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Surface waves and free oscillations in a regionalized earth model

J. H. Woodhouse and T. P. Girnius* *Department of Geological Sciences, Harvard University, Cambridge, Massachusetts 02138, USA*

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Summary. The linearized equation is derived which relates observed long-period seismic waveforms to the aspherical perturbations of a spherically symmetric earth model. This is accomplished by formulating the theory of spectral splitting in the time domain. It is shown to be possible greatly to simplify the resulting equations in a way which makes it apparent that for each modal multiplet the 'scattered' field depends only upon three local functionals of earth structure. The effect of regional structural variations may then be quantified in a manner analogous to that assumed in the 'pure path technique', but without making the usual asymptotic approximations. These results are used to investigate the validity of the asymptotic result for the locations of the centroids of spectral peaks in individual recordings, for a regionalized model of the Earth. A technique is suggested for retrieving information about geographical structural variations from low-frequency waveform data.

1 Introduction

The effect of lateral heterogeneity on the dispersion of long-period surface waves is known to be substantial in the Earth and has been the subject of many observational studies (e.g. Toksöz & Anderson 1966; Dziewonski & Landisman 1970; Dziewonski 1971). The interpretation of such dispersion data has invariably been performed under the assumption that the observed dispersion characteristics are those appropriate for a model with the structure which is the average of the true structure along the great circle path for which the data were obtained. For a given regionalization of the Earth this ray theoretical approximation enables one to invert for the dispersion characteristics of each region, assumed uniform, and thus to quantify the variations in structure among the regions. We shall refer to this approach as the 'pure path technique'.

Another approach to the treatment of the effects of lateral heterogeneity is through the theory of spectral splitting of free oscillations (Dahlen 1974; Woodhouse & Dahlen 1978; Jordan 1978). This may be referred to as the splitting technique, though it has not been applied, in full, to the interpretation of actual data.

* Present address: Department of Applied Mathematics, California Institute of Technology, Pasadena, California 91125, USA.

It has been shown by Jordan (1978) that a convenient connection between the two techniques may be established through the spectral 'location parameter', namely the difference in frequency between the centroid of a spectral peak corresponding to a given (fundamental) normal mode, observed for a particular source and receiver, and the corresponding eigenfrequency of a reference earth model. In particular, Jordan (1978) has shown that asymptotically, in the limit of large angular order, the location parameter for a particular source and receiver is simply the average along the great circle path of $\delta\omega_{\text{local}}$, which is defined at each point of the globe as the frequency shift appropriate for a spherical model possessing the structure locally beneath that point. This result provides the theoretical justification for the pure path technique and is intuitively very appealing. In order to obtain it, however, severe approximations must be made. In particular it must be assumed that the wavelengths characteristic of the lateral variations in structure are much longer than those of the mode under consideration, a condition which is probably not satisfied in the Earth.

In an important recent contribution Silver & Jordan (1981) have measured the spectral location parameter for fundamental spheroidal modes with angular orders in the range $5 \leqslant l \leqslant 43$, for a large number of paths, making use of data from the IDA (International Deployment of Accelerometers) network. A notable feature of their results is the extreme roughness of the location parameter as a function of angular order. It was conjectured by Silver & Jordan that this may signal the inadequacy of the pure path approximation; one of the conclusions we are able to draw in this paper is that this is indeed the case, and thus a direct interpretation of these data in terms of $\delta\omega_{\text{local}}$ is invalid. A more constructive outcome of the present work is that an interpretation which does not rely upon the asymptotic result can be obtained in a very similar way, and with little additional complication.

It will be shown that the result from the theory of splitting for the spectral location parameter may be recast, without further assumptions, into a form similar to that assumed in the pure path approximation. The difference is that instead of depending only upon $\delta\omega_{\text{local}}$, the location parameter also depends upon two other local functionals of earth structure and, in addition, the path averages must be performed using not the geometrical path lengths in each region, but three other parameters which are fairly readily calculated, and for which formulae will be given. Thus it is possible to quantify the effect on the location parameter of regional structural variations in a manner which is similar to the pure path technique, but which does not involve the approximations which that technique entails.

In the following sections the theoretical development is presented, along with the results of some illustrative calculations. By considering the effect of splitting in the time domain, a comparatively simple expression is obtained which gives the linearized relationship between observed time series and lateral heterogeneities, and the non-asymptotic behaviour of the location parameter is investigated. In a concluding section a strategy for the inverse problem is suggested.

2 Preliminaries

Consider a spherically symmetric, non-rotating, perfectly elastic, isotropic (SNREI) reference earth model with distributions of density, bulk modulus and shear modulus $\rho_0(r)$, $\kappa(r)$, $\mu(r)$ and with displacement eigenfunctions $\mathbf{s}_k(\mathbf{x})$ with associated eigenfrequencies ω_{0k}. We shall use the symbol k to denote the parameters (n, q, l) which characterize a mutliplet; these are radial order, mode type (toroidal or spheroidal) and angular order, respectively. The azimuthal order is denoted by m and $\mathbf{x}$ is the position vector in a

Cartesian system of coordinates with the origin at the centre of the model. Spherical polar coordinates will be defined through

$$\mathbf{x} = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta). \tag{1}$$

We shall also assume a spherically symmetric reference model for the intrinsic attenuation parameters $Q_\mu(r, \omega)$, $Q_\kappa(r, \omega)$ and will denote by ω_k the complex eigenfrequencies modified by the effects of attenuation and the associated physical dispersion, which may be calculated by means of perturbation theory. The perturbations in the eigenfunctions and in the excitation coefficients due to this perturbation will be neglected.

We shall consider an indigenous source represented by a glut rate distribution $\dot{\mathbf{\Gamma}}(\mathbf{x}, t)$ (Backus & Mulcahy 1976). At times after the source has ceased to act the low-frequency elastic displacement field, relative to the final configuration, may be written (Gilbert 1971; Gilbert & Dziewonski 1975)

$$\mathbf{s}(\mathbf{x}, t) = \sum_k \sum_{m=-l}^{l} a_k^m \mathbf{s}_k^m(\mathbf{x}) \exp(i\omega_k t) \tag{2}$$

where the real part is understood. The excitation coefficients are

$$a_k^m = -\mathbf{e}_k^{m*} : \mathbf{M} \tag{3}$$

where $\mathbf{e}_k^m$ is the strain in the (k, m)th singlet, evaluated at the source and $\mathbf{M}$ is the moment tensor:

$$\mathbf{M} = \int_V \int_{-\infty}^{\infty} \dot{\mathbf{\Gamma}}(\mathbf{x}, t)\, d^3\mathbf{x}\, dt. \tag{4}$$

It has been assumed in (3) that the eigenfunctions have been normalized according to (Gilbert & Dziewonski 1975)

$$\int_V \rho_0 \omega_{0k}^2 \mathbf{s}_k^{m*}\, d^3\mathbf{x} = \delta_{kk'}\delta_{mm'} \tag{5}$$

and in (4) and (5) the volume of integration is the whole earth.

A particular seismogram may be obtained by operating on (2) with the 'instrument vector' $\mathbf{v}$, namely a unit vector in the direction of motion sensed by the instrument; $\mathbf{v}$ may also incorporate a time operator or, in the frequency domain, a function of frequency characterizing the instrument response. In addition it may include operators associated with the truncation and editing of the time series. We may then write

$$a_k^m = S_k^m(\theta_s, \phi_s) \tag{6}$$

$$\mathbf{v} \cdot \mathbf{s} = \sum_{km} R_k^m(\theta_r, \phi_r) S_k^m(\theta_s, \phi_s) \exp(i\omega_k t) \tag{7}$$

with

$$R_k^m(\theta_r, \phi_r) = \sum_{N=-1}^{1} R_{kN} Y_l^{Nm}(\theta_r, \phi_r) \tag{8}$$

$$S_k^m(\theta_s, \phi_s) = \sum_{N=-2}^{2} S_{kN} Y_l^{Nm*}(\theta_s, \phi_s) \tag{9}$$

where Y_l^{Nm} are the generalized spherical harmonics of Phinney & Burridge (1973), θ_r, ϕ_r, θ_s, ϕ_s are the spherical coordinates of receiver and source and $R_{kN} = R_k^N(0,0)$, $S_{kN} = S_k^N(0,0)$

Table 1. The coefficients R_{kN} and S_{kN}.

