Supplementary Information: Snowflake phononic topological insulator at the nanoscale

I. FINITE ELEMENTS SIMULATIONS

The results shown in Fig. 1-3 of the main text have been obtained by solving the eigenvalue problem

$$\text{div} \left[ C : (\text{grad} \psi + (\text{grad} \psi)^T) \right] = -2\omega^2 \psi,$$  \hspace{1cm} (1)

using the COMSOL finite element solver. $\psi$ denotes the complex three-dimensional wave function related to the mechanical displacement field $u = \text{Re} \left[ \psi \cdot e^{i\omega t} \right]$ of the crystal, $C$ is the elasticity tensor, and $\rho$ the mass density. Here $: \text{short-hand for the tensor product}$

$$[C : \text{grad}\psi]_{ij} = C_{ijkl} \partial_j \psi_k.$$  

For concreteness, we envisioned a phononic crystal slab of thickness $220\text{ nm}$ as in state-of-the-art optomechanical devices [1].

II. FULL BAND STRUCTURES FOR VARIOUS CENTRAL SNOWFLAKE RADII

Fig. 1 of the main text just shows the frequency range of the band structure which is relevant for the emerging edge states at the domain interface for the parameters $\Delta r = 0$ and $\Delta r = 200 \text{ nm}$ (Fig. 1d,e of the main text). To give a larger overview Fig. 1 displays the same band structures again but in an extended frequency range. In addition to that, band structures for various other central snowflake radii $r_c = r + \Delta r$ are depicted. We are especially interested in the size of the band gap separating the two Dirac cones. As expected, the finite element simulations show that the band gap separating the Dirac cones closes when all snowflakes have the same radius, $r_c = r = 1800 \text{ nm}$. For larger central snowflake radii, $r_c > r (\Delta r > 0)$, the band gap increases monotonically with the radius. In this case, a limit to the band gap size is set by the geometry [when $r_c \leq 2500 \text{ nm}$ because for $r_c = 2500 \text{ nm}$ the size of the bridge connecting the triangles vanishes]. On the other hand, for smaller snowflake radii, $r_c < r (\Delta r < 0)$, the band gap depends non-monotonically on the radius and reaches a maximum for $\Delta r \approx -200 \text{ nm}$ ($r_c = 1600\text{ nm}$). We attribute this non-monotonic behavior to the breakdown of the four-band approximation.

III. ZIGZAG VS ARMCHAIR DOMAIN WALL

Edge states that are not topologically protected crucially depend on the details of the boundary of the system. For example, a graphene strip supports edge states in the presence of zig-zag edges but not armchair edges. In our present system, our effective Hamiltonian has a topological bulk and, thus, edge states will appear at any smooth domain wall. In principle, one could nevertheless expect a different behavior for sharp domain walls. In Fig. 2, we show the band structure for two different strips supporting sharp domain walls of the zig-zag type (panel a) and armchair type (panel b), respectively. The domain walls are created by choosing opposite values of $\Delta r$ in the two domains while keeping the six-fold rotational symmetry within each unit cell. Comparing the band structure for different domain wall configurations, cf. (a) and (b), we observe a qualitatively different behavior only for the bands depicted with faded color which correspond to edge states localized on the physical boundaries. This is not surprising because the mapping to our effective topological Hamiltonian Eq. (2) completely breaks down at a sharp interface with the vacuum. On the other hand, as far as the remaining bands are concerned there is no clear difference between the two domain walls configurations. This includes the domain wall bands (the bands with linear dispersion crossing in the vicinity of the $\Gamma$-point). Most importantly, in both cases no minigap could be resolved within the numerical accuracy of our finite elements simulations. [Such a minigap could be reasonably expected to appear due to the sharp breaking of the six-fold symmetry at the domain wall.]

