*Geophys. J. Int.* (1991) **105**, 649–657

# The relative amplitudes of mantle heterogeneity in *P* velocity, *S* velocity and density from free-oscillation data

X.-D. Li,* D. Giardini† and J. H. Woodhouse‡
*Department of Earth and Planetary Sciences, Harvard University, Cambridge, Massachusetts* 02138, *USA*

Accepted 1990 December 21. Received 1990 December 10; in original form 1990 February 9

## SUMMARY

Splitting functions retrieved from spectra of the free oscillations are sensitive to the lateral variations in *P* velocity ($\alpha$), *S* velocity ($\beta$), and density ($\rho$) simultaneously. In this study they are used to constrain the values of the ratios $d \ln \alpha / d \ln \beta$ and $d \ln \rho / d \ln \alpha$ for the lower mantle. Assuming that the upper mantle structure is obtainable from model M84A (this is not a crucial assumption as experiments indicate), the optimal value of $d \ln \alpha / d \ln \beta$ inferred from the modal data is 0.44 and $d \ln \alpha / d \ln \beta$ lies in the interval (0.39, 0.60) with 75 per cent confidence, strongly discriminating against the value (0.8) often used. The constraints on density structure of the current data are insufficient to yield new definitive results. The analysis demonstrates, however, that the value of $d \ln \rho / d \ln \alpha$ could be estimated from a larger set of modal data.

**Key words:** free oscillations, mantle heterogeneity, tomography.

## 1 INTRODUCTION

Lateral heterogeneity in seismic velocities and density in the mantle reflects variations in temperature and, possibly, in chemical composition. Thus knowledge of the relationship among the perturbations in *P* velocity ($\alpha$), *S* velocity ($\beta$), and density ($\rho$), in conjunction with experimental results on rock properties at mantle conditions, has the potential to discriminate between different mineralogies and different hypotheses concerning the cause of heterogeneity. Based upon laboratory experiments on the change in rock properties with temperature, Anderson *et al.* (1968) have concluded $d \ln \alpha / d \ln \beta = 0.8$ and $d \ln \rho / d \ln \alpha = 0.5$ and these values have been often adopted for the Earth's mantle (e.g., Forte & Peltier 1987; Ritzwoller, Masters & Gilbert 1988). However, other authors have questioned the validity of these values for the lower mantle (Anderson 1987; Yeganeh-Haeri, Weidner & Ito 1989), where temperature and pressure are simultaneously high and many material properties are still unknown.

Many previous attempts to estimate $d \ln \alpha / d \ln \beta$ from seismic data have been based upon the comparison *P* and *S* station corrections (e.g., Hales & Doyle 1967; Wichens & Buchbinder 1980), which largely reflect upper mantle heterogeneity. Such studies generally concern limited areas for which *S* arrival times have been carefully reread for selected earthquakes. Souriau & Woodhouse (1985) have addressed the problem by making a worldwide comparison between the predicted *S*-wave delays of the upper mantle model M84C (Woodhouse & Dziewonski 1984) with *P* station corrections (Dziewonski & Anderson 1983). These studies generally indicate low values (0.42–0.75) of $d \ln \alpha / d \ln \beta$ for the upper mantle, a result which has been ascribed to the existence of partial melting in the upper mantle (Hales & Doyle 1967).

Regional studies of *ScS–S* and *PcP–P* differential traveltimes (e.g., Jordan & Lynn 1974; Lay 1983) have also suggested a low value (~0.5) of $d \ln \alpha / d \ln \beta$ in the lower mantle, at least for those regions studied.

In recent years, large-scale 3-D mantle models have been developed for both $\alpha$ and $\beta$. Direct comparison of these models immediately yields estimates on the value of $d \ln \alpha / d \ln \beta$. For example, Morelli & Dziewonski (1987) have developed a *P* velocity model V.3 for the lower mantle based on *P* traveltime residuals [this model is very similar to the model L02.56 of Dziewonski (1984)]; by modelling the waveforms of *SH* body waves, Woodhouse & Dziewonski (1986) have constructed a model of shear velocity heterogeneity in the lower mantle (this model will be referred to as SW).

As pointed out by Dziewonski & Woodhouse (1987), the value of $d \ln \alpha / d \ln \beta$ in the lower mantle determined from these two models is also low (<0.5). This is a particularly

* Now at Seismological Laboratory, Caltech, Pasadena, California 91125, USA.
† Now at Istituto Nazionale di Geofisica, Roma, Italy.
‡ Now at Department of Earth Sciences, Oxford University, Oxford OX1 3PR, UK.

interesting result, since partial melting is a less likely explanation for the lower mantle than for the upper mantle. Two possible explanations involving the physics of mantle minerals at lower mantle conditions have been proposed by Anderson (1987) and by Yeganeh-Haeri *et al.* (1989). Owing to the imperfect, and different, resolution of these two models, however, there remains the possibility that one or both of them are underestimates or overestimates of the true heterogeneity of the lower mantle. It is also possible that the magnitude of heterogeneity is frequency dependent, and thus the results of comparing models based upon data at different frequencies could be biased by such an effect.

In this study, we approach the problem by making use of the spectral splitting of the Earth's free oscillations, which is sensitive to the perturbations in $\alpha$ and $\beta$ simultaneously and thus provides constraints on $d \ln \alpha / d \ln \beta$ from the same kind of data.

Based upon the splitting of normal modes, Giardini, Li & Woodhouse (1987, 1988) reported evidence that lateral variations in $P$ velocity are proportional to those in $S$ velocity in the lower mantle. In these studies the optimal value for $d \ln \alpha / d \ln \beta$, assumed constant in the lower mantle, was found to be approximately 0.4. This result was somewhat preliminary and potentially is open to the criticism (Ritzwoller *et al.* 1988) that a larger value could be accommodated if a suitable model of core–mantle boundary (CMB) topography were introduced. Hence the evidence for the small value of $d \ln \alpha / d \ln \beta$ given in Giardini *et al.* (1987) was not unequivocal.

It is our purpose in this paper to treat the problem more completely, and to determine confidence intervals for the derived values. We make use of the splitting coefficients (Giardini *et al.* 1988; Li, Giardini & Woodhouse 1991) of 17 modes which are primarily or solely sensitive to mantle structure—which we term *mantle modes*. The selected modes may be partitioned into two categories: (1) modes whose sensitivity in the lower mantle is mainly to $S$ velocity heterogeneity and (2) modes primarily sensitive to $P$ velocity structure. In order to eliminate possible contamination from CMB topography, we group the CMB-sensitive modes into pairs. For each pair, a combined set of splitting coefficients is constructed by taking linear combinations of the splitting coefficients of the individual modes in such a way that the resulting coefficients have vanishing sensitivity to CMB topography. The combined splitting coefficients are then used to constrain the value of $d \ln \alpha / d \ln \beta$ with no contamination from CMB structure. Since the splitting functions currently retrieved are insufficient to yield independent heterogeneous models of $\alpha$ and $\beta$, from which the approach to the problem of estimating $d \ln \alpha / d \ln \beta$ would be very straightforward, we perform the analysis in the data (splitting coefficient) space with the help of pre-existing heterogeneous mantle models. The amplitude ratios of the observed splitting functions and the synthetic splitting functions predicted by a particular mantle model are first estimated for $\alpha$-sensitive modes and $\beta$-sensitive modes, respectively. The value of $d \ln \alpha / d \ln \beta$ is then obtained by comparison of these two ratios. In such a scheme, the estimate of $d \ln \alpha / d \ln \beta$ is constrained only by the modal data under the assumption that the mantle model predicts the heterogeneity pattern of reality, and is independent of the amplitude of the mantle model used. (However, the amplitude will influence the discussion on physical dispersion effects in Section 5.) Experiments are performed by using two different mantle models and yield very similar results for $d \ln \alpha / d \ln \beta$, indicating that the uncertainty in the patterns of the used models is a minor factor in inferring the value of $d \ln \alpha / d \ln \beta$.

