\chi = 0}^{t} \epsilon(\chi) R_0 dt_0 \\ &= \frac{\rho}{R} \int_{\chi = 0}^{R} \epsilon(\chi) R_0 Q_0^{-1/2} dR_0. \end{aligned} \tag{20}$$

If we make $\epsilon(\chi)$ constant, the integral becomes divergent for the type III: We have already pointed out (section 5) that, in this case, no radiation can take place while the radius is infinite. As to the types I and II, the intensity is smaller than the expression which results from R_0 being replaced by R, i.e., $I < \rho\epsilon t$. And, in particular, for the type I, it is *a fortiori* smaller than the values $\rho\epsilon \times 1.8 \times 10^9$ (years) obtained by Zwicky from linear extrapolation.

Zwicky's argument by which he shows that, provided the emission of cosmic rays is a general property of matter,[9] the range of 2×10^9 light years is much too short to account for their intensity remains, therefore, entirely valid. The explanation may be either that the range is longer (that is to say, the expansion of the universe is not of the type I), or the cosmic rays are emitted under special conditions. One possibility is that suggested by Lemaitre that they are the messengers of some super-radioactive materials which existed once but have long since disappeared in the solar system. Mathematically this would mean that $\epsilon(\chi)$ in eq. (20) has to increase with χ much more rapidly than R_0 decreases. We remind, however, of the remarks made on this question in section 3. It was argued that if we admit for the cosmic rays very long ranges, say, of the order of 10^{11} light years, we must admit them also for visible light and this would be in contradiction with the comparative darkness of the night sky. This objection would not hold, however, if the main part of the absorbing matter in the interstellar spaces were in the form of dust. As an example, let us consider the case of N dust grains per cm.^{3} each of the radius $a = 10^{-2}$ cm., consisting of some dark material which absorbs most of the light falling on it and has the specific gravity d. A pencil of light rays would be completely absorbed after traveling the distance $L_v = 1/\pi N a^2$ cm. On the other hand, the total mass of the dust particles is $4\pi a^3 Nd/3$ per cm.^{3}, and if a cosmic ray has the range $l = 100$ cm. in water, it will have in the dust range $L_c = 3l/4\pi a^3 Nd$. The ratio $L_c/L_v = 3l/4ad$ is of the order 1000 if d is about 7.5. Visible light can, therefore, easily have an absorption thousands of times higher than cosmic rays so that the bulk of the two kinds of radiation falling upon the earth is drawn from quite different parts of the universe. If we take $L_v = 10^{10}$ light years, the density of disperse dark matter in this example works out to be 10^{-29} g./cm.^{3}, an entirely reasonable value.

[1] F. Zwicky, *Phys. Rev.*, **43**, 147 (1933).

[2] A. Friedmann, *Zeitschr. Physik*, **10**, 377 (1922).

[3] R. C. Tolman, *Proc. Nat. Acad. Sci.*, **16**, 320 (1930).

[4] R. C. Tolman, *Phys. Rev.*, **38**, 1758 (1931).

[5] R. C. Tolman and M. Ward, *Ibid.*, **39**, 835 (1933).

[6] Compare the remarks on this point in the beginning of section 5.

[7] Compare L. Charlier, "Die Mechanik des Himmels," Vol. I, pp. 85–90; Leipzig (1902).

[8] Compare R. C. Tolman, *Proc. Nat. Acad. Sci.*, **17**, 153 (1931).

[9] As an average over very large material systems, not for every kind of matter separately.

[10] The writer discovered a paper by H. P. Robertson (*Rev. of Modern Physics*, **5**, 62, 1933) which contains a classification of types of expansion similar to that of our section 4.