IV. DERIVATION OF EFFECTIVE HAMILTONIAN UP TO QUADRATIC TERMS

The effective Dirac Hamiltonian Eq. (4) of the main text is obtained by projecting the elasticity equations (1) onto a fixed in-plane quasi-momentum $k = (k_x, k_y)$ close to the $\Gamma$ point. The resulting Hamiltonian is further projected onto a Hilbert space spanned by the appropriate set of four normal modes $|\psi_{\sigma, \tau, k}\rangle$ labeled by the indices $\sigma = \pm 1$, $\tau = \pm 1$. In position space, we have

$$\psi_{\sigma, \tau, k}(x, z) = e^{i k x} r \psi_{\sigma, \tau}(x, z),$$

where $r = (x, y)$ and $z$ are the plane and out-of-plane coordinates, respectively. The states $\psi_{\sigma, \tau}(x, z)$ at the $\Gamma$ point are chosen to be eigenstates of the three-fold rotations, and of the discrete translational symmetry $T_a$ associated to the original unit cell (in the original Wigner-Seitz cell their quasi-momenta lie at the $K$ and $K'$ points for $\tau = 1$ and $\tau = -1$, respectively) fulfilling Eq. (1) of the main text. We note that the mirror symmetry is not strictly speaking necessary. In a crystal where this symmetry is not present, one could use the alternative definition

$$|\psi_{\sigma, \tau}\rangle = \hat{T} |\psi_{-\sigma, -\tau}\rangle = \hat{R}_{\tau} |\psi_{\sigma, -\tau}\rangle.$$

This alternative definition of the basis $|\psi_{\sigma, \tau}\rangle$ has the disadvantage of not fixing the relative phase of the states...
Figure 1. Band structures for various central radii \( r_c \) of the snowflake phononic crystal slab, along a path passing the high symmetry points of the Brillouin zone (red, Fig. 1e of the main text). Here, the modes symmetric to the \( xy \)-plane are depicted with darker colors. The lower row shows the region around the Dirac cones of interest, with the gray shaded area indicating the complete band gap arising due to the breaking of the discrete translational symmetry \( T_a \).

with opposite \( \sigma \) (because the time-reversal symmetry \( T \) is anti-unitary). After fixing appropriately this relative phase, one arrives nevertheless at the same effective Hamiltonian Eq. (2), see Ref. [2] for the case \( g = 0 \). A different choice of the relative phase corresponds to a gauge transformation that will change the effective Hamiltonian Eq. (2) but, obviously, not the physical properties, e. g. the presence of edge states or the Kramers degeneracy.

The effective Hamiltonian, including also the quadratic terms, is most conveniently derived using a different set of eigenstates,

\[
|p_k^+\rangle = \frac{1}{\sqrt{2}} \left( -|\psi_{1,1,k}\rangle + |\psi_{1,-1,k}\rangle \right),
\]

\[
|p_k^-\rangle = \frac{1}{\sqrt{2}} \left( |\psi_{-1,1,k}\rangle - |\psi_{-1,-1,k}\rangle \right),
\]

\[
|d_k^+\rangle = \frac{1}{\sqrt{2}} \left( |\psi_{-1,1,k}\rangle + |\psi_{1,-1,k}\rangle \right),
\]

\[
|d_k^-\rangle = \frac{1}{\sqrt{2}} \left( |\psi_{1,1,k}\rangle + |\psi_{-1,-1,k}\rangle \right). \tag{2}
\]

At the \( \Gamma \) point, the states \(|p_{k=0}^\pm\rangle\) and \(|d_{k=0}^\pm\rangle\) are of the \( p-\) and \( d-\) type as it can be readily verified by using \( \hat{R}_{x/3} = \)
\( \hat{R}_{-2\pi/3} \hat{R}_x \) where \( \hat{R}_{-2\pi/3} |\psi_{\sigma,\tau}\rangle = \exp[i\sigma 2\pi/3] |\psi_{\sigma,\tau}\rangle \), to find the expected behavior for \( p\) - and \( d\)-type orbital under 60-degree rotations, \( \hat{R}_{\pi/3} |p^\pm\rangle = \exp[i\pi/3] |p^\pm\rangle \), \( \hat{R}_{\pi/3} |d^\pm\rangle = \exp[\pm 2\pi i/3] |d^\pm\rangle \). From the definitions of the \( \tau \)'s and \( \sigma \)'s matrices, it also readily follows that the \( p\)- and \( d\)-type orbitals are eigenstates of the gap-opening operator \( \tau_x \) and of the conserved pseudo-spin \( \hat{S} = \tau_x \hat{\sigma}_z \), \( |d^\pm\rangle = |\pm p^\pm\rangle \), \( \hat{S}|p^\pm\rangle = |p^\pm\rangle \), and \( \hat{S}|d^\pm\rangle = \mp|p^\pm\rangle \). From Eq. (2) of the main text one can also deduce the behavior under reflections \( \hat{M}_{\pm z} |p^\pm\rangle = -|p^\pm\rangle \), \( \hat{M}_{\pm z} |d^\pm\rangle = |d^\pm\rangle \). Below, we denote the eigenstates defined in (2) by the more compact alternative notation \( |m, k\rangle \) where \( m = \pm 1, \pm 2 \) is the six-fold quasi-angular momentum carried by the pseudo-spin of the phonons, \( |\pm 1, k\rangle = |p^+_k\rangle \) and \( |\pm 2, k\rangle = |d^+_k\rangle \).