Another important geophysical problem is that of the relationship between heterogeneity in density and in seismic velocities. The calculation of the geoid and of plate motions using 3-D mantle models (Richards & Hager 1984; Forte & Peltier 1987) clearly requires positive values of the ratios $d \ln \rho / d \ln \alpha$ and $d \ln \rho / d \ln \beta$, consistent with a thermal origin for mantle heterogeneity, but constrain these parameters only weakly owing to trade-offs with viscosity structure and to other uncertainties. Consequently, independent estimates of these ratios would lead to more accurate estimates of mantle viscosity. The study of the Earth's free oscillations provides the sole means of placing seismological constraints on the interior distribution of density anomalies. In this paper, we demonstrate the possibility of estimating the value of $d \ln \rho / d \ln \alpha$ and examine the power of resolution of the current data.

## 2 SPLITTING FUNCTIONS AND EARTH STRUCTURE

Knowing the source parameters of earthquakes, the seismic displacement fields are solely determined by the structure and the rotation of the Earth. As a result of first-order perturbation theory, the dependence on the earth structure, for an isolated multiplet can be fully described by a function defined on the surface of a sphere, termed the splitting function of the multiplet (Woodhouse & Giardini 1985). Giardini *et al.* (1988) have presented the theory and the inversion results for the splitting coefficients, $c_{st}$, of a number of long-period modes, which constitute the data for the present study.

Splitting function coefficients are related only to the spherical harmonic components of the aspherical structure of the Earth of the same degree, $s$, and order, $t$. For the mantle modes concerned in this study, we may write

$$c_{st} = \int_{r_C}^{a} \left( A_s \frac{\delta \alpha_{st}}{\alpha} + B_s \frac{\delta \beta_{st}}{\beta} + R_s \frac{\delta \rho_{st}}{\rho} \right) dr + \sum_d H_s^d \delta h_{st}^d, \qquad (1)$$

where $a$ and $r_C$ are the radii of the surface and of the CMB, respectively; the summation is over all discontinuities; $\delta \alpha_{st}$, $\delta \beta_{st}$, and $\delta \rho_{st}$ are the spherical harmonic coefficients of the perturbations in $P$ velocity, $S$ velocity, and density, respectively; $\alpha$, $\beta$ and $\rho$ are the values of $P$ velocity, $S$ velocity, and density, from the spherical reference model (the PREM model of Dziewonski & Anderson 1981); $\delta h_{st}^d$ are the spherical harmonic coefficients of the undulation of the $d$th discontinuity normalized by its unperturbed radius; $A_s(r)$, $B_s(r)$, $R_s(r)$ and $H_s^d$ are differential kernels of the multiplet, and are obtainable from the spherical reference earth model [Woodhouse & Dahlen (1978); explicit formulae using the current notation are given by Li *et al.* (1991), equations (7), (8) (9) and (36)].

## 3 PROPORTIONALITY OF $\alpha$ AND $\beta$ HETEROGENEITIES IN LOWER MANTLE

In terms of their sensitivities to the lower mantle heterogeneity, modes may be partitioned into two groups: $\alpha$-sensitive modes and $\beta$-sensitive modes. The splitting functions of $\beta$-sensitive modes and the $P$ velocity model V.3 (Morelli & Dziewonski 1987) may be used to constrain the value of $d \ln \alpha / d \ln \beta$, and $\alpha$-sensitive modes can be used to determine the extent to which the magnitude of heterogeneity in the model V.3 is consistent with modal splitting; this will be quantified in terms of a multiplying factor by which the perturbations of V.3 need to be amplified or deamplified in order to obtain agreement with the modal results. This factor could also be interpreted as a measure of the frequency dependence of heterogeneity, since V.3 is based upon traveltime anomalies of waves having periods of approximately 1 s, whereas the modal periods are more than two orders of magnitudes greater. By estimating and making use of this factor, a correction can be made to eliminate the potential bias in estimating the value of $d \ln \alpha / d \ln \beta$. Parallel but independent analyses can be made by comparing the splitting data and the $S$ velocity model (Woodhouse & Dziewonski 1986); the characteristic period of the data used in constructing this model is of the order 60 s.

In four experiments 1(a), 1(b), 2(a) and 2(b) described below we seek to estimate the values of ratios which we denote by $\alpha^T/\beta^M$, $\alpha^M/\alpha^T$, $\alpha^M/\beta^S$ and $\beta^M/\beta^S$, respectively, where the symbol $\alpha^T/\beta^M$ indicates the ratio of the relative perturbations in $\alpha$ as constructed from the traveltime model (V.3) and in $\beta$ as required by the modal data, $\beta^M/\beta^S$ denotes the ratio of the relative perturbations in $\beta$ as required by the modal data and in $\beta$ as constructed from the $SH$-waveform model (SW), etc. These four ratios are to be used to draw our final, 'debiased', conclusion on the value of $d \ln \alpha / d \ln \beta$ in the lower mantle.

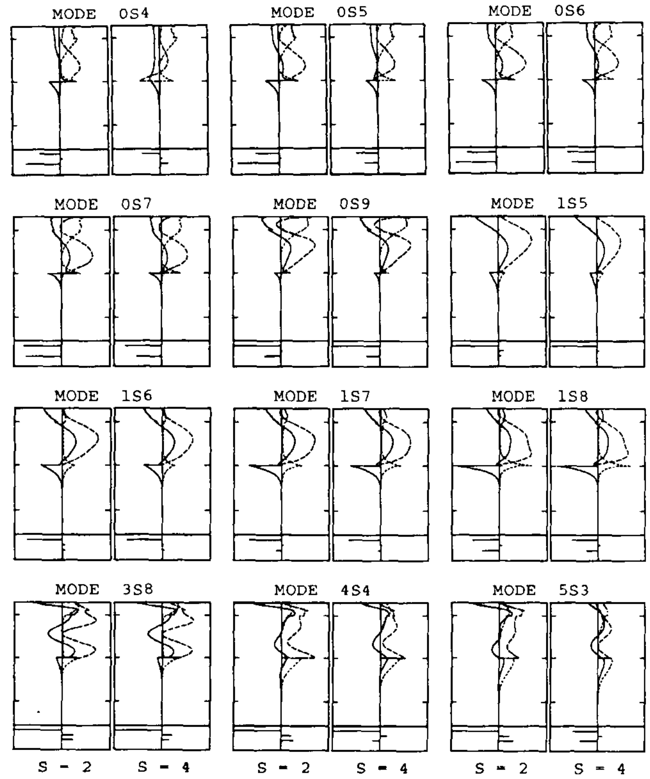


**Figure 1.** Differential kernels (see equation 1) for spherical harmonic degrees 2 and 4 of some mantle modes. The sensitivities of splitting functions, as a function of depth, to the perturbations in $\alpha$ [$A_s(r)$, dotted lines], $\beta$, [$B_s(r)$, dashed lines], and $\rho$[$R_s(r)$, solid lines] are plotted in the upper panels. On side margins, the 670 discontinuity, the core–mantle boundary (CMB), and the inner core boundary are marked from the top down. The horizontal scale is $\pm 4/a$, where $a$ is the radius of the Earth. In the lower panels, are plotted the sensitivies of the splitting coefficients to undulations in discontinuities ($H_s$), for the free surface, the 670 discontinuity, and the CMB, with the scale running from $-1$ to $+1$.

