RADIO-WAVE PROPAGATION AND ELECTROMAGNETIC SURFACE WAVES

BY PAUL S. EPSTEIN

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA

Communicated March 31, 1947

1. *A Discrepancy and Its Resolution.*—The problem of the propagation of radio-waves along the surface of the (plane) earth was first treated in a celebrated paper by A. Sommerfeld (1909).[1] Let the (r, φ)-plane of a cylindric system of coördinates coincide with the plane of the earth surface and let the z-axis point vertically upward. Sommerfeld considered a vertically oscillating dipole (radio-antenna) in the air, close to the origin ($z = 0$, $r = 0$) and asked what secondary waves were produced by the discontinuity due to the presence of the partially conducting earth. His investigation led to the result that the Hertzian vector Π describing the electromagnetic field can be divided, in either medium (air and earth), into two parts: $\Pi = Q + P$, where the part Q has the character of *space-waves*, which at large distances from the origin are proportional to R^{-1}, if $R = (r^2 + z^2)^{1/2}$. On the other hand, P is a *surface-wave:* at large distances it becomes proportional to $r^{-1/2}$ and is restricted to the vicinity of the earth surface.

Ten years later (1919) the problem was reëxamined by H. Weyl[2] who used a somewhat different mathematical approach. He obtained a solution which was identical with Sommerfeld's space waves Q but which did not contain the surface wave P.

The reason for this discrepancy has never been satisfactorily explained. Since Weyl's method seemed mathematically simpler his result was favored by public opinion in numerous papers by other authors. Finally, in 1935, Sommerfeld himself conceded that the surface wave has no reality.[3] Referring to F. Noether,[4] he attributed this to an inaccuracy in the evaluation of his general solution. This evaluation consisted in carrying out a contour integration in which a pole of the integrand yielded the surface-wave P and two branch-cuts accounted for the space-wave Q. According to Noether's explanation the pole is so close to one of the branch-cuts that the method of integration used by Sommerfeld, possibly, was not sufficiently reliable.

The question has not only historical interest but retains even now some actuality because Sommerfeld's method was recently applied by C. Y. Fu[5] to the analysis of the propagation of seismic waves. In this case, the singularities, instead of nearly coalescing, as in Sommerfeld's problem, are rather far apart so that the evaluation does not present any difficulties. If Noether's suggestion is the complete explanation for the absence of surface waves in the electrodynamic disturbances, it should not apply to seismic waves. In other words, an oscillating elastic dipole within the earth would produce, among other things, seismic surface waves. It is, therefore, important to decide what the true nature of the discrepancy is and whether the above explanation is exhaustive.

We shall show in the next section that Noether's explanation is both insufficient and unnecessary. The difficulties already arise (and can be resolved) when only the general solution is considered, and before any evaluation is attempted. It seems that the resolution of the discrepancy has been delayed so long because of the mental attitude of all involved which led them to take it for granted that one of the conflicting solutions must contain a mathematical error. This is not so: from the mathematical point of view, both Sommerfeld's and Weyl's solutions are unimpeachable. However, they represent two different physical phenomena. On the one hand, Weyl's solution (Q) corresponds just to the wave of the oscillating dipole with its secondary space waves due to reflexion and transmission. On the other hand, Sommerfeld's solution ($Q + P$) is the superposition of two independent physical systems as follows: (1) the oscillating dipole with its secondaries (Q), (2) an electrodynamic surface wave (P). These two systems stand in no causal relation to each other, their yoking together in one mathematical expression is purely accidental. The fact that the space-waves Q, as evaluated by Sommerfeld, are identical with those found by Weyl does not bear out Noether's suggestion that the evaluation is at fault. On the contrary, Sommerfeld's evaluation seems to be entirely adequate.