N	$R_{kN} = R_k^N(0,0)$	$S_{kN} = S_k^N(0,0)$
0	$k_0 U v_r$	$-k_0[\partial_r U M_{rr} + \tfrac{1}{2}F(M_{\theta\theta} + M_{\phi\phi})]$
± 1	$k_1(V \pm iW)(\mp v_\theta - iv_\phi)$	$k_1(X \mp iZ)(\pm M_{r\theta} - iM_{r\phi})$
± 2		$-k_2 r_s^{-1}(V \mp iW)[(M_{\theta\theta} - M_{\phi\phi}) \mp 2iM_{\theta\phi}]$
	U, V, W are scalar eigenfunctions for the kth mode, evaluated at Earth's surface	U, V, W are scalar eigenfunctions for the kth mode, evaluated at the source radius r_s

$$k_n \equiv \frac{1}{2^n}\left[\frac{2l+1}{4\pi}\cdot\frac{(l+n)!}{(l-n)!}\right]^{1/2}$$

$$F = r_s^{-1}[2U - l(l+1)V]$$
$$X = \partial_r V + r_s^{-1}(U - V)$$
$$Z = \partial_r W - r_s^{-1}W$$

$M_{rr}, M_{r\theta}$ etc. are components of the moment tensor.
v_r, v_θ, v_ϕ are components of the instrument vector.

are given explicitly in Table 1 in terms of the spherical components of the instrument vector and moment tensor, respectively. The formulae (7) to (9) are equivalent to the corresponding formulae of Gilbert & Dziewonski (1975). They have advantages for this work since they display explicitly the dependence upon source and receiver locations and also show the symmetry between source and receiver parameters; the instrument vector operator **v** is involved in a way analogous to the moment tensor **M**, which may also be an operator if a time dependence different from a step function is desired.

The results of first-order splitting theory (Dahlen 1968, 1974; Woodhouse & Dahlen 1978), neglecting quasi-degenerate coupling, can be summarized as follows. For a model which is slightly aspherically perturbed from the reference model a splitting matrix $\mathbf{H}^{(k)}$ may be defined for each multiplet k. This matrix is of dimension $(2l+1) \times (2l+1)$ and is a linear functional of the model perturbations. If $\mathbf{U}^{(k)}$ is the matrix whose columns are the eigenvectors of $\mathbf{H}^{(k)}$ we have

$$\mathbf{H}^{(k)}\mathbf{U}^{(k)} = \mathbf{U}^{(k)}\mathbf{\Omega}^{(k)} \tag{10}$$

where $\mathbf{\Omega}^{(k)}$ is the diagonal matrix of eigenvalues. Then the eigenfunctions of the perturbed model are

$$\mathbf{u}_k^j(\mathbf{x}) = \sum_m U_{mj}^{(k)} \mathbf{s}_k^m(\mathbf{x}) \qquad (j = 1, 2, \ldots, 2l+1) \tag{11}$$

and the associated eigenfrequencies are $\omega_k + \Omega_{jj}^{(k)}$.

3 The excitation of split singlets and time-dependent splitting theory

To obtain the excitation of the split eigenfunctions $\mathbf{u}_k^j(\mathbf{x})$ by a given source it is necessary to expand the equivalent body force density (Backus & Mulcahy 1976) in terms of $\mathbf{u}_k^j(\mathbf{x})$. This is most easily achieved by making use of the known expansion in terms of $\mathbf{s}_k^m(\mathbf{x})$. Suppose for an arbitrary vector field $\mathbf{f}(\mathbf{x})$ we have the expansion

$$\mathbf{f}(\mathbf{x}) = \sum_{km} f_k^m \mathbf{s}_k^m(\mathbf{x}). \tag{12}$$

Then from (11)

$$\mathbf{f}(\mathbf{x}) = \sum_{kj}\left(\sum_{m} U_{jm}^{(k)-1} f_k^m\right)\mathbf{u}_k^j(\mathbf{x}) \tag{13}$$

where the superscript -1 denotes the matrix inverse. It follows that if a_k^m are the excitation coefficients for the unperturbed model, then in the perturbed model the excitation coefficients are

$$b_k^j = \sum_m U_{jm}^{(k)-1} a_k^m\,; \tag{14a}$$

i.e. corresponding to (2) we have

$$\mathbf{s}(\mathbf{x}, t) = \sum_{kj} b_k^j \mathbf{u}_k^j(\mathbf{x}) \exp\left[i(\omega_k + \Omega_{jj}^{(k)})t\right]. \tag{14b}$$

Concerning the excitation of split singlets given by (14a, b) we make the following parenthetical remarks. Dahlen (1981) has recently considered the excitation of the normal modes of non-rotating model with an aspherical distribution of the quality factors Q_μ, Q_κ. This result is subsumed in (14a, b) since, as Dahlen has shown, $\mathbf{H}^{(k)}$ possesses in this case the symmetry

$$H_{mm'}^{(k)} = (-1)^{m+m'} H_{-m'-m}^{(k)}. \tag{15}$$

From this it follows that the columns of $\mathbf{U}^{(k)}$ may be normalized in such a way that

$$U_{jm}^{(k)-1} = (-1)^m U_{-mj}^{(k)} \tag{16}$$

and in this case (14a, b) reduce to the corresponding results of Dahlen (1981). However, in the case where the model is also rotating (15) is not longer valid and no further simplification of (14a, b) is possible unless the distributions of Q_μ, Q_κ are spherically symmetric. In this latter case it may be shown that the trace free part of $\mathbf{H}^{(k)}$ is Hermitean, and consequently the columns of $\mathbf{U}^{(k)}$ may be normalized in such a way that $\mathbf{U}^{(k)}$ is unitary, i.e.

$$U_{jm}^{(k)-1} = U_{mj}^{(k)*}. \tag{17}$$

Equations (14a, b) are generally valid and are also more convenient in the present application.

Using equation (11) in (14a, b) we may also write for the displacement field in the aspherical model

$$\mathbf{s}(\mathbf{x}, t) = \sum_{km'm} P_{mm'}^{(k)} a_k^{m'} \mathbf{s}_k^m(\mathbf{x}) \exp(i\omega_k t) \tag{18}$$

$$= \sum_{km} a_k^m(t) \mathbf{s}_k^m(\mathbf{x}) \exp(i\omega_k t) \tag{19}$$

where we have defined

$$P_{mm'}^{(k)} = P_{mm'}^{(k)}(t) = \sum_j U_{mj}^{(k)} \exp(i\Omega_{jj}^{(k)}t)\, U_{jm'}^{(k)-1} \tag{20}$$

$$a_k^m(t) = \sum_{m'} P_{mm'}^{(k)} a_k^{m'}. \tag{21}$$

In view of (10) equation (20) may also be written in matrix form

$$\mathbf{P}^{(k)}(t) = \exp\{i\mathbf{H}^{(k)}t\}. \tag{22}$$

If the $(2l+1)$ quantities a_k^m, and similarly $a_k^m(t)$ $(-l \leq m \leq l)$ are arranged as column vectors then we find that $\mathbf{a}_k(t)$ satisfies the system of ordinary differential equations

$$\frac{d}{dt}\,\mathbf{a}_k(t) = i\mathbf{H}^{(k)}\mathbf{a}_k(t) \tag{23}$$

subject to the initial condition

$$\mathbf{a}_k(0) = \mathbf{a}_k, \tag{24}$$

and that $\mathbf{P}^{(k)}(t)$ is the matrizant or propagator matrix of this system of equations (Coddington & Levinson 1955).

Equations (19), (23), (24) have the simple interpretation that initially the unperturbed singlets are excited just as they would be in the reference model; as time goes on, however, scattering and attenuation lead to the exchange of energy between singlets and to their eventual decay, this process being governed by the system of differential equations (23). This time dependent formulation of splitting theory besides being theoretically and numerically useful also lends physical insight into the process of scattering. When calculating the eigenfunctions of an aspherical model it is found, as may be expected, that the eigenfunctions given by (11) are highly sensitive to the nature and location of the heterogeneity. It may therefore be conjectured that if an experiment is designed to resolve the individual split mode shapes much information could be gained about the nature of the Earth's deviations from spherical symmetry. The reformulation above, however, indicates that such an experiment is doomed to failure since when the singlets are recombined to give an observable seismogram the heterogeneity enters only through $\mathbf{P}^{(k)}(t)$ given by (22), and it is clear that this will not display undue sensitivity.