### Experiment 1(a): the value of $\alpha^T/\beta^M$

Modes ${}_0S_4$, ${}_0S_5$, ${}_0S_6$, ${}_0S_7$, ${}_0S_9$, ${}_1S_5$, ${}_1S_6$, ${}_1S_7$, ${}_1S_8$, ${}_3S_8$, ${}_4S_4$ and ${}_5S_3$ are, in the lower mantle, principally sensitive to the heterogeneity in $S$ velocity with some minor sensitivity to the heterogeneity in density. The differential kernels $A_s(r)$, $B_s(r)$ and $R_s(r)$ (see equation 1) of these 12 modes for $s = 2$ and 4 are plotted in Fig. 1. At the bottom of each panel in Fig. 1 we also show the sensitivity to topographic perturbations [coefficients $H_s$ in equation (1)] of the three major discontinuities—the surface, the 670 discontinuity, and the core–mantle boundary (CMB). The strong sensitivity to CMB perturbations of most of these modes is evident.

Considering the fluid outer core to be laterally homogeneous (Stevenson 1987), we may assume that the splitting of these modes is from three sources: (1) the $\beta$ heterogeneity in lower mantle (since both the relative perturbation in $\rho$ and its associated kernels are smaller than those of $\beta$, we neglect the contributions of $\rho$ heterogeneity in the lower mantle); (2) upper mantle structure (including crustal structure); and (3) the topography of the CMB.

Since we are concerned with the relationship between perturbations in the velocities of the mantle, it is convenient to eliminate the splitting effects of CMB topography from the analysis. This can be done by grouping the 12 modes into six pairs and forming six 'hybrid modes' A, B, C, D, E and F. Each hybrid mode is designed so that its sensitivity to the CMB structure of degree $s = 2$ vanishes. Specifically, hybrid mode A is a combination of modes ${}_5S_3$ and ${}_0S_6$, weighted by factors 0.941 and 0.338 respectively, i.e., the splitting function coefficients and the differential kernels of mode A are the combinations of the corresponding splitting function coefficients and differential kernels of modes ${}_5S_3$ and ${}_0S_6$, with weights 0.941 and 0.338 respectively. We can write this symbolically as $\mathrm{A} = 0.941{}_5S_3 + 0.338{}_0S_6$. In this notation the other hybrid modes are $\mathrm{B} = 0.949{}_3S_8 + 0.314{}_0S_4$, $\mathrm{C} = 0.989{}_1S_5 + 0.147{}_1S_8$, $\mathrm{D} = 0.993{}_1S_7 - 0.116{}_0S_5$, $\mathrm{E} = 0.943{}_4S_4 + 0.332{}_0S_7$ and $\mathrm{F} = 0.990{}_1S_6 + 0.141{}_0S_9$. The differential kernels for degrees $s = 2$ and 4 of these six hybrid modes are shown in Fig. 2. The sensitivities to the CMB structure of $s = 2$ vanish, as intended. For $s = 4$ the sensitivities to CMB topography are very small for all the modes except modes A and E (this is because the sensitivies for $s = 2$ and for $s = 4$ are the same in the limit of large angular order $l$). In this experiment we use the splitting function coefficients of degree $s = 2$ of all six hybrid modes and the coefficients of $s = 4$ of modes B, C, D and F.

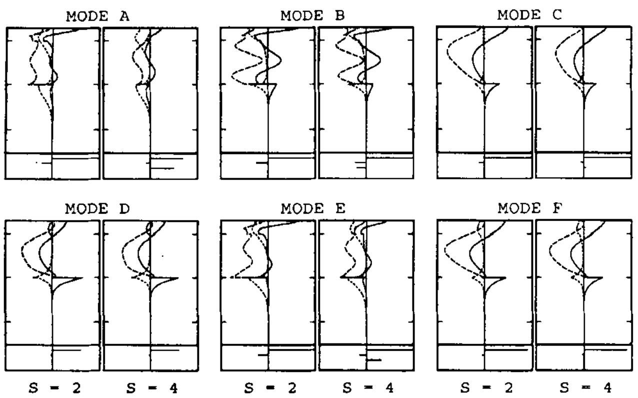


**Figure 2.** Differential kernels for spherical harmonic degrees 2 and 4 of 6 artificial modes. Each artificial mode is a combination of two real modes, combined in such a way that it has zero sensitivity to the undulation of the core–mantle boundary for degree 2. In the notation defined in text, these modes are A = $0.941{}_5S_3 + 0.338{}_0S_6$, B = $0.949{}_3S_8 + 0.314{}_0S_4$, C = $0.989{}_1S_5 + 0.147{}_1S_8$, D = $0.993{}_1S_7 - 0.116{}_0S_5$, E = $0.943{}_4S_4 + 0.332{}_0S_7$, and F = $0.990{}_1S_6 + 0.141{}_0S_9$. See also caption to Fig. 1.

For each of the selected splitting function coefficients, we may write

$$c_{st} = c_{st}^{\beta} + \bar{c}_{st} + \varepsilon_{st} \tag{2}$$

where $c_{st}$ are splitting coefficients from modal inversion (available from table 2 of Li *et al.* 1991), $c_{st}^{\beta}$ are synthetic splitting coefficients due to lower mantle $\beta$ heterogeneity, $\bar{c}_{st}$ are the contributions from the upper mantle and $\varepsilon_{st}$ are error terms. Now let us assume that the relative perturbations in $\alpha$ and in $\beta$ are proportional to each other with a constant proportionality coefficient $\mathcal{P} \equiv d \ln \alpha / d \ln \beta$. Under this assumption we have by virtue of (1).

$$\begin{aligned} c_{st}^{\beta} &= \int_{r_C}^{r_{670}} B_s(r)(\delta\beta_{st}/\beta)\, dr \\ &= \int_{r_C}^{r_{670}} B_s(r)(\delta\alpha_{st}^T/\alpha)/\mathcal{P}_T\, dr = c_{st}^{\beta T}/\mathcal{P}_T \end{aligned} \tag{3}$$

with

$$c_{st}^{\beta T} = \int_{r_C}^{r_{670}} B_s(r)(\delta\alpha_{st}^T/\alpha)\, dr, \tag{4}$$

where $r_{670}$ and $r_C$ are the radii of the 670 discontinuity and the CMB respectively, $\delta\alpha_{st}^T$ are coefficients of $\alpha$ heterogeneity calculated from traveltime model V.3, and $\mathcal{P}_T$ is an estimate of $\mathcal{P}$ to be found (with the subscript $T$ standing for that the traveltime earth model is used).

In order to estimate $\mathcal{P}_T$ from (2) and (3), we need to make use of an upper mantle model to calculate $\bar{c}_{st}$. We explore two different strategies: (1) we take the $S$ velocity model M84A (Woodhouse & Dziewonski 1984), which includes crustal effects, and assume $d \ln \rho / d \ln \beta = 0.25$ and $d \ln \alpha / d \ln \beta = 0.5$; and (2) since the pattern of heterogeneity in the upper mantle is not highly correlated with that in the lower mantle (see Dziewonski 1984; Woodhouse & Dziewonski 1984) we simply set $\bar{c}_{st} = 0$ and consequently incorporate upper mantle contributions into the error terms, $\varepsilon_{st}$. The comparison of results using each of these strategies