2. *Mathematical Proof.*—Sommerfeld starts from the representation of the z-component of the Hertzian function for an oscillating dipole, $\Pi_0 = \exp. (ikR)/R$ in the form of the integral

$$\Pi_0 = \frac{1}{2}\int H(\lambda r) \exp. (\mp \sigma z)\sigma^{-1}\lambda d\lambda, \qquad (1)$$
$$\sigma = (\lambda^2 - k^2)^{1/2}, \ \sigma' = (\lambda^2 - k'^2)^{1/2}.$$

We designate here by k and k' the wave-numbers of the upper and lower medium (air and earth), while H denotes Hankel's cylindric function of the first kind and of order zero. The upper sign of the exponent refers to the case $z > 0$, the lower to $z < 0$, while the signs of the roots must be chosen so as to make the real parts of σ and σ' positive. The path of integration (L) in the λ-plane is the real axis from $-\infty$ to $+\infty$.

The Hertzian function determines the field components in the following way

$$E = \Pi k^2 + \nabla(\nabla \cdot \Pi), \qquad H = -ik^2 c\omega^{-1}\nabla \times \Pi, \qquad (2)$$

where c is the velocity of light, ω the frequency, and the time factor exp.

$(-i\omega t)$ is omitted. This leads to the border conditions at the surface of the earth ($z = 0$),

$$k^2\Pi - k'^2\Pi' = 0, \qquad \partial(\Pi - \Pi')/\partial z = 0, \tag{3}$$

where Π is the total Hertzian function in the upper medium, Π' in the lower.

To satisfy the border conditions, Sommerfeld assumes the existence of a *reflected* Hertzian function in the upper medium, $\Pi_R = \Pi - \Pi_0$, and of a *transmitted* one Π', in the lower, which differ from the expression (1) mainly by the respective factors $f(\lambda)$ and $f'(\lambda)$ in the integrands. These expressions satisfy the wave equations $\nabla^2\Pi + k^2\Pi = 0$ and $\nabla^2\Pi' + k'^2\Pi' = 0$, while the factors can be chosen so as to fulfill the border conditions. Thus the total Hertzians in the two media become

$$\Pi = {}^1/_2 \int H(\lambda r) \exp. (-\sigma z)\sigma^{-1}[1 + f(\lambda)]\lambda d\lambda, \tag{4}$$

$$\Pi' = {}^1/_2 \int H(\lambda r) \exp. (\sigma' z)\sigma^{-1} f'(\lambda)\lambda d\lambda, \tag{5}$$

the paths of integration being the same as in Π_0. With the help of the border conditions the factors are found to be

$$1 + f(\lambda) = (k^2/k'^2)f'(\lambda) = (k^2\sigma' - k'^2\sigma)/(k^2\sigma' + k'^2\sigma). \tag{6}$$

The singularities of the integrands of (4) and (5) are represented in Fig. 1. They consist of the branch points, $\lambda = k$, $\lambda = k'$, and of the pole, $\lambda = k_0$, of the expression (6). The path of integration (L) can be displaced and is equivalent to two loops Q_1 and Q_2 around the branch cuts and to the residuum at the pole. As we mentioned in Section 1, the loops represent the space-waves $Q = (Q_1 + Q_2)$, the residuum the surface-wave P.

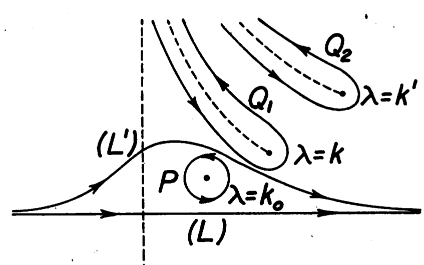

FIGURE 1.