Numerically equation (19) provides an efficient way of calculating time series in a heterogeneous earth. Suppose that $\mathbf{H}^{(k)}$ is known; then for a given series of time points 0, Δt, $2\Delta t \ldots$ the 'envelope' $\mathbf{a}_k(t)$ is given by

$$\mathbf{a}_k(0) = \mathbf{0},\ \mathbf{a}_k(n\Delta t) = \mathbf{P}^{(k)}(\Delta t)\mathbf{a}_k[(n-1)\Delta t]. \tag{25}$$

Thus when $\mathbf{P}^{(k)}(\Delta t)$ is calculated $\mathbf{a}(t)$ may be evaluated at the expense of one matrix multiplication per time step. Since $\mathbf{a}^{(k)}(t)$ varies on a scale very much slower than $\exp(i\omega_k t)$ the time step can be relatively long and interpolation to intermediate points will often be sufficiently accurate. $\mathbf{P}^{(k)}(\Delta t)$ may be calculated either by direct numerical integration of

$$\frac{d}{dt}\,\mathbf{P}^{(k)}(t) = i\mathbf{H}^{(k)}\mathbf{P}^{(k)}(t),\ \mathbf{P}^{(k)}(0) = \mathbf{1} \tag{26}$$

on the time interval $[0, \Delta t]$ or from the power series corresponding to (22); it may also be calculated by means of the singular value decomposition of the complex matrix $\mathbf{H}^{(k)}$, but in general this is probably not the most efficient way to proceed. Some examples of such calculations and their comparison with accelerometer data for some of the gravest modes of free oscillation will be described elsewhere.

4 The linearized equations and the spectral location parameter

The closing remarks of Section 3 are only incidental to the present application, since here we shall be concerned with the case in which $\mathbf{P}^{(k)}(t)$ is well approximated by the first two terms in its Taylor series:

$$\mathbf{P}^{(k)}(t) = \exp\{i\mathbf{H}^{(k)}t\} \simeq 1 + i\mathbf{H}^{(k)}t. \tag{27}$$

A condition which guarantees that this is the case for all times t is

$$\operatorname{Im}(\omega_k) \gg \max_j |\operatorname{Re}\Omega_{jj}^{(k)}|. \tag{28}$$

Physically this inequality corresponds to the case in which the mode decays so rapidly that scattering does not have time greatly to modify the envelope $\mathbf{a}_k(t)$. If this condition is violated there will be a finite time interval $[0, T]$ for which (27) remains a good approximation. Let $\Omega^{(k)}$ be a bound on $|\operatorname{Re}\Omega_{jj}^{(k)}|$. Then the time interval for which (27) is valid may be estimated from

$$T = T(k, \beta) = \sup\{T \in \mathbb{R} : (t \in [0, T]) \Rightarrow [\![\exp(-\alpha_k t)\,|\exp(i\Omega^{(k)}t) - 1 - i\Omega^{(k)}t| \leqslant \beta]\!]\} \tag{29}$$

where $\mathbb{R}$ is the set of real numbers, $\alpha_k = \operatorname{Im}(\omega_k)$ and β is a measure of the desired accuracy, which may be chosen to correspond to an assumed level of noise. We do not attempt to make the argument here more rigorous since (29) will be most useful simply for rough estimates, based on *a posteriori* estimates of $\Omega^{(k)}$. The time interval for which the subsequent analysis is useful will depend upon the frequency interval of the data to which it is applied.

Substituting, then, from (27) into (18) and making use of (6) we obtain the following result for a particular seismogram in the aspherical model:

$$\mathbf{v}\cdot\mathbf{s} = \sum_{km} R_k^m(\theta_r, \phi_r) S_k^m(\theta_s, \phi_s)(1 + i\lambda_k t)\exp(i\omega_k t) \tag{30}$$

where λ_k is the spectral location parameter of Jordan (1978):

$$\lambda_k \equiv \frac{\sum_{mm'} R_k^m(\theta_r, \phi_r) H_{mm'}^{(k)} S_k^{m'}(\theta_s, \phi_s)}{\sum_m R_k^m(\theta_r, \phi_r) S_k^m(\theta_s, \phi_s)}. \tag{31}$$

Equation (30) is simply a form of the Born approximation for seismic scattering when scattering between different multiplets is neglected, and could have been obtained independently of the splitting formalism used here. The advantage of the present approach is that information is gained on the conditions under which it is applicable and, in addition, the formulae for the matrix elements $H_{mm'}^{(k)}$, are in the literature (Woodhouse & Dahlen 1978). This connection with the Born approximation makes it fairly clear how the above analysis could be extended to include scattering between multiplets (i.e. quasi-degenerate coupling) but the details will be postponed to a later contribution.

5 Kernels for splitting matrix elements and the location parameter

The question which naturally arises, and has been considered in some detail by Jordan (1978) is: how do lateral variations in earth structure reflect themselves in the values of λ_k? The question may be answered simply by writing down the complicated expressions for the matrix elements $H_{mm'}^{(k)}$, in terms of the model perturbations and substituting into (31). In this section, however, we point out that a major simplification of the resulting expressions is possible, and leads to the result that the structural variations reflect themselves in λ_k simply through three local functionals of earth structure, which will be designated

$\delta\omega^{(i)}(\theta, \phi)$ $(i = 0, 1, 2)$, of which the first is $\delta\omega_{\text{local}}$. This result follows from (31) without making any further approximations.

Let us suppose that the aspherical model differs from the reference model by virtue of perturbations

$$\delta\rho_0^e + \delta\rho_0, \quad \delta\mu^e + \delta\mu, \quad \delta\kappa^e + \delta\kappa, \quad \delta\phi_0^e + \delta\phi_0, \quad \delta\partial_r\phi_0^e + \delta\partial_r\phi_0 \tag{32}$$

in density, shear modulus, bulk modulus, gravitational potential and the radial derivative of gravitational potential respectively, where superscript e indicates the contribution from the Earth's hydrostatic ellipticity of figure under the standard assumptions (Dahlen 1968). Each of these perturbations is a function of the three spatial coordinates. In addition the locations of internal surfaces of discontinuity and the free surface are perturbed by normal displacements $h^e(\theta, \phi) + h(\theta, \phi)$, these quantities being defined on each surface of discontinuity in the reference model. Correspondingly we may write

$$H_{mm'}^{(k)} = \hat{H}_{mm'}^{(k)} + \tilde{H}_{mm'}^{(k)} \tag{33}$$

$$\lambda_{mm'}^{(k)} = \hat{\lambda}_k + \tilde{\lambda}_k \tag{34}$$

where $\hat{H}_{mm'}^{(k)}$, $\hat{\lambda}_k$ are the contributions arising from rotation and ellipticity. These quantities may be calculated exactly for the reference earth model using the formulae given by Woodhouse & Dahlen (1978) and will not be of concern to us here. Let us denote the five perturbations in (32) by the algebraic vector

$$\delta\mathbf{m} = \delta\mathbf{m}(r, \theta, \phi) = (\delta\rho_0, \delta\mu, \delta\kappa, \delta\phi_0, \delta\partial_r\phi_0). \tag{35}$$

Both $\delta\mathbf{m}$ and h may be expanded in spherical harmonics (we use the convention of Edmonds 1960):

$$\delta\mathbf{m} = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \delta\mathbf{m}_s^t Y_s^t(\theta, \phi) \tag{36}$$

$$\delta h = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} \delta h_s^t Y_s^t(\theta, \phi). \tag{37}$$

Using the formulae of Woodhouse & Dahlen (1978) we may then write

$$\begin{aligned}\tilde{\mathbf{H}}_{mm'}^{(k)} = \sum_{st} \gamma_s^{mm't} \Bigg\{ & \int \delta\mathbf{m}_s^t \cdot [\![\mathbf{M}_k^{(0)}(r) + s(s+1)\mathbf{M}_k^{(1)}(r) + [s(s+1)]^2\mathbf{M}_k^{(2)}(r)]\!] r^2\, dr \\ & - \sum_d h_s^t [\![H_k^{(0)} + s(s+1)H_k^{(1)} + [s(s+1)]^2 H_k^{(2)}]\!]_-^+ \Bigg\}\end{aligned} \tag{38}$$

where $\mathbf{M}_k^{(i)}(r)$, $H_k^{(i)}(r)$ are given by known expressions in terms of scalar eigenfunctions of the reference model and, what is important here, are independent of s, t. Specific formulae are summarized in Table 2. The coefficients $\gamma_s^{mm't}$ are

$$\gamma_s^{mm't} = \int_\Omega Y_l^{m*} Y_s^t Y_l^{m'}\, d\Omega \tag{39}$$

where integration is over the surface of the unit sphere. In (38) the summation over d indicates a summation over all surfaces of discontinuity.