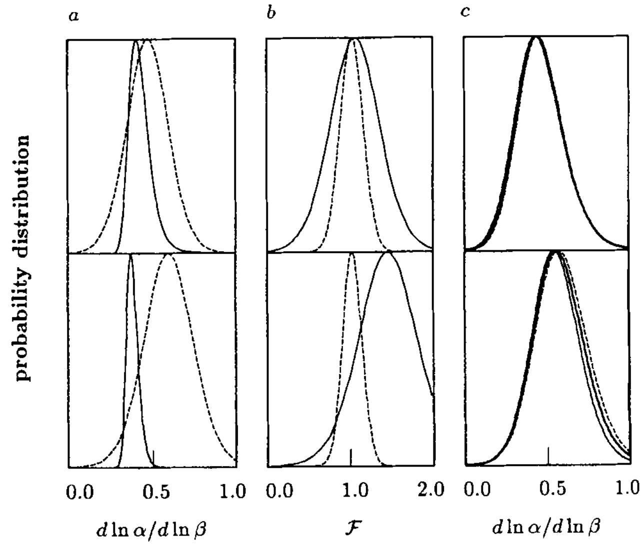


**Figure 3.** Statistical distributions of estimators of $d \ln \alpha / d \ln \beta$ for the lower mantle. The results in the upper panels are those obtained with a correction for the upper mantle heterogeneity, while the results in the lower panels are derived without upper mantle corrections. The distributions are constrained by the splitting functions of mantle modes, together with two existing heterogeneous mantle models—the thin solid curves represent the results using model V.3 and the dashed curves are for the results using model SW. (a) Distributions of estimators of $\mathcal{P} = d \ln \alpha / d \ln \beta$ in Experiments 1(a) and 2(a) (see text). (b) Distributions of the estimator $\mathcal{F}$, which characterizes the potential overestimate ($\mathcal{F} < 1$) or underestimate of the existing models with respect to the modal results. (c) Distributions of the estimator $\mathcal{P} = d \ln \alpha / d \ln \beta$, obtained after correcting for the biases due to the potential overestimates, with respect to the effect of modal results, of the earth models used. The heavy solid curves are the averages of the results represented by the thin solid curves and the dashed curves.

(see below) serves to quantify the influence of upper mantle structure on the results.

If we now assume that $\varepsilon_{st}$ are independent, normal random variables with same variance, the statistical distribution of the parameter $\mathcal{P}_T$ in (3) can be derived. The details of this derivation are given in the Appendix A.

The two solid curves in Fig. 3(a) show the distributions of $\mathcal{P}_T$: the upper panel represents the result by using M84A for the upper mantle correction, and the lower panel is for the case in which no upper mantle correction is made. The maximum-likelihood values are 0.40 and 0.37, respectively.

#### Experiment 1(b): correction for $\alpha^M/\alpha^T$

The distributions of $\mathcal{P}_T$ obtained in Experiment 1(a) could be biased due to the potential overestimate or underestimate of V.3 with respect to the modal data. This experiment is designed to give an estimate on the mutual overestimate or underestimate. In Fig. 4 we plot the differential kernels of modes ${}_4S_3$, ${}_5S_4$, ${}_5S_5$, ${}_5S_6$ and ${}_6S_{10}$. In the lower mantle the contributions to the splitting coefficients of degree $s = 2$ of these modes are mainly from $\alpha$ heterogeneity, as indicated by their kernels. However considering that the amplitude of $\beta$ heterogeneity could be much larger (by a factor up to 1/0.39, see Section 5) than that of $\alpha$ heterogeneity, it is

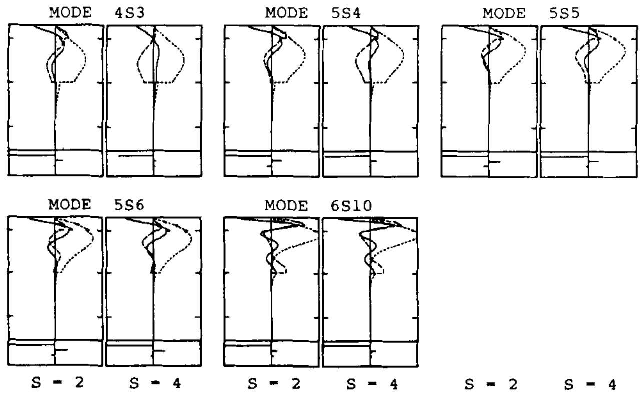


**Figure 4.** Differential kernels of spherical harmonic degrees 2 and 4 for 5 modes which are mainly sensitive to the $\alpha$ heterogeneity in the lower mantle. See also caption to Fig. 1.

appropriate to also take the contribution of $\beta$ heterogeneity into account. We here use the synthetic splitting coefficients predicted from the $\beta$ heterogeneity model SW for this purpose. For the following reasons, the effect of the uncertainty in model SW is small and may be incorporated into the error terms. First model SW gives, as Experiment 2(b) indicates, almost the same amplitude of $\beta$ heterogeneity as the modal data require. Second, since the synthetic splitting coefficients contributed from $\beta$ heterogeneity themselves are small (the rms of them are 0.20 of the rms of the observed data) and are poorly correlated with the observed data (correlation coefficient = 0.21) and with the contribution of $\alpha$ heterogeneity (correlation coefficient = $-0.24$), their uncertainty may be regarded as a second-order effect.

Now we attempt to constrain the factor $\mathcal{F}_\alpha^{M/T}$ which characterizes the potential overestimate or underestimate of V.3 with respect to the modal splitting functions. We make use of the splitting coefficients of $s = 2$ of modes ${}_4S_3$, ${}_5S_4$, ${}_5S_5$ and ${}_5S_6$. We exclude mode ${}_6S_{10}$ here since it is sensitive mostly to the top part of the lower mantle where the resolution of V.3 is relatively poor (A. M. Dziewonski, personal communication); however this mode is employed in Experiment 2(a) below. Based upon the discussion above, we may write according to (1)

$$c_{st} = \int_{r_C}^{r_{670}} A_s(r)(\mathcal{F}_\alpha^{M/T}\delta\alpha_{st}^T/\alpha)\,dr + c_{st}^\beta + \bar{c}_{st} + \varepsilon_{st}, \qquad s = 2, \tag{5}$$

where the coefficients $\delta\alpha_{st}^T$ are again taken from traveltime model V.3, $c_{st}^\beta$ are small contributions of lower mantle $\beta$ heterogeneity obtained from model SW, and $\bar{c}_{st}$ are the contributions from the upper mantle. As previously for $\mathcal{P}_T$, we make use of (5) to find the probability distributions of $\mathcal{F}_\alpha^{M/T}$, which are given by the two solid curves in Fig. 3(b). Again the upper panel shows the result obtained using M84A for the upper mantle correction and lower panel is for the case in which no upper mantle correction is made.

We may regard the distributions of $\mathcal{P}_T$ shown in Fig. 3(a) as conditional distributions with the condition $\mathcal{F}_\alpha^{M/T} = 1$. The unconditional value of $d \ln \alpha / d \ln \beta$, then, is given by $\bar{\mathcal{P}}_T = \mathcal{P}_T \mathcal{F}_\alpha^{M/T}$. Using the distributions of $\mathcal{P}_T$ and $\mathcal{F}_\alpha^{M/T}$ previously found, we may calculate the distributions of $\bar{\mathcal{P}}_T$ (see Appendix B); the results are given by the thin solid curves in Fig. 3(c). The estimate so determined is independent of the amplitude of the traveltime model V.3, and is constrained by the modal data under the assumption that the pattern of V.3 represents reality.

### Experiment 2(a): the value of $\alpha^M/\beta^S$

The above analysis is based on the comparison of modal splitting coefficients with those calculated using the $P$ velocity model V.3 of Morelli & Dziewonski (1987). A parallel, but independent, analysis can be performed by comparing observed splitting functions with those predicted using the $S$ velocity model SW (Woodhouse & Dziewonski 1986).

Denoting the $\beta$ heterogeneity coefficients from $SH$-waveform model SW as $\delta\beta_{st}^S$, we can estimate the value of $\mathcal{P} = d \ln \alpha / d \ln \beta$ by using the $s = 2$ splitting coefficients of modes ${}_4S_3$, ${}_5S_4$, ${}_5S_5$, ${}_5S_6$ and ${}_6S_{10}$ [see the argument in Experiment 1(b)]:

$$c_{st} = \int_{r_C}^{r_{670}} A_s(r)(\mathcal{P}_S\delta\beta_{st}^S/\beta)\,dr + c_{st}^\beta + \bar{c}_{st} + \varepsilon_{st}, \qquad s = 2, \tag{6}$$

where $\mathcal{P}_S$ is an estimate of $\mathcal{P}$ by using $SH$-waveform model and other notations are the same as in (5). The estimated probability distributions of $\mathcal{P}_S$ are shown by the dashed curves in Fig. 3(a).