It should be noticed, however, that the integrand of the expression (1) representing the original dipole does not have any singularity at the point $\lambda = k_0$. Hence, the path of integration of this integral can be displaced into the curve (L') passing above the point $\lambda = k_0$. Since all that is required of the expressions (4) and (5) is that they satisfy the respective wave equations and the border-conditions, the integrals in them can be also conducted over the path (L'). Thus, in addition to the L-solution discussed above we have a second solution which we shall call the L'-solution. It is given by the integrals (4) and (5) conducted over the path L'. The evaluation of the L'-solution can be effected in the same way, by displacing the path of integration upward. It is thus equivalent to the two loops about the branch cuts (which represent the space waves) and it does *not* include the residuum of the pole (i.e., it does not contain the surface-wave P).

The existence of the second solution was heretofore overlooked. Drawing attention to it is the essential contribution of this article. From a mathematical point of view, the existence of two different solutions is not at all surprising because it is well known that the integral of the equation $\nabla^2\Pi + k^2\Pi$, for given border conditions, is not unique.[6] Since the L-solution and the L'-solution each satisfy the differential equations and the border conditions of the problem, it follows that their difference must also satisfy the same conditions and represent a third possible solution. This is given by the integrals (4) and (5) conducted over the paths $+L$ and $-L'$, which are equivalent to a circuit about the pole $\lambda = k_0$ or to the residuum in this pole. We know already that this residuum represents the surface wave P of Sommerfeld. We find, in this way, that the surface wave satisfies independently all the conditions of the problem and can exist for itself without any connection with the oscillating pole. This result is not entirely new since in another connection Sommerfeld himself had recognized the independent existence of *surface waves*.[7]

However, Sommerfeld considered there only plane waves while here we have to do with a circular surface-wave. Therefore, it will be well to say a word about this case. The simplest expressions for the z-components of the Hertzian functions Π_s and Π_s', (in the two media) of a circular surface-wave are as follows

$$\Pi_s = AH(\lambda r) \exp. (-\sigma z), \qquad \Pi_s' = BH(\lambda r) \exp. (\sigma' z), \qquad (7)$$

where A and B are two constant coefficients; the other components being zero.

The border conditions (3) take then the form

$$k^2A - k'^2B = 0, \qquad \sigma A + \sigma' B = 0.$$

The two equations are compatible only when their determinant vanishes,

$$k'^2\sigma + k^2\sigma' = 0. \qquad (8)$$

Hence, the value which the parameter λ must be given in the expressions (7) is the root of the eq. (8). This root, $\lambda = k_0$, is identical with the pole of the functions (6). Therefore, it is easy to see that the expressions (7) become identical with the residua of the functions (4) and (5) for the pole $\lambda = k_0$, when the constants A, B are suitably chosen. In other words, they are identical with Sommerfeld's surface wave P.

From the physical point of view, the most interesting of our results is the existence of the L'-solution. As this solution contains the oscillating dipole but not the surface wave, it shows conclusively that the latter wave is not a part of the dipole radiation and is not generated by the dipole. This is true not only for electrodynamic but also for elastic oscillating dipoles which also do not generate (seismic) surface waves. Yet, apart from dipoles, the surface waves can exist and in seismology they are regularly observed in connection with earthquakes. The question how they are generated is an important one but it lies outside the scope of this article.

[1] Sommerfeld, A., *Annalen. Physik*, **28**, 665 (1909).

[2] Weyl, H., *Annalen. Physik*, **60**, 481 (1919).

[3] Sommerfeld, A., contribution to "*Frank-Mises, Differential- und Integral-gleichungen. Zweiter Teil,*" p. 932 (Braunschweig, 1935).

[4] Noether, F., *Funktionentheorie und ihre Anwendungen in der Technik*, 165 (1931). The reference was not accessible to the author.

[5] Fu, C. Y., *Geophysics*, **12**, 57 (1947).

[6] Sommerfeld attempted a proof of the uniqueness (reference 1, pp. 680–682). However, this proof breaks down for wave functions which possess source singularities, as both the oscillating dipole and the surface wave do. In this respect the wave equation is in no way different from the Laplacean equation.

[7] Frank-Mises, *loc. cit.*, 877, 878, 930.