The effective Hamiltonian can be expanded in powers of the momentum, \( \hat{H}_k = \hat{H}_k^{(0)} + \hat{H}_k^{(1)} + \hat{H}_k^{(2)} + \mathcal{O}(k^3) \),

where the matrix elements of \( \hat{H}_k^{(0)} \) are independent of \( k \), those of \( \hat{H}_k^{(1)} \) are linear in \( k \), and those of \( \hat{H}_k^{(2)} \) quadratic in \( k \). Next, we will show how each term \( \hat{H}_k^{(i)} \), \( i = 0, 1, 2 \), is constrained by the symmetries of the problem. In the proof below, we will also assume the additional condition \( |\psi_{\sigma,\tau}\rangle = |\tau\psi_{-\sigma,-\tau}\rangle \), when plugged into Eq. (2), yields \( \hat{T}|m, k\rangle = -m, -k\rangle \) for the eigenstates of the 60-degree rotations. We note that this condition sets a constraint on the sum of the phases of the two eigenstates \( |\psi_{\sigma,\tau}\rangle \) and \( |\psi_{-\sigma,-\tau}\rangle \). Since all relative phases for the bases \( |\psi_{\sigma,\tau}\rangle \) were already fixed by Eq. (1) of the main text, the additional condition corresponds to fix a global phase for the basis \( |\psi_{\sigma,\tau}\rangle \). This will clearly not change the form of the final Hamiltonian but makes the proof less cumbersome. Thus, the additional condition does not lead to any loss of generality of the proof.

We start from \( \hat{H}_k^{(0)} \) whose most general form is

\[ \hat{H}_k^{(0)} = \sum_{m,m'} h_{m,m',k}^{(0)} |m, k\rangle \langle m', k| \]

where \( h_{m,m'}^{(0)} \) is hermitian. We enforce the rotational symmetry by requiring

\[ \hat{R}_{\pi/3} \hat{R}_k \hat{R}_{\pi/3}^\dagger \hat{R}_{\pi/3} \]

which should hold at each order in \( k \). Taking into account that \( \hat{R}_{\pi/3} |m, k\rangle = \exp[-im\pi/3] |m, Rk\rangle \) where \( Rk \) is the rotated quasi-momentum,

\[ \hat{R} = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \]

we find

\[ \sum_k \hat{R}_{\pi/3} \hat{H}_k^{(0)} \hat{R}_{\pi/3}^\dagger = \sum_{m,m'} \sum_k c^{(m'-m)\pi/3} h_{m,m',k}^{(0)} |m, \hat{R}k\rangle \langle m', \hat{R}k| \]

\[ = \sum_{m,m'} \sum_k c^{(m'-m)\pi/3} h_{m,m',k}^{(0)} |m, k\rangle \langle m', k| \]

\[ = \sum_{m,m'} \sum_k h_{m,m',k}^{(0)} |m, k\rangle \langle m', k| \]

The last equality follows from the six-fold rotational symmetry Eq. (5) and it implies that \( h_{m,m'}^{(0)} = 0 \) for \( m \neq m' \). In other words, the states carrying different quasi-angular momentum are not coupled in \( \hat{H}_k^{(0)} \). We can further constrain the matrix elements \( h_{m,m'}^{(0)} \) by taking advantage of the time-reversal symmetry. Under the time-reversal symmetry we have

\[ \sum_k \hat{H}_k^{(0)} \hat{T}^\dagger = \sum_{m,m'} \sum_k h_{m,m',k}^{(0)} \langle m, -k| \langle -m, -k| \]