### Experiment 2(b): correction for $\beta^M/\beta^S$

To estimate the potential overestimate or underestimate of SW with respect to the modal data, we use the splitting function coefficients of hybrid modes A, B, C, D, E and F (see Experiment 1a). The coefficients of spherical harmonic degree 4 of modes A and E are omitted as in Experiment 1(a). We write, for the selected splitting coefficients:

$$c_{st} = \int_{r_C}^{r_{670}} B_s(r)(\mathcal{F}_\beta^{M/S}\delta\beta_{st}^S)\,dr + \bar{c}_{st} + \varepsilon_{st}, \tag{7}$$

where $\mathcal{F}_\beta^{M/S}$ characterizes the potential overestimate or underestimate of SW and $\delta\beta_{st}^S$ are coefficients of $\beta$ heterogeneity calculated form $SH$-waveform model SW. The distribution of $\mathcal{F}_\beta^{M/S}$ derived from (7) are shown as dashed curves in Fig. 3(b).

A 'debiased' estimate of the ratio $d \ln \alpha / d \ln \beta$ from the comparison between SW and splitting coefficients is $\bar{\mathcal{P}}_S = \mathcal{P}_S / \mathcal{F}_\beta^{M/S}$, where $\mathcal{P}_S$ is that estimated in Experiment 2(a). The probability distributions of $\bar{\mathcal{P}}_S$ are given by the dashed lines in Fig. 3(c).

### Estimate of $d \ln \alpha / d \ln \beta$ in the lower mantle

The estimator $\bar{\mathcal{P}}_T$ obtained in Experiment 1 and the estimator $\bar{\mathcal{P}}_S$ obtained in Experiment 2 are based upon two different assumptions—taking the pattern of model V.3 or of model SW as reality. However the distributions of $\bar{\mathcal{P}}_T$ and $\bar{\mathcal{P}}_S$ are almost the same, indicating that the constraints on the results are mainly from the modal data. Since these distributions are so similar we simply use their average in stating the conclusions of this experiment. This averaged

**Table 1.** Confidence intervals for estimating $\mathcal{P} = d \ln \alpha / d \ln \beta$ from seismic data alone.

| confidence level | 95% | 90% | 75% | 60% |
|---|---|---|---|---|
| M84A used as correction | $0.19 < \mathcal{P} < 0.76$ | $0.23 < \mathcal{P} < 0.70$ | $0.30 < \mathcal{P} < 0.61$ | $0.34 < \mathcal{P} < 0.56$ |
| no upper-mantle corrections | $0.27 < \mathcal{P} < 0.89$ | $0.32 < \mathcal{P} < 0.82$ | $0.39 < \mathcal{P} < 0.73$ | $0.43 < \mathcal{P} < 0.68$ |

distribution is represented by the thick solid lines in Fig. 3(c) (again the upper panel shows the result obtained using M84A for the upper mantle correction, and the lower panel is for the case in which no upper mantle correction is made).

Minimum-length confidence intervals for $\mathcal{P} = d \ln \alpha / d \ln \beta$, derived from the distributions of Fig. 3, are given in Table 1. Again, results are given for the two cases—with and without the upper mantle correction—and for confidence levels of 95, 90, 75 and 60 per cent.

Comparing the two results—with and without the upper mantle correction, we may conclude that the uncertainty (which is presumably smaller than the difference between the two cases) in our knowledge of upper mantle heterogeneity does not crucially influence the estimate of $d \ln \alpha / d \ln \beta$ in the lower mantle. It is clearly more reasonable to assume that upper mantle heterogeneity is that of M84A than to assume that the upper mantle is homogeneous. For this reason we adopt the results obtained using the upper mantle correction, which may be summarized as follows: the ratio $d \ln \alpha / d \ln \beta$ in the lower mantle takes an optimal value of 0.44 and lies between 0.30 and 0.61 with 75 per cent confidence.

## 4 RELATIONSHIP BETWEEN HETEROGENEITIES IN DENSITY AND IN VELOCITIES

For each of the mantle modes, we may assume that the splitting coefficients are composed of three parts: (1) the contribution from upper mantle structure, (2) the contribution from lower mantle heterogeneity in $\alpha$ and $\beta$, and (3) the contribution from lower mantle $\rho$ heterogeneity. Assuming $d \ln \rho / d \ln \alpha = \mathcal{R}$, we may write according to (1)

$$c_{st} = \bar{c}_{st} + c_{st}^{\mathrm{vel}} + \int_{r_C}^{r_{670}} R_s(r) \mathcal{R} (\delta \alpha_{st} / \alpha) \, dr + \varepsilon_{st}, \tag{8}$$

where $\bar{c}_{st}$ denote the contributions from the upper mantle, which may be calculated as above, and $c_{st}^{\mathrm{vel}}$ are the contributions from the lower mantle velocity structure. The third term on the right side of (8) is the contribution from the heterogeneity in density and $\varepsilon_{st}$ are error terms. $c_{st}^{\mathrm{vel}}$ and $\delta \alpha_{st}$ in (8) may be evaluated by using the model V.3, together with multiplying factors determined in the preceding section; we take $\mathcal{P} = d \ln \alpha / d \ln \beta = 0.44$ [the maximum-likelihood value from the heavy solid line in the upper panel of Fig. 3(c)]. Equation (8) provides a constraint on the parameter $\mathcal{R} = d \ln \rho / d \ln \alpha$, which can be estimated by the method outlined in the Appendix A. We have calculated the probability distribution of the estimator of $\mathcal{R}$, using 25 mantle modes (Giardini *et al.* 1988; Li *et al.* 1991): ${}_0S_3$, ${}_0S_4$, ${}_0S_5$, ${}_0S_6$, ${}_0S_7$, ${}_0S_9$, ${}_1S_3$, ${}_1S_4$, ${}_1S_5$, ${}_1S_6$, ${}_1S_7$, ${}_1S_8$, ${}_2S_4$, ${}_2S_5$, ${}_2S_6$, ${}_2S_8$, ${}_3S_1$, ${}_3S_8$, ${}_4S_3$, ${}_4S_4$, ${}_5S_3$, ${}_5S_4$, ${}_5S_5$, ${}_5S_6$ and ${}_6S_{10}$. The result is given by the solid line in Fig. 5. In order to investigate the effect of the uncertainty in the value taken for $\mathcal{P}$, we have repeated the experiment with the exception that we use $\mathcal{P} = 0.40$ [this value is reasonable upon our information, see Fig. 3(c)]. The result is shown by the dashed line in Fig. 5.

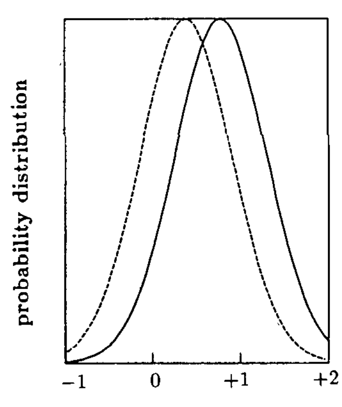


**Figure 5.** Statistical distributions of the estimator $\mathcal{R} = d \ln \rho / d \ln \alpha$. The solid curve is the result obtained assuming $d \ln \alpha / d \ln \beta = 0.44$ and the dashed curve corresponds $d \ln \alpha / d \ln \beta = 0.40$.

The breadth of the distributions shown in Fig. 5 leads us to conclude that the constraint of the current data set on the parameter $\mathcal{R} = d \ln \rho / d \ln \alpha$ is not strong enough to provide a useful estimate. The weaker constraint on $\rho$ heterogeneity of the modal data is notable in Figs 1 and 4. The differential kernels for $\rho$ are in general much smaller than those for $\alpha$ or $\beta$, especially in the lower mantle. In addition, the kernels for $\rho$ often change signs with depth, resulting in more chances for cancellations of contributions from different depths. Despite of the poor resolution of the current data set, the possibility of estimating the value of $d \ln \rho / d \ln \alpha$ from seismic data has been demonstrated, the result being not contradictory to our previous information. The constraints on $\mathcal{R}$ should be improved as more high-quality, very long-period seismic data become available.