Table 2. Elements of kernel vectors $\mathbf{M}_k^{(i)}$ and interface coefficients $H_k^{(i)}$.

	$i=0$	$i=1$	$i=2$
$M_{1k}^{(i)}$	$\tfrac{1}{2}\omega[8\pi G\rho_0 U^2 - g_0 U(F+2r^{-1}U) + 2U\partial_r\phi_1 - \omega^2 U^2 - l(l+1)(\omega^2V^2 + \omega^2W^2 - 2r^{-1}V\phi_1 - r^{-1}g_0UV)]$	$\tfrac{1}{4}\omega[\omega^2V^2 + \omega^2W^2 - 2r^{-1}V\phi_1 - r^{-1}g_0UV]$	
$M_{2k}^{(i)}$	$\tfrac{1}{2}\omega\{\tfrac{1}{3}(2\partial_rU - F)^2 + l(l+1)r^{-2}[(r\partial_rV - V + U)^2 + (r\partial_rW - W)^2 + (l-1)(l+2)(V^2+W^2)]\}$	$-\tfrac{1}{4}\omega r^{-2}[\![(r\partial_rV - V + U)^2 + (r\partial_rW - W)^2 + [4l(l+1) - 2](V^2+W^2)]\!]$	$\tfrac{1}{4}\omega r^{-2}(V^2+W^2)$
$M_{3k}^{(i)}$	$\tfrac{1}{2}\omega(\partial_rU + F)^2$		
$M_{4k}^{(i)}$		$\tfrac{1}{4}\omega r^{-2}\rho_0[2U^2 + rU\partial_rV - V(r\partial_rU + 2rF - U)]$	
$M_{5k}^{(i)}$	$-\omega\rho_0 UF$	$\tfrac{1}{4}\omega\rho_0 r^{-1}UV$	
$H_k^{(i)}$	$\tfrac{1}{2}\omega\rho_0 r^2[8\pi G\rho_0U^2 - g_0U(F+2r^{-1}U) + 2U\partial_r\phi_1 - \omega^2U^2 - l(l+1)(\omega^2V^2 + \omega^2W^2 - 2r^{-1}V\phi_1 - r^{-1}g_0UV)] + \tfrac{1}{2}\omega\mu\{-\tfrac{1}{3}r^2(2\partial_rU - F)(2\partial_rU + F) + l(l+1)[(l-1)(l+2) \times (V^2+W^2) - (r\partial_rV - V + U)(r\partial_rV + V - U) - (r\partial_rW - W)(r\partial_rW + W)]\} + \tfrac{1}{2}\omega\kappa r^2(\partial_rU + F) \times (-\partial_rU + F)$	$\tfrac{1}{4}\omega\rho_0 r[r\omega^2(V^2+W^2) - 2V\phi_1 - g_0UV] + \tfrac{1}{4}\omega\mu[\![\tfrac{4}{3}rV(2\partial_rU - F) + (r\partial_rV - V + U) \times (r\partial_rV + V - U) + (r\partial_rW - W) \times (r\partial_rW + W) - [4l(l+1) - 2] \times (V^2+W^2)]\!] + \tfrac{1}{2}\omega\kappa rV(\partial_rU + F)$	$\tfrac{1}{4}\omega\mu(V^2+W^2)$

Equation (38) may be significantly simplified by noting that the summations over s, t can be performed explicitly. Writing ∇_1 for the gradient operator on the unit sphere:

$$\nabla_1 = \hat{\boldsymbol{\theta}}\partial_\theta + \hat{\boldsymbol{\phi}} \operatorname{cosec}\theta\,\partial_\phi \tag{40}$$

and noting that

$$-\nabla_1^2 Y_s^t = s(s+1) Y_s^t \tag{41}$$

we may use (39), together with Green's theorem on the sphere to write

$$\sum_{st} \left[\frac{s(s+1)}{2l(l+1)}\right]^i \gamma_s^{mm't} \delta \mathbf{m}_s^t(r) = \int_\Omega \delta \mathbf{m}(r,\theta,\phi) k_i^{mm'}(\theta,\phi)\, d\Omega \tag{42}$$

and similarly for H_s^t where the kernels are

$$k_i^{mm'}(\theta,\phi) = \frac{(-\nabla_1^2)^i Y_l^{m*} Y_l^{m'}}{[2l(l+1)]^i}; \quad (i = 0, 1, 2). \tag{43}$$

the factors $[2l(l+1)]^i$ have been introduced to normalize the kernels in a convenient but somewhat arbitrary way. Equation (38) becomes

$$\tilde{H}_{mm'}^{(k)} = \sum_{i=0}^{2} \int_\Omega \delta\omega_k^{(i)}(\theta,\phi) k_i^{mm'}(\theta,\phi)\, d\Omega \tag{44}$$

where the three local functionals

$$\delta\omega_k^{(i)}(\theta,\phi) \equiv [2l(l+1)]^i \left\{\int_0^a \delta\mathbf{m}(r,\theta,\phi)\cdot \mathbf{M}_k^{(i)}(r) r^2\, dr - \sum_d h(\theta,\phi)[H_k^{(i)}]_-^+\right\} \tag{45}$$

have been defined. It is remarkable that the angular dependence of $\delta\omega_k^{(i)}$ is sampled by only the three universal functions $k_i^{mm'}(\theta,\phi)$, and this circumstance greatly simplifies our conception of the effect of lateral heterogeneity upon observed seismograms. The location parameter defined in (31) may now be written

$$\tilde{\lambda}_k = \sum_{i=0}^{2} \int_\Omega \delta\omega_k^{(i)}(\theta,\phi) K_i(\theta,\phi)\, d\Omega \tag{46}$$

with kernels

$$K_i(\theta,\phi) = \frac{\sum_{mm'} R_k^m(\theta_r,\phi_r) k_i^{mm'}(\theta,\phi) S_k^{m'}(\theta_s,\phi_s)}{\sum_m R_k^m(\theta_r,\phi_r) S_k^m(\theta_s,\phi_s)}. \tag{47}$$

For explicit calculations of $K_i(\theta,\phi)$ the formalism of Phinney & Burridge (1973) is again useful. It may be shown that

$$k_i^{mm'}(\theta,\phi) = \frac{2l+1}{4\pi} \sum_{N=-i}^{i} a_i^N Y_l^{Nm*}(\theta,\phi) Y_l^{Nm'}(\theta,\phi) \tag{48}$$

where a_i^N are given in Table 3. Substituting into (47) and making use of (8), (9), together with the addition theorem for generalized spherical harmonics (or, equivalently, making use

Table 3. Coefficients a_i^M for the evaluation of path fraction kernels.