Thus, from \( \sum_k \hat{H}_k^{(0)} \hat{T}^\dagger = \sum_k \hat{H}_k^{(0)} \) it follows that \( h_{m,m}^{(0)} = h_{m,m}^{(0)} \). No further constraint follows from the mirror symmetries. In order to arrive at Eq. (2) of the main text, we also have to express the allowed terms in terms of the \( \tau \)'s and \( \sigma \)'s Pauli matrices. This is readily done by rewriting Eq. (4) with the constraint \( h_{m,m}^{(0)} = h_{m,m}^{(0)} \) in terms of the basis \( |\psi_{\sigma,\tau,k}\rangle \) using Eq. (2). In this basis, we find

\[ \hat{H}_k^{(0)} = \omega + g\tau_x \]

where \( \omega = \left( h_{11}^{(0)} + h_{22}^{(0)} \right)/2 \) is the frequency of the tip of the cones for the unperturbed lattice \( \Delta \tau = 0 \), and \( g = \left( h_{22}^{(0)} - h_{11}^{(0)} \right)/2 \) is the mass.

Next we follow a similar procedure with \( \hat{H}_k^{(1)} \) whose most general form is

\[ \hat{H}_k^{(1)} = \sum_{m,m'} k \cdot h_{m,m',k}^{(1)} |m, k\rangle \langle m', k| \]

Here, for each fixed \( m \) and \( m' \), \( m, m' = \pm 2, \pm 1 \), \( h_{m,m'}^{(1)} \) is a 2-vector, and \( h_{m,m'}^{(1)} = h_{m,m'}^{(1)} \) to ensure that \( \hat{H}_k^{(1)} \) is hermitian. Under a rotation by 60 degrees we find

\[ \sum_k \hat{R}_{\pi/3} \hat{H}_k^{(1)} \hat{R}_{\pi/3}^\dagger = \sum_{m,m'} \sum_k k \cdot h_{m,m',k}^{(1)} e^{i(m'-m)\pi/3} |m, \hat{R}k\rangle \langle m', \hat{R}k| \]

\[ = \sum_{m,m'} \sum_k k \cdot \hat{H}_k^{(1)} e^{i(m'-m)\pi/3} |m, k\rangle \langle m', k| \]