## 5 DISCUSSION

It has been shown using seismic data that the value of $d \ln \alpha / d \ln \beta$ in the lower mantle is in the interval (0.30, 0.61) with 75 per cent confidence. On other grounds, however, a value as low as 0.30 is implausible, since it would lead to the situation in which heterogeneity in bulk modulus ($\kappa$) were negatively correlated with that in shear modulus ($\mu$). Using the PREM model and assuming that $d \ln \rho / d \ln \kappa$ is a stable parameter with the change in $d \ln \mu / d \ln \kappa$, it is readily shown that in order to avoid such behaviour the ratio $d \ln \alpha / d \ln \beta$ should be greater than 0.39; for this limiting value, the heterogeneity in bulk modulus would be (almost) zero. If this restriction is accepted, confidence intervals for $d \ln \alpha / d \ln \beta$ need to be recalculated. The new results for 95, 90, 75 and 60 per cent confidence are listed in Table 2. Adopting the estimate with the upper mantle correction (see

**Table 2.** Confidence intervals for estimating $\mathcal{P} = d \ln \alpha / d \ln \beta$ under the constraint $d \ln \kappa / d \ln \mu > 0$.

| confidence level | 95% | 90% | 75% | 60% |
|---|---|---|---|---|
| M84A used as correction | $0.39 < \mathcal{P} < 0.78$ | $0.39 < \mathcal{P} < 0.70$ | $0.39 < \mathcal{P} < 0.60$ | $0.39 < \mathcal{P} < 0.54$ |
| no upper-mantle corrections | $0.39 < \mathcal{P} < 0.86$ | $0.39 < \mathcal{P} < 0.79$ | $0.42 < \mathcal{P} < 0.70$ | $0.45 < \mathcal{P} < 0.66$ |

the previous section), the value of $d \ln \alpha / d \ln \beta$ is confined to the relatively narrow interval (0.39, 0.60) with 75 per cent confidence. The value of 0.8 based upon the experimental work of Anderson *et al.* (1968) is ruled out with 96 per cent confidence.

The evidence presented here that the heterogeneity as seen by free oscillations is of similar magnitude to that determined using other data sets corresponding to waves of much shorter periods has interesting consequences for the possible lateral variations in attenuation. It is not unreasonable to assume, under the hypothesis that lateral heterogeneity is due to lateral variations in temperature, that there exists proportionality between variations in seismic velocity and in attenuation parameters. Let us suppose that

$$\delta q_\alpha = -\gamma_\alpha \, \delta \ln \alpha, \tag{9}$$

where $\delta q_\alpha$ represents the heterogeneity in inverse quality factor $[q_\alpha = (1 - E)Q_\kappa^{-1} + EQ_\mu^{-1}$ with $E = \frac{4}{3}(\beta/\alpha)^2]$. Then because of physical dispersion (e.g., Liu, Anderson & Kanamori 1976) $\delta \ln \alpha$ is necessarily frequency dependent. Perturbing the approximate relation (Nowick & Berry 1972):

$$\frac{d \ln \alpha}{d \ln \omega} = \frac{1}{\pi} q_\alpha, \tag{10}$$

where $\omega$ is frequency, we find

$$\frac{d\delta \ln \alpha}{d \ln \omega} = \frac{1}{\pi} \delta q_\alpha = -\frac{\gamma_\alpha}{\pi} \delta \ln \alpha, \tag{11}$$

which may be regarded as a differential equation for the dependence of heterogeneity ($\delta \ln \alpha$) upon frequency. In order to solve this equation, we may, for example, assume that $\gamma_\alpha$ is independent of frequency. In this case we obtain

$$\frac{(\ln \alpha)_{\omega_1}}{(\ln \alpha)_{\omega_2}} = \left(\frac{\omega_1}{\omega_2}\right)^{-\gamma\alpha/\pi}, \tag{12}$$

or

$$\delta q_\alpha = -\pi \frac{\ln \mathcal{F}}{\ln (\omega_2/\omega_1)} \delta \ln \alpha, \tag{13}$$

where $\mathcal{F} = (\delta \ln \alpha)_{\omega_1} / (\delta \ln \alpha)_{\omega_2}$. If we assume that the measurement of $\mathcal{F}_\alpha^{M/T}$ performed in Experiment 1(b) gives an estimate on $\mathcal{F}$, we may take $\omega_1 \sim 2\pi/500$ s, $\omega_2 \sim 2\pi/1$ s, and $\mathcal{F} \leq 1.70$, this last value being the upper limit of $\mathcal{F}$ with 90 per cent confidence [calculated from the solid curve in the upper panel of Fig. 3(b) under the constraint* $\mathcal{F}_\alpha^{M/T} > 1$]. Taking $\pm 0.1$ per cent to be the typical level of heterogeneity in $\alpha$ (in spherical harmonic degrees 2 and 4) these values give $|\delta q_\alpha| \leq 2.68 \times 10^{-4}$. A reasonable alternative assumption in solving (11) is that the heterogeneity $\delta q_\alpha$ is independent of frequency, since the average $Q$ value of the Earth's mantle depends weakly upon frequency in the seismic frequency band (Knopoff 1964; Anderson 1967). In this case, we obtain $|\delta q_\alpha| \leq 3.54 \times 10^{-4}$. Naturally, a similar argument can be made for $\beta$ and $q_\beta$. For the comparison of SW with the modal data (Experiment 2b) we may take $\omega_1 \sim 2\pi/500$ s, $\omega_2 \sim 2\pi/60$ s, $\mathcal{F} \leq 1.29$, and $|\delta \ln \beta| \sim 0.2$ per cent, yielding $|\delta q_\beta| \leq (7.55$ or $8.59) \times 10^{-4}$ if $\gamma_\beta$ or $\delta q_\beta$ is assumed to be independent of frequency. Assuming that attenuation is entirely in shear ($Q_\kappa^{-1} = 0$), we have, approximately, $\delta Q_\mu^{-1} = \delta q_\beta = 2.56 \, \delta q_\alpha$, and thus the bounds derived from the comparison of the modal data with the two different models are practically the same, namely that heterogeneity in $Q_\mu^{-1}$ (in degrees 2 and 4) is no more than $\pm 9 \times 10^{-4}$, approximately 30 per cent of the spherically symmetric lower mantle $Q_\mu^{-1}$ (from PREM). This result depends upon the assumption that there is no systematic underestimate in the inversions for models V.3 or SW with respect to the modal results. As to the validation of this assumption, however, we still need more evidence. At modal frequencies, the uncertainty in the magnitude of the splitting functions is large enough to influence the discussion here. At high frequencies, the traveltime data from ISC (International Seismology Center) have been inverted for models of the lower mantle heterogeneity by using different algorithms and parametrizations (e.g., Morelli & Dziewonski 1987; Hager & Clayton 1989; Inoue *et al.* 1990), and although it is not straightforward to compare models with different resolutions, the difference in the amplitudes of these models is evident. With more accurate information on the frequency dependence of heterogeneity, it will be possible to obtain more reliable constraints on the magnitude of the heterogeneity in attenuation.