	a_1^M	a_2^M	a_3^M
$M = 0$	1	1	$\frac{3}{2}$
$M = \pm 1$		$-\frac{1}{2}$	$[2l(l+1)]^{-1} - 1$
$M = \pm 2$			$\frac{1}{4} - [2l(l+1)]^{-1}$

of a coordinate system in which (θ, ϕ) is loated at the pole) we find

$$K_i(\theta, \phi) = \frac{1}{p_k} \cdot \frac{2l+1}{4\pi} \cdot \sum_{m=-i}^{i} a_i^m R_k^m(\theta_{rp}, \phi_{rp}, \pi - \phi_{pr}) S_k^m(\theta_{sp}, \phi_{sp}, \pi - \phi_{ps}) \tag{49}$$

with

$$\left.\begin{aligned} R_k^m(\theta, \phi, \gamma) &= \sum_{N=-1}^{1} R_{kN} Y_l^{Nm}(\theta, \phi) \exp(iN\gamma) \\ S_k^m(\theta, \phi, \gamma) &= \sum_{N=-2}^{2} S_{kN} Y_l^{Nm*}(\theta, \phi) \exp(-iN\gamma) \end{aligned}\right\} \tag{50}$$

and where θ_{pr}, ϕ_{pr}, ϕ_{ps} denote the angular distance and azimuth of the point (θ, ϕ) from the point (θ_r, ϕ_r) and the azimuth of the point (θ, ϕ) from the point (θ_s, ϕ_s) with others defined analogously. Azimuth is here measured from south to east; i.e. it is the supplement of conventional azimuth. The normalization constant p_k in (49) is

$$\left.\begin{aligned} p_k &= \sum_{m=-2}^{2} R_k^m(\theta_{sr}, 0, \pi - \phi_{sr}) S_{km} \exp(im\phi_{rs}) \\ &= \sum_{m=-1}^{1} R_{km} \exp(-im\phi_{sr}) S_k^m(\theta_{rs}, 0, \pi - \phi_{rs}), \end{aligned}\right\} \tag{51}$$

these two equivalent representations corresponding to the coordinate systems in which the source is at the pole or the receiver at the pole, respectively; both are equivalent to the expressions given by Gilbert & Dziewonski (1975) for the amplitude of the kth mode observed by a given instrument.

The kernels $K_i(\theta, \phi)$ given by (49) are readily calculated. Indeed for an azimuthally symmetric source and a vertical instrument $K_0(\theta, \phi)$ is simply proportional to the product of Legendre polynomials: $P_l(\cos\theta_{ps})\ P_l(\cos\theta_{pr})$. It may be shown from (47), (48) that they are normalized according to

$$\int_\Omega K_i(\theta, \phi)\, d\Omega = \delta_{0i}. \tag{52}$$

The general character of the kernels is illustrated in Figs 1–3. In each of these figures the kernels are plotted in a three-dimensional representation where the horizontal plane corresponds to a Mercator projection of the Earth's surface. The coordinate system is chosen so that both source and receiver lie on the 'equator'; in each case the source is located at the rightmost (i.e. 'eastmost') end of the plots and the receiver 108° to the 'west'. In each plot a peak is located at the source and receiver and their antipodes; the receiver is assumed to be a vertical component instrument. These kernels represent the sensitivity of

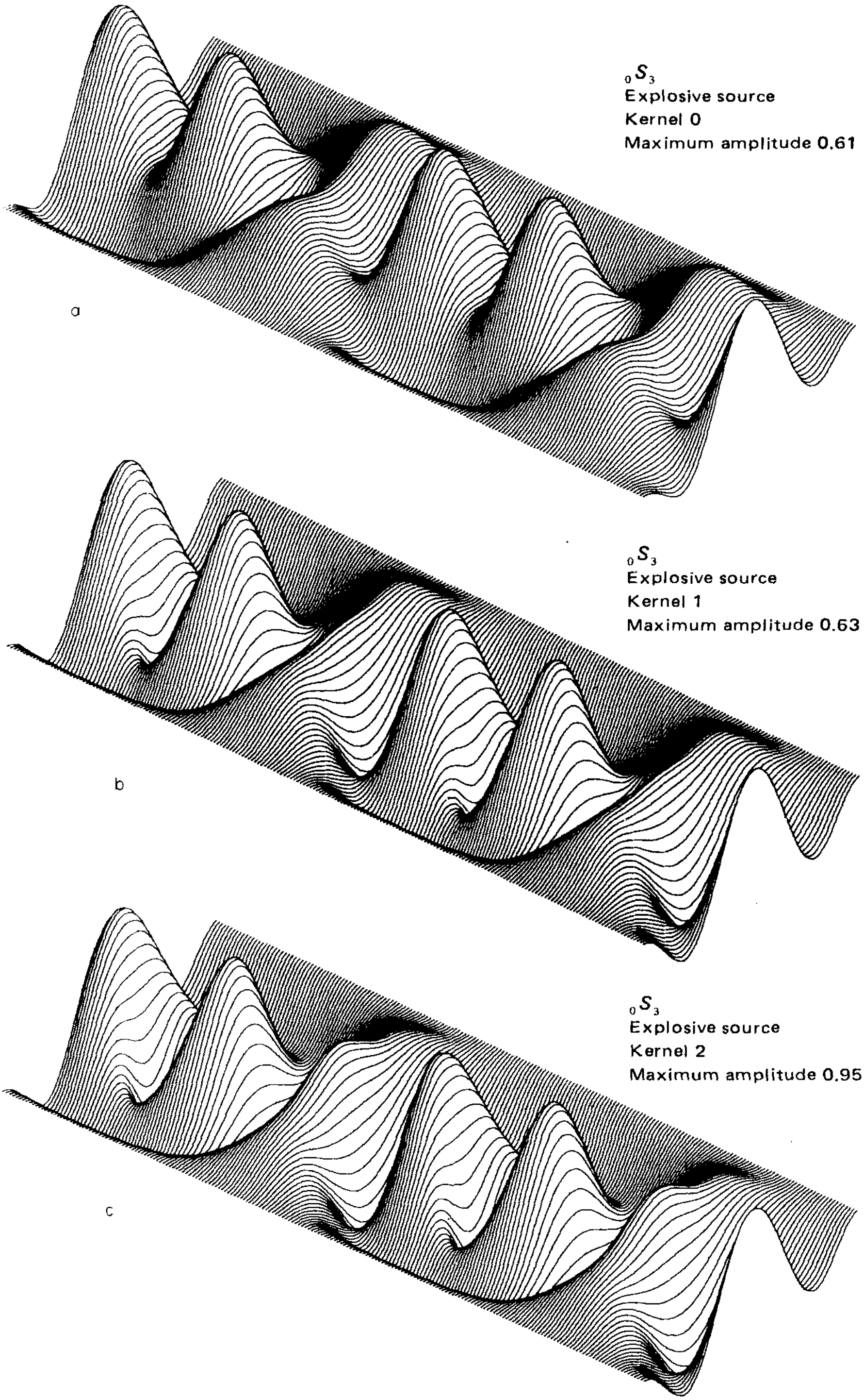

Figure 1. (a, b, c) The three kernels $K_i(\theta, \phi)$ are shown, for the mode $_0S_3$, in a three-dimensional representation in which the horizontal plane represents a Mercator projection of the Earth's surface, with the 'equator' chosen to be the great circle path between a source and receiver placed 108° apart. The source is explosive and located at the 'eastmost' (i.e. rightmost) end of the plot, with a vertical instrument placed 108° to the 'west'.