\[ = \sum_{m,m'} \sum_k k \cdot h_{m,m',k}^{(1)} |m, k\rangle \langle m', k| \]
Thus, we find the constraint
\[ \hat{H}_{m,m'}^{(1)} e^{i(m-m')\pi/3} = h_{m,m'}^{(1)}. \]  
(7)
This implies that \( h_{m,m'}^{(1)} = 0 \), or it is an eigenvector of the matrix \( \hat{R} \) with eigenvalue \( \exp[i(m-m')\pi/3] \). Since \( \hat{R} \) has eigenvalues \( \exp[\pm i\pi/3] \), \( h_{m,m'}^{(1)} \) can be different from zero only for \( \Delta m = m - m' = \pm 1 \). In other words, the linear terms in the quasi-momentum induce transitions that change the quasi-angular momentum by one unit. We note that the allowed transitions do not flip the spin \( S \). This underlies the block structure of the Dirac Hamiltonian (2) of the main text. By solving Eq. (7), we find
\[ h_{1,1}^{(1)} = (h_{2,1}^{(1)})^* = h_{+}^{(1)} \begin{pmatrix} 1 \\ i \end{pmatrix}, \]
\[ h_{1,-1}^{(1)} = (h_{2,-1}^{(1)})^* = h_{-}^{(1)} \begin{pmatrix} 1 \\ i \end{pmatrix}, \]
(8)
where \( h_{\pm}^{(1)} \) are two complex numbers. The Hamiltonian is further constrained by the remaining symmetries. By applying the time-reversal symmetry we find
\[ \sum_{k} \hat{T} \hat{H}_{k}^{(1)} \hat{T}^{-1} = \sum_{k} \sum_{m m'} k \cdot (h_{m,m'}^{(1)})^* - m, -k \langle m', -k | = \sum_{k} \sum_{m m'} k \cdot (-h_{-m,-m'})^* |m, k \rangle \langle m', k | = \sum_{k} \sum_{m m'} k \cdot h_{m,m'}^{(1)} |m, k \rangle \langle m', k |. \]
(9)
From the last equality (that enforces the \( \hat{T} \) symmetry)
\[ h_{m,m'}^{(1)} = (-h_{m,-m'})^* = -h_{-m',-m'}. \]
Using Eqs. (8), it follows
\[ h_{+}^{(1)} = -h_{-}^{(1)}. \]
(10)
We can constrain even more \( h_{m,m'}^{(1)} \) by using the mirror symmetry \( M_{xx} \). We find,
\[ \sum_{k} M_{xx} \hat{H}_{k}^{(1)} \hat{M}_{xx} = \sum_{k} \sum_{m m'} (-)^{m+m'} k \cdot h_{m,m'}^{(1)} |m, \hat{M} k \rangle \langle -m', \hat{M} k | = \sum_{k} \sum_{m m'} (-)^{m+m'} k \cdot \hat{M} h_{m,-m'}^{(1)} |m, k \rangle \langle m', k | = \sum_{k} \sum_{m m'} k \cdot h_{m,m'}^{(1)} |m, k \rangle \langle m', k |, \]
(11)
where
\[ \hat{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
From the last equality in Eq. (11), we find
\[ h_{1,2}^{(1)} = -\hat{M} h_{-1,-2}^{(1)}, \]
and from Eq. (8)
\[ h_{+}^{(1)} = -h_{-}^{(1)*}. \]
From the above equation and Eq. (10), it follows that \( h_{+}^{(1)} = -h_{-}^{(1)} \) and both matrix elements are real. As already outlined for the term \( H_{k}^{(0)} \), we express also \( \hat{H}_{k}^{(1)} \) in terms of the \( \sigma \)'s and \( \tau \)'s Pauli matrices to obtain
\[ \hat{H}_{k}^{(1)} = v\tau_x(k_x\hat{\sigma}_x - k_y\hat{\sigma}_y), \]
(12)
where \( v = -h_{+}^{(1)} \). Thus, putting together Eqs. (3,6,12) we find Eq. (2) of the main text.

Next, we go a step further, calculating the term \( H_{k}^{(2)} \) containing the quadratic contributions,
\[ H_{k}^{(2)} = \sum_{m m'} (\hat{h}_{m,m'}^{(2)} k \cdot \hat{M} k, \langle m', k | \]
where for each fixed \( m, m' \), \( m, m' \in \pm 2, \pm 1 \), \( \hat{h}_{m,m'}^{(2)} \) is a \( 2 \times 2 \) complex matrix that can be chosen to be symmetric. Moreover, \( \hat{h}_{m,m'}^{(2)} = (\hat{h}_{m,m'}^{(2)})^* \) to ensure that \( H_{k}^{(2)} \) is hermitian. By applying a rotation by 60-degrees we find
\[ \sum_{k} \hat{R}(\pi/3) H_{k}^{(2)} \hat{R}^\dagger (\pi/3) = \sum_{k} \sum_{m m'} k \hat{h}_{m,m'}^{(2)} k_{\tilde{m},m'} e^{i(m-m')\pi/3} |m, \hat{R} k \rangle \langle m', \hat{R} k | = \sum_{k} \sum_{m m'} (\hat{R}^{-1} k) \hat{h}_{m,m'}^{(2)} (\hat{R}^{-1} k) e^{i(m-m')\pi/3} |m, k \rangle \langle m', k | = \sum_{k} \sum_{m m'} k \hat{R} \hat{h}_{m,m'}^{(2)} k_{\tilde{m},m} e^{i(m-m')\pi/3} |m, \hat{R} k \rangle \langle m', k |, \]
\[ = \sum_{k} \sum_{m m'} k \hat{h}_{m,m'}^{(2)} k_{\tilde{m},m} |m, k \rangle \langle m', k |, \]
Using the last equality (that follows from the six-fold rotational symmetry of \( H_{k}^{(2)} \)) we find
\[ e^{i(m-m')\pi/3} \hat{R} \hat{h}_{m,m'}^{(2)} \hat{R}^{-1} = \hat{h}_{m,m'}^{(2)}. \]
(13)
For each fixed \( m, m' \), \( m, m' \in \pm 2, \pm 1 \), this is just a homogeneous system of four linear equations of four variables (the entries of the matrix \( \hat{h}_{m,m'}^{(2)} \)). For the combinations of \( m, m' \) where all equations are independent, it has only a trivial solution \( \hat{h}_{m,m'}^{(2)} = 0 \). Thus, a finite matrix element \( \hat{h}_{m,m'}^{(2)} \neq 0 \) is allowed only if the system of linear equations defined in Eq. (13) is not independent. It turns out that this is the case for \( \Delta m = (m'-m) \mod 6 = 0, \pm 2 \). We note that the transitions that change the quasi-angular momentum by two units flip the spin \( S \). Such spin-flipping transitions lift
the degeneracy of states with opposite spin. By solving Eq. (13) [taking also into account the constraints on $h_{m,m}^{(2)}$, discussed above] we find,