It is also of interest to investigate the frequency dependence of $d \ln \alpha / d \ln \beta$ and to test whether physical dispersion can reconcile the discrepancy between the values of $d \ln \alpha / d \ln \beta$ obtained here and those from laboratory. In the literature (e.g., Liu *et al.* 1976; Kanamori & Anderson 1977) band-limited constant $Q$ models are often used. For simplicity, we may approximate such models by

$$q(\omega) = \begin{cases} \bar{q}, & \omega_L < \omega < \omega_U, \\ 0, & \text{elsewhere}, \end{cases} \tag{14}$$

where $\bar{q}$ is independent of frequency and $\omega_L$ and $\omega_U$ are the low-frequency and high-frequency cut-offs, respectively. Then we have, according to (11),

$$|\delta \ln \alpha(\omega)| - |\delta \ln \alpha(\omega')| = -\frac{1}{\pi} |\delta \bar{q}_\alpha| \ln \frac{\omega}{\omega'}, \tag{15}$$

where $\omega$ and $\omega'$ are two different frequencies in the seismic frequency band and $\delta \ln \alpha / \delta \bar{q}_\alpha < 0$ has been assumed.

* Under the assumption that there is no systematic underestimate in the inversions for models V.3 or SW with respect to the modal results, $\mathcal{F}_\alpha^{M/T}$ and $\mathcal{F}_\beta^{M/S}$ should be confined to be larger than unity since the magnitude of heterogeneity should decrease with frequency.

Similarly

$$|\delta \ln \beta(\omega)| - |\delta \ln \beta(\omega')| = -\frac{1}{\pi}|\delta \bar{q}_\beta| \ln \frac{\omega}{\omega'}. \tag{16}$$

Equations (15) and (16) indicate that $|\delta \ln \alpha|$ and $|\delta \ln \beta|$ are linear functions of $\ln \omega$ with negative slope $-|\delta \bar{q}_\alpha|/\pi$ and $-|\delta \bar{q}_\beta|/\pi$, respectively. Since $|\delta \bar{q}_\alpha| = E\,|\delta \bar{q}_\beta| \simeq 0.39\,|\delta \bar{q}_\beta|$ (see above), $|\delta \ln \alpha|$ decreases more slowly than $|\delta \ln \beta|$ with increasing frequency. As previously we may estimate $|\delta \bar{q}_\alpha|$ by measuring $\delta \ln \alpha$ at $\omega = 2\pi/500$ s and $\omega' = \omega_T = 2\pi/1$ s, which are the characteristic frequencies of modal data and traveltime data, respectively. Thus we have

$$|\delta \bar{q}_\alpha| = |\delta \ln \alpha| \frac{\pi(1 - 1/F)}{\ln(\omega_T/\omega)}, \tag{17}$$

where $F \equiv (\delta \ln \alpha)_\omega/(\delta \ln \alpha)_{\omega_T}$. If we can extrapolate the laboratory value, 0.8, of $d \ln \alpha/d \ln \beta$ (Anderson *et al.* 1968) to the cut-off frequency $\omega_U$ and evaluate $\omega'$ in (15) and (16) at the cut-off frequency $\omega_U$, we obtain by virtue of (15), (16), (17), and the relation $|\delta \bar{q}_\alpha| = E\,|\delta \bar{q}_\beta|$

$$\frac{d \ln \alpha}{d \ln \beta} = \frac{0.8}{1 + \Delta}, \tag{18}$$

where

$$\Delta = \frac{\ln(\omega_U/\omega)}{\ln(\omega_T/\omega)} \left(\frac{0.8}{E} - 1\right)\left(1 - \frac{1}{F}\right). \tag{19}$$

In order to evaluate (18) we require the value of the cut-off frequency, $\omega_U$, which is poorly known. Sipkin & Jordan (1979) have reported that when $\omega > 2\pi/10$ s, $Q$ appears to increase rapidly with frequency. If we take $2\pi/1$ s as the cut-off frequency $\omega_U$, $F = 4.56$ is required to explain the optimal value (0.44) of $d \ln \alpha/d \ln \beta$ of this study. But for the upper limit (0.6) of the minimum-length 75 per cent confidence interval for $d \ln \alpha/d \ln \beta$, we only need $F = 1.47$. Thus our current knowledge on $F$ cannot completely rule out the possibility that the discrepancy between the values of $d \ln \alpha/d \ln \beta$ obtained here and those inferred from laboratory experiments is due to physical dispersion. However since a low value ($\sim$0.5) of $d \ln \alpha/d \ln \beta$ is also indicated at high frequency ($\sim 2\pi/10$ s) (Jordan & Lynn 1974; Lay 1983), other physical mechanisms, such as those proposed by Anderson (1987) and Yeganeh-Haeri *et al.* (1989), are probably required to reconcile the discrepancy.

## ACKNOWLEDGMENTS

We would like to thank A. Morelli and A. M. Dziewonski for allowing us to make use of their unpublished model V.3, and J. H. Woodhouse and A. M. Dziewonski for their unpublished model SW. Discussions with D.-K. Li, on statistics, have been helpful in this study. We also appreciate the thoughtful comments by J. Park, P. Lognonne and an anonymous reviewer of the manuscript. This research was carried out with the support of the National Science Foundation under the grants EAR87-08622 and EAR86-18829.

## REFERENCES

Anderson, D. L., 1967. The anelasticity of the mantle, *Geophys. J. R. astr. Soc.*, **14**, 135–164.

Anderson, D. L., 1987. A seismic equation of state II. Shear properties and thermodynamics of the lower mantle, *Phys. Earth planet. Inter.*, **45**, 307–323.

Anderson, O. L., Schreiber, E., Liebermann, R. C. & Soga, M., 1968. Some elastic constant data on minerals relevant to geophysics, *Rev. Geophys.*, **6**, 491–524.

Dziewonski, A. M., 1984. Mapping the lower mantle: Determination of laterial heterogeneity in *P* velocity up to degree and order 6, *J. geophys. Res.*, **89**, 5929–5952.

Dziewonski, A. M. & Anderson, D. L., 1981. Preliminary reference earth model (PREM), *Phys. Earth planet. Inter.*, **25**, 297–356.

Dziewonski, A. M. & Anderson, D. L., 1983. Travel times and station corrections for *P*-waves at teleseismic distances, *J. geophys. Res.*, **88**, 3295–3314.

Dziewonski, A. M. & Woodhouse, J. H., 1987. Global images of the Earth's interior, *Science*, **236**, 37–48.

Forte, A. M. & Peltier, W. R., 1987. Plate tectonics and aspherical Earth structure: the importance of poloidal–toroidal coupling, *J. geophys. Res.*, **92**, 3645–3679.

Giardini, D., Li, X.-D. & Woodhouse, J. H., 1987. Three-dimensional structure of the Earth from splitting in free oscillations spectra, *Nature*, **325**, 405–411.

Giardini, D., Li, X.-D. & Woodhouse, J. H., 1988. Splitting functions of long-period normal modes of the Earth, *J. geophys. Res.*, **93**, 13 716–13 742.

Hager, B. H. & Clayton, R. W., 1989. Constraints on the structure of mantle convection using seismic observations, flow models, and the geoid, *Mantle Convection*, ed. Peltier, W. R., Gordon and Breach, New York.

Hales, A. L. & Doyle, H. A., 1967. *P* and *S* travel time anomalies and their interpretation, *Geophys. J. R. astr. Soc.*, **13**, 403–415.

Inoue, H., Fukao, Y., Tanabe, K. & Ogata, Y., 1990. Whole mantle *P*-wave travel time tomography, *Phys. Earth planet. Inter.*, **59**, 294–328.

Jordan, T. H. & Lynn, W. S., 1974. A velocity anomaly in the lower mantle, *J. geophys. Res.*, **79**, 2679–2685.

Kanamori, H. & Anderson, D. L., 1977. Importance of physical dispersion in surface wave and free oscillation problems: review, *Rev. Geophys. Space Phys.*, **15**, 105–112.

Kendall, M. & Stuart, A., 1977a. *The Advanced Theory of Statistics*, 4th edn, vol. 2, pp. 377–386, Charles Griffin & Company Limited, High Wycombe.