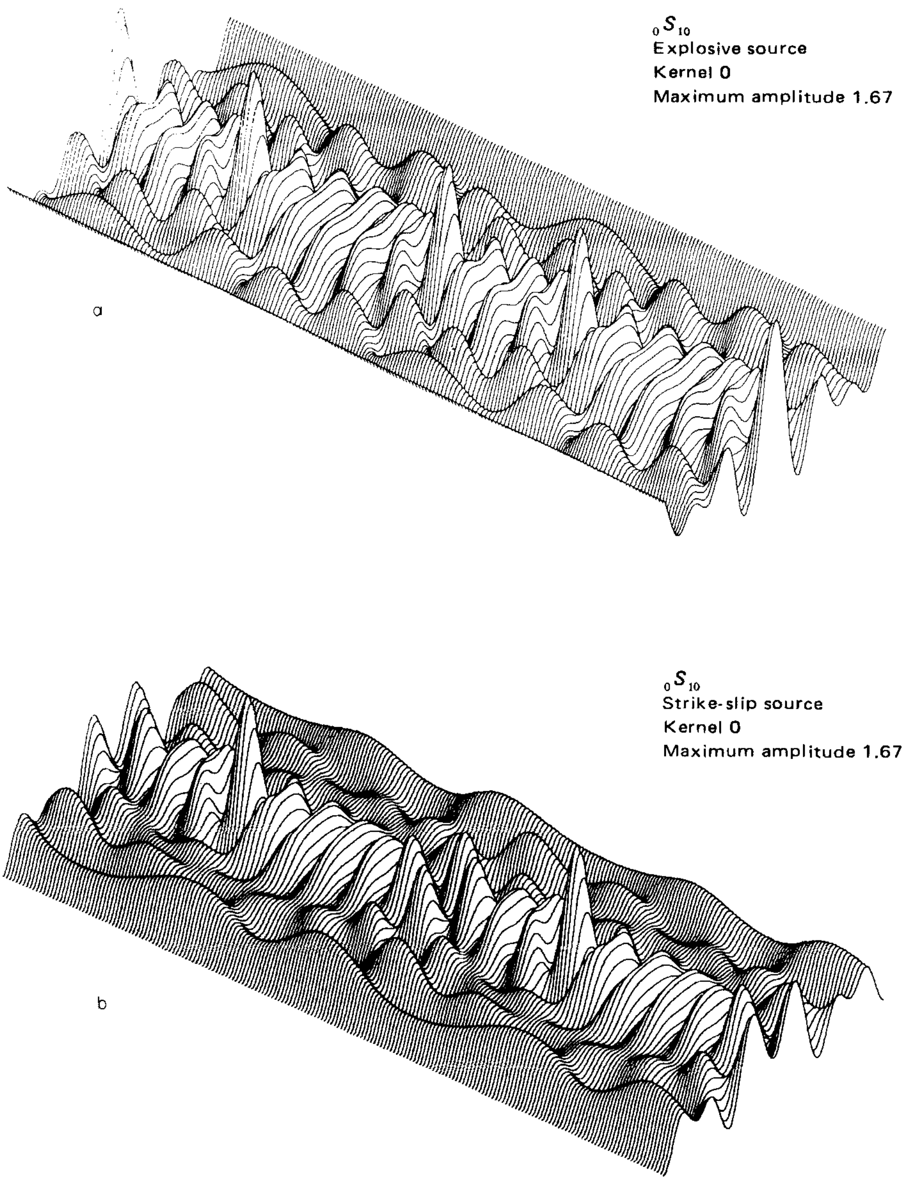

Figure 2. (a, b) As for Fig. 1 except that the kernel K_0 is represented for two different sources. In (a) the source is explosive and in (b) it is a strike-slip fault with strike inclined at 45° to the 'equator'.

eigenfrequency measurements from single records to geographical structural variations. In Fig. 1(a, b, c) the three kernels are plotted for $_0S_3$ and an explosive source. At such low angular orders very little corresponding to a geometrical interpretation can be seen. It is noteworthy that the kernels K_1, K_2 have a similar shape to K_0 except that they possess deep negative valleys. We have checked numerically that K_0 integrates to unity and K_1, K_2 to zero over the Earth's surface, as required by equation (49). In Fig. 2(a, b) are depicted the kernels K_0 corresponding first to an explosive source and second to vertical strike-slip with the fault plane inclined at 45° to the equator. It will be seen that the kernels are

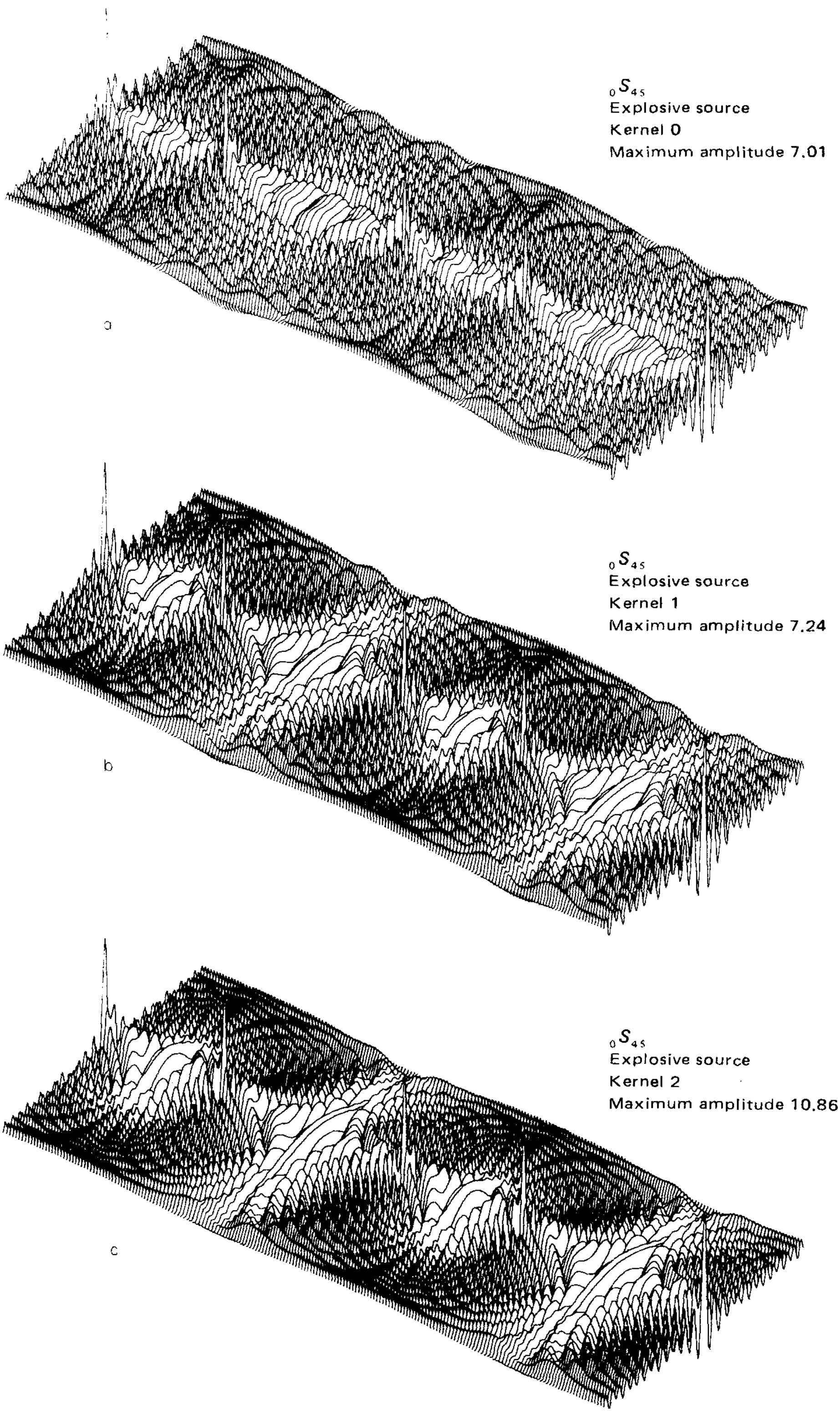

Figure 3. (a, b, c) As for Fig. 1 but for the mode $_0S_{45}$.

rather similar close to the equator, except near the source, but have 'side lobes' which are quite different. In both these plots the ridge along the equator shows that the asymptotic property of the kernels derived by Jordan (1978) is beginning to emerge; when integrated against a smooth perturbation the oscillations tend to cancel in the integral, but the systematic equatorial peak gives approximately the great circle average. Fig. 3(a, b, c) shows the same plots as for Fig. 1, this time for the mode ${}_0S_{45}$. The equatorial peak in Fig. 3(a) (actually better described as a stationary phase point) is clearly defined but it is notable that it is still very broad and the kernel does not become uniformly small towards the poles as may have been expected on the basis of the asymptotic result. Of course for receivers close to the source or its antipodes the kernels would not be localized along any great circle, and we may expect progressive degradation of the asymptotic result as these configurations are approached. The analysis in terms of the path fraction kernels K_i remains valid in this case.

6 Path fractions in a regionalized earth

Consider an earth model comprising a number of regions $\mathscr{R}_j$ $(j=1, 2 \dots J)$ characterized by the functions $n_j = n_j(\theta, \phi)$. For a strict regionalization we could define:

$$n_j(\theta, \phi) = \begin{cases} 1 & (\theta, \phi) \in \mathscr{R}_j \\ 0 & (\theta, \phi) \notin \mathscr{R}_j \end{cases} \tag{53}$$

but more generally we can allow n_j to be arbitrary functions of (θ, ϕ) subject to

$$\sum_{j=1}^{J} n_j(\theta, \phi) = 1. \tag{54}$$

The functions used for the calculations which follow are of this kind and are defined to be constant in $5° \times 5°$ squares and to represent the fraction of the square occupied by terrains of the jth tectonic type. They are based upon the regionalization of Mauk (1977). If each region, or regional type is assumed to be laterally uniform we may write (equation 45)

$$\delta\omega_k^{(i)}(\theta, \phi) = \sum_{j=1}^{J} n_j(\theta, \phi)\delta\omega_{kj}^{(i)} \tag{55}$$

where $\delta\omega_{kj}^{(i)}$ is independent of (θ, ϕ) and is given by (45) in terms of the perturbations in structural parameters $\delta\mathbf{m}_j(r)$, h_j which characterize the jth regional type. In this connection one caveat must be made. The perturbations in gravitational potential and its derivative will not be uniform in each region even if the perturbation $\delta\rho$ does have this property. This is because the gravitational potential in one region will depend upon the density distribution in the others. For this reason (55) is only strictly valid if the density is unperturbed or is perturbed in a spherically symmetric way. If the density is perturbed aspherically (55) will correctly reflect the inertial effects of the perturbation, but not the influence which the consequent perturbation in gravitational forces exerts on the oscillations. We do not consider this a serious shortcoming, since it is probably a very small effect for all modes, and in any case the relative density variations are likely, on physical grounds, to be much smaller than those of μ, Q_μ, κ, Q_κ.

Adopting equation (55) and substituting into (46) we may write

$$\tilde{\lambda}_k = \sum_{i=0}^{2} \sum_{j=1}^{J} \delta\omega_{kj}^{(i)} p_{kj}^{(i)} \tag{56}$$

with

$$p_{kj}^{(i)} \equiv \int_{\Omega} n_j(\theta, \phi) K_i(\theta, \phi)\, d\Omega. \tag{57}$$

The quantity $p_{kj}^{(i)}$ may be thought of as the path fraction of the ith kind for the jth region, by analogy with the asymptotic result; in fact if the path fraction of the zeroth kind is replaced by the geometrical path fraction, and if the other two path fractions ($i = 1, 2$) are neglected, then (56) reduces to the asymptotic result of Jordan (1978). The significance of (56) in the present context is that it may be used as the basis for a 'pure path' approach to long-period data without making approximations beyond the neglect of the gravitational effects mentioned above, along with the approximations inherent in linearized perturbation theory itself. By virtue of (52), (54) they satisfy

$$\sum_{j=1}^{J} p_{kj}^{(i)} = \delta_{0i}. \tag{58}$$

The principal complication in using (56) is that the path fractions depend upon the moment tensor of the source, as well as the orientation of the instrument and the locations of both source and receiver. The dependence of the path fraction kernels on the source mechanism is illustrated in Fig. 2(a, b); it is not known at this time whether this effect could greatly affect the interpretation, but in any case the determination of source parameters from long-period digital data is becoming increasingly routine and the necessity of knowing the source should not be a serious drawback. For sources very close to the surface it is often impossible to resolve certain components of the moment tensor, but for the present application all that is needed is a source giving a reasonable fit to the data; the components which are indeterminate are those associated with synthetic seismograms of very small amplitude, and for this reason they are components which are not needed in the application envisaged here.

Knowing the source and receiver, the path fractions are probably most easily calculated by first expanding the functions $n_j(\theta, \phi)$ in spherical harmonics:

$$n_j(\theta, \phi) = \sum_{s=0}^{\infty} \sum_{t=-s}^{s} n_{js}^{t} Y_s^t(\theta, \phi). \tag{59}$$

The path fractions are then given by:

$$p_{kj}^{(i)} = \frac{1}{p_k} \sum_{\substack{s=0 \\ (s \text{ even})}}^{2l} \sum_{m=-l}^{l} \sum_{m'=-l}^{l} R_k^{m'}(\theta_r, \phi_r) S_k^{m*}(\theta_s, \phi_s) \left[\frac{s(s+1)}{2l(l+1)}\right]^i \times (-1)^{m'} (2l+1) \left(\frac{2s+1}{4\pi}\right)^{1/2} \begin{pmatrix} l & l & s \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l & s \\ -m' & m & m'-m \end{pmatrix} n_{js}^{m'-m} \tag{60}$$

where the Wigner $3-j$ symbols (Edmonds 1960) have been introduced. There would seem to be no particular advantage in referring this calculation to any special system of coordinates since for each new system the coefficients n_{js}^{t} must be rotated.

We have performed some calculations to illustrate the behaviour of these path fractions and to compare $p_{kj}^{(0)}$ with the geometrical path fractions obtained by averaging $n_j(\theta, \phi)$ along the appropriate great circle paths.

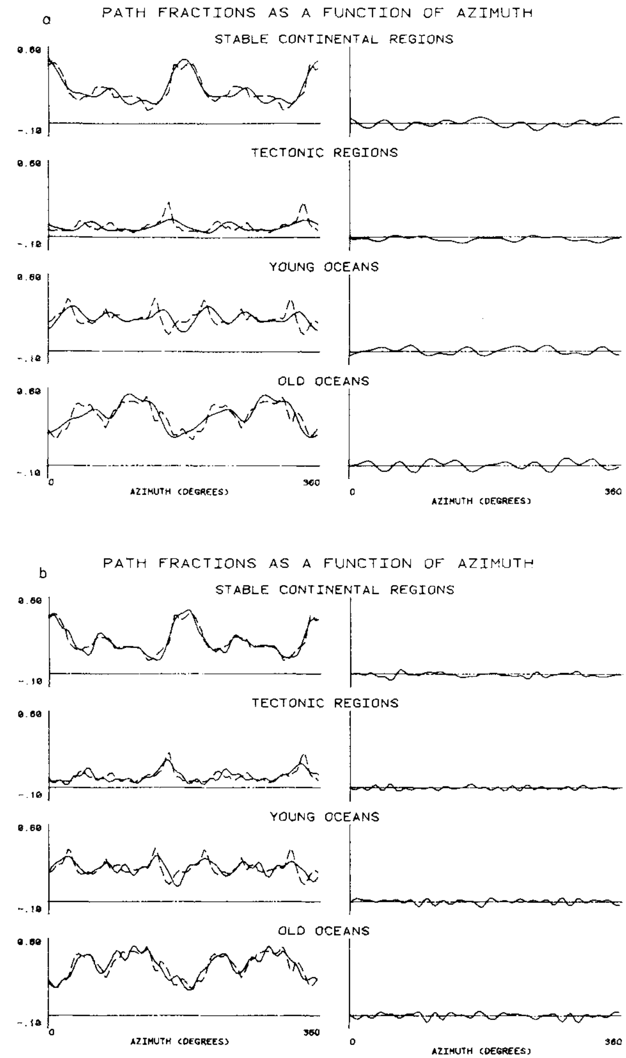

Figure 4. (a, b, c, d) Path fractions are plotted as a function of azimuth of receiver from source. The source is explosive at 0° N, 90° E and the receivers are vertical component instruments placed at a uniform distance of 89° in the case of (a, b and c), and at a distance of 27° in the case of (d). The figures are for the modes ${}_0S_{10}$, ${}_0S_{20}$, ${}_0S_{45}$, ${}_0S_{48}$, respectively. The dashed line is the same in each figure and represents the geometrical path fractions. The regionalization adopted is after Mauk (1977). Mauk's regions have been combined as follows: *stable continental regions*: Archean shield, Proterozoic shield, Precambrian undeformed, early Palaeozoic orogeny, late Palaeozoic orogeny, post-Precambrian undeformed, shelf sediments; *tectonic regions*: Mesozoic orogeny, Cenozoic folding, Cenozoic volcanics, Mesozoic volcanics, intermontane basin fill, island arcs; *young oceans*: anomaly 0–5 (0–10 Myr), anomaly 5–6 (10–20 Myr), anomaly 6–13 (20–38 Myr); *old oceans*: anomaly 13–25 (38–63 Myr), late Cretaceous seafloor (63–100 Myr), early Cretaceous seafloor (100–140 Myr), seafloor older than 140 Myr.