Kendall, M. & Stuart, A., 1977b. *The Advanced Theory of Statistics*, 4th edn, vol. 2, pp. 147–152, Charles Griffin & Company Limited, High Wycombe.

Knopoff, L., 1964. Q, *Rev. Geophys. Space Phys.*, **2**, 625–660.

Lay, T., 1983. Localized velocity anomalies in the lower mantle, *Geophys. J. R. astr. Soc.*, **72**, 483–516.

Li, X.-D., Giardini, D. & Woodhouse, J. H., 1991. Large-scale three-dimensional even-degree structure of the Earth from splitting of long-period normal modes, *J. geophys. Res.*, **96**, 551–577.

Liu, H.-P., Anderson, D. L. & Kanamori, H., 1976. Velocity dispersion due to anelasticity; implications for seismology and mantle composition, *Geophys. J. R. astr. Soc.*, **47**, 41–58.

Morelli, A. & Dziewonski, A. M., 1987. The harmonic expansion approach to the retrieval of deep earth structure, in *Seismic Tomography*, pp. 251–274, ed. Nolet, G., Reidel, Hingham, MA.

Nowick, A. S. & Berry, B. S., 1972. *Anelastic Relaxation in Crystalline Solids*, pp. 677, Academic Press, New York.

Richards, M. A. & Hager, B. H., 1984. Geoid anomalies in a dynamic Earth, *J. geophys. Res.*, **89**, 5987–6002.
Ritzwoller, M., Masters, G. & Gilbert, F., 1988. Constraining aspherical structure with low-degree interaction coefficients: application to uncoupled multiplets, *J. geophys. Res.*, **93**, 6369–6396.
Sipkin, S. A. & Jordan, T. H., 1979. Frequency dependence of $Q_{ScS}$, *Bull. seism. Soc. Am.*, **69**, 1055–1079.
Souriau, A. & Woodhouse, J. H., 1985. A worldwide comparison of predicted *S*-wave delays from a three-dimensional upper mantle model with *P*-wave station corrections, *Phys. Earth planet. Inter.*, **39**, 75–88.
Stevenson, D. J., 1987. Limits on lateral density and velocity variations in the Earth's outer core, *Geophys. J. R. astr. Soc.*, **88**, 311–319.
Wichens, A. J. & Buchbinder, G. G. R., 1980. *S*-wave residuals in Canada, *Bull. seism. Soc. Am.*, **70**, 809–822.
Woodhouse, J. H. & Dahlen, F. A., 1978. The effect of a general aspherical perturbation on the free oscillations of the Earth, *Geophys. J. R. astr. Soc.*, **53**, 335–354.
Woodhouse, J. H. & Dziewonski, A. M., 1984. Mapping the upper mantle: Three-dimensional modeling of Earth structure by inversion of seismic waveforms, *J. geophys. Res.*, **89**, 5953–5986.
Woodhouse, J. H. & Dziewonski, A. M., 1986. Three dimensional mantle models based on mantle wave and long period body wave data, *EOS, Trans. Am. geophys. Un.*, **67**, 307.
Woodhouse, J. H. & Giardini, D., 1985. Inversion for the splitting function of isolated low order normal mode multiplets, *EOS Trans. Am. geophys. Un.*, **66**, 300.
Yeganeh-Haeri, A., Weidner, D. J. & Ito, E., 1989. Elasticity of $MgSiO_3$ in the perovskite structure, *Science*, **243**, 787–789.

## APPENDIX A

Suppose we wish to estimate $\mathcal{C}$, or $\mathcal{C}^{-1}$, from the equation:

$$y_i = \mathcal{C}x_i + \varepsilon_i, \qquad i = 1, 2, \ldots, N, \tag{20}$$

where $y_i$ and $x_i$ are splitting function coefficients, derived from seismic spectra or calculated from existing earth models; and $\varepsilon_i$ are errors, assumed to be independent random variables with the same (unknown) variance. The least-squares estimator $\hat{\mathcal{C}}$ (Kendall & Stuart 1977a) of $\mathcal{C}$ is given by

$$\hat{\mathcal{C}} = \sum_{i=1}^{N} x_i y_i \Big/ \sum_{i=1}^{N} x_i^2. \tag{21}$$

This estimator itself is a random variable owing to the randomness of $\varepsilon_i$. Kendall & Stuart (1977a) show that the statistic

$$\tau = (\hat{\mathcal{C}} - \mathcal{C})\left(\sigma^2 \Big/ \sum_{i=1}^{N} x_i^2\right)^{-1/2} \tag{22}$$

has a Student's *t*-distribution with $N-1$ degrees of freedom if the distributions of $\varepsilon_i$ are normal, where

$$\sigma^2 = \sum_{i=1}^{N} (y_i - \hat{\mathcal{C}}x_i)^2/(N-1) \tag{23}$$

is the estimator of the variance of $\varepsilon_i$. Equation (22) indicates that the fiducial probability distribution (Kendall & Stuart 1977b) of parameter $\mathcal{C}$ is given by

$$df_{\mathcal{C}} \propto \left(\sigma^2 \Big/ \sum_{i=1}^{N} x_i^2\right)^{-1/2} t(\tau; N-1)\, d\mathcal{C} \propto t(\tau; N-1)\, d\mathcal{C} \tag{24}$$

or

$$df_{1/\mathcal{C}} \propto t(\tau; N-1)\mathcal{C}^2\, d\frac{1}{\mathcal{C}} \tag{25}$$

where $t(\tau; N-1)$ is the probability density of the *t*-distribution with argument $\tau$ and $N-1$ degrees of freedom and $\tau = \tau(\mathcal{C})$ is given in (22).

## APPENDIX B

Suppose we are given the probability distributions, $f_p(p)$ and $f_q(q)$, of two independent positive random variables $p > 0$ and $q > 0$, respectively. And $f_p(p)$ and $f_q(q)$ possess the following properties:

$$\lim_{p\to 0} f_p(p) = \lim_{q\to 0} f_q(q) = 0. \tag{26}$$

We here derive the probability distributions of their product $x = pq$ and ratio $y = p/q$. From the independence of variables $p$ and $q$, we may calculate the probability that $x$ lies in the interval $[x_0 - \Delta x, x_0 + \Delta x]$:

$$P(x_0 - \Delta x \le x \le x_0 + \Delta x) = \int_0^\infty f_p(p) P\left(\frac{x_0 - \Delta x}{p} \le q \le \frac{x_0 + \Delta x}{p}\right) dp, \tag{27}$$

where $P(e)$ is the probability that the event $e$ occurs. By definition we have

$$\begin{aligned} f_x(x_0) &= \lim_{\Delta x\to 0} \frac{1}{2\Delta x} P(x_0 - \Delta x \le x \le x_0 + \Delta x) \\ &= \int_0^\infty f_p(p) \frac{1}{p} \lim_{\Delta x\to 0} \frac{1}{2\Delta x/p} P\left(\frac{x_0 - \Delta x}{p} \le q \le \frac{x_0 + \Delta x}{p}\right) dp \\ &= \int_0^\infty f_p(p) \frac{1}{p} f_q\left(\frac{x_0}{p}\right) dp, \end{aligned} \tag{28}$$

or simply

$$f_x(x) = \int_0^\infty \frac{1}{p} f_p(p) f_q\left(\frac{x}{p}\right) dp = \int_0^\infty \frac{1}{q} f_q(q) f_p\left(\frac{x}{q}\right) dq. \tag{29}$$

Using the same kind of technique we obtain

$$f_y(y) = \int_0^\infty \frac{p}{y^2} f_p(p) f_q\left(\frac{p}{y}\right) dp = \int_0^\infty q f_q(q) f_p(yq)\, dq. \tag{30}$$

For normalized $f_p(p)$ and $f_q(q)$ [i.e., $\int f_p(p)\, dp = \int f_q(q)\, dq = 1$], it is easily verified that $f_x(x)$ and $f_y(y)$ are also normalized.