In Fig. 4(a, b, c) results are shown for ${}_0S_{10}$, ${}_0S_{20}$, ${}_0S_{45}$. A hypothetical, explosive source has been placed on the equator at 90° E and the path fractions calculated for a large number of receivers lying on a circle about the source of radius 89°; azimuth is anticlockwise from south, as before. The dashed lines represent the geometrical path fractions and in the left and right panels of each figure are shown $p_{kj}^{(0)}, p_{kj}^{(1)}$, respectively. The regionalization is after

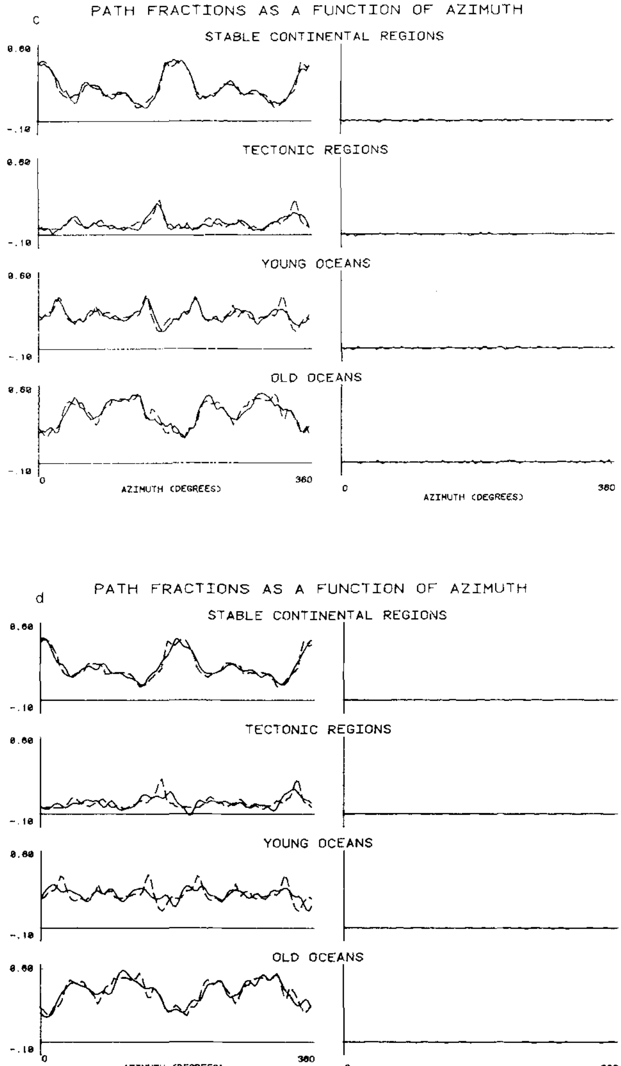

Figure 4 (c, d)

Mauk (1977) whose 20 regions have been combined into the four categories shown. The categories are detailed in the figure caption, but are not of immediate concern for this illustrative calculation. These are 'best case' examples, in that the receiver is near an antinode and at a distance close to 90° where the asymptotic result may be expected to be most nearly valid. Nevertheless it is clear that significant deviations occur which are particularly important for small regions (see 'Tectonic regions' in particular). Fig. 4(d) shows a similar plot for ${}_0S_{48}$ observed at 27° where the deviations are more pronounced. Results would be similar for more realistic sources but would be complicated by large excursions close to azimuthal radiation pattern modes. At nodes the denominator in (47) vanishes and the path fractions are singular. Of course the spectral peaks would be small near such nodes and the largest excursions would not be observed. A similar phenomenon will be discussed later in relation to Fig. 5. This behaviour is simply an artefact of the definition of λ_k, and the contribution to the seismograms is always finite (see equation 30). For this reason we

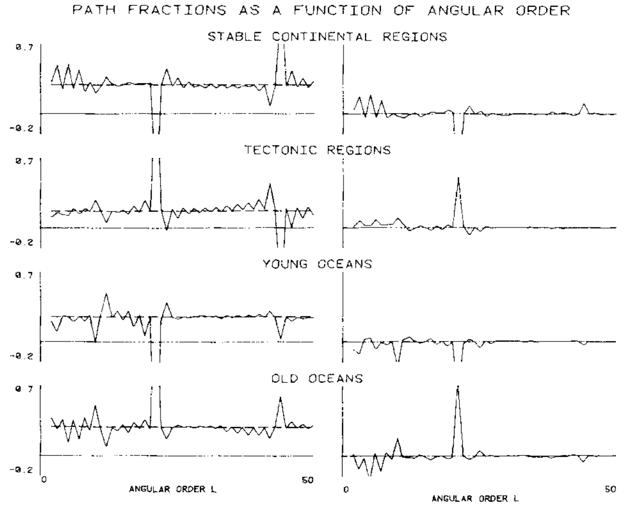

Figure 5. Path fractions as a function of angular order for fundamental spheroidal modes and a path between the Kermadecs and Naña, Peru. The source used was a double couple, with an orientation corresponding to the tectonic setting.

shall suggest in Section 7 a different approach to the inversion of the long-period data which does not involve the measurement of spectral location parameters.

Fig. 5 shows the path fractions as a function of angular-order for an earthquake in the Kermadecs observed at NNA Peru (distance $\Delta \simeq 93^\circ$). Again the largest excursions occur at values of l for which the receiver is close to a node. This particular path was chosen because Silver & Jordan (1981, fig. 9) have displayed their results for the location parameter corresponding to this path. The pattern of rapid oscillations (asymptotically behaving like $\tan((l+½)\Delta - \pi/4))$ predicted by the theory is clearly seen in the data and represents evidence that the asymptotic interpretation is invalid. Indeed in the asymptotic interpretation this roughness will translate into a roughness of $\delta\omega_{\mathrm{local}}$ as a function of l, for each region – an impossibility for any reasonable earth models since $\delta\omega_{\mathrm{local}}$ must be analytic in l. Again this erratic behaviour may be regarded as an artefact of the definition of the location parameter. A further undesirable effect is the aliasing introduced by sampling this oscillating function at integer values of l, which is capable of producing spurious trends in the results.

7 Conclusions

It has been shown that a simplification of splitting theory is possible, and this has been used to investigate the sensitivity of spectral measurements to lateral variations in earth structure. The natural application of these results is in elucidation of the Earth's deviations from spherical symmetry.

Equation (30) gives a direct linear relationship between the observed seismograms and the laterally heterogeneous model perturbations and although we have focused most attention on fundamental modes, (30) remains valid for overtones and forms a natural basis for the attempt to invert waveform data for regional structural parameters. It could also form the basis for investigating certain statistical properties of the Earth's lateral heterogeneity. It does not seem appropriate here to present a method of inversion in great algebraic detail; the implementation of the method is a subject of current research, and experience gained with the data will doubtless influence the development of the specific algorithms adopted.

Dziewonski & Steim (1982) have recently developed a technique for the interpretation of great circle surface wave dispersion which has proved successful in obtaining smooth regional dispersion curves directly from the long-period waveforms. One essential element in the technique is the use of the partial derivatives of the predicted waveforms (or spectra) with respect to some chosen parameters characterizing regional structural perturbations, in order iteratively to refine the estimates of these parameters. All intermediate steps, such as the isolation of individual orbits and the evaluation of correlation functions are eliminated. A technique similar in philosophy has been used by Dziewonski, Chou & Woodhouse (1981) to locate an earthquake and determine its moment tensor simultaneously. Also Woodhouse (1979) has shown that travel-time data may be inverted directly without the usual intermediate steps of smoothing the observed travel times or of estimating the intercept time as a function of ray parameter.

Similarly equation (30) may be used to calculate the partial derivatives of the waveforms or spectra with respect to parameters characterizing regional structural variations. Using equations (56) and (45) this can be achieved at a small fraction of the computational effort which would appear to be necessary using the conventional formulation of splitting theory, even though the two formulations are essentially equivalent in the regime where the approximation (27) is valid – which we believe to include a large part of the low-frequency seismic spectrum. Where (27) is not valid no linearized theory is correct and an iterative scheme will be necessary in which perturbations of the splitting matrix away from some non-zero value will have to be considered. The linearized scheme outlined here is the first step in such a procedure.

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