$$
\hat{h}_{m,m}^{(2)} = \alpha_m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
\hat{h}_{-1,1}^{(2)} = (\hat{h}_{1,-1}^{(2)})^* = \beta_1 \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},
$$

$$
\hat{h}_{2,-2}^{(2)} = (\hat{h}_{-2,2}^{(2)})^* = \beta_2 \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},
$$

where $\alpha_m$ ($m = \pm 1, \pm 2$) are real numbers, while $\beta_1$ and $\beta_2$ are complex numbers. However, a straightforward calculation along the same line as the one displayed in Eq. (11) shows that the mirror symmetry sets the additional constraints $\alpha_m = \alpha_{-m}$ and $\beta_{1/2} \in \text{Re}$. Finally, we rewrite also $H_{k}^{(2)}$ in terms of the Pauli matrices to obtain

$$
H_{k}^{(2)} = (\tilde{\alpha} + \delta \alpha \sigma_x)(k_x^2 + k_y^2)
- (\tilde{\beta} + \delta \beta \sigma_y)[(k_x^2 - k_y^2)\sigma_x + 2k_xk_y\sigma_y]
$$

where $\tilde{\alpha} = (\alpha_1 + \alpha_2)/2$, $\delta \alpha = (\alpha_2 - \alpha_1)/2$, and likewise for $\tilde{\beta}$ and $\delta \beta$.

V. FULL FINITE-ELEMENT SIMULATION OF HELICAL PROPAGATION

In this section, we present full numerical ab-initio results for the propagation of helical excitations in a finite-size geometry that supplement and confirm the tight-binding results discussed in the main text.

To motivate the approach followed here, we note that it is challenging to generate helicity-selective excitations in an experiment (especially at the nanoscale). In theory, the most straightforward approach consists in tailoring the driving force to imprint a finite quasi-angular momentum to the vibrations. This could be achieved by interfering two lasers one of them causing orbital angular momentum and, thus, creating a pattern of phase delays between the sinusoidal radiation pressure forces acting on different neighboring triangular membranes.

In practice, a simpler approach consists in exciting both valley polarizations at the same time (by a simple sinusoidal force acting on a single membrane), relying on the valley-dependent propagation direction to separate the two polarizations, and subsequently confirming this separation using a suitable geometry. Such a scheme can be implemented using a beam-splitter-like setup such as the one proposed in [3] for acoustic waves at the macroscale. As we will show here, a setup of this type can also enable valley-polarized states of elastic waves at the nanoscale. This setup requires the silicon slab to be divided into four regions (Fig. 3), where opposite sections belong to the same domain (either $\Delta r < 0$ or $\Delta r > 0$), which basically results in two intersecting domain interfaces. A driving force with a properly chosen frequency excites the structure at one point near a domain interface in order to launch edge states propagating left- and rightwards along this particular domain interface (cf. Fig. 3a,b). In the case of non-helical edge states there is no distinction between left- and right-movers such that after hitting the center of the beamsplitter the elastic waves will propagate in all directions along the domain interfaces (Fig. 3b). However, in the case of helical states only the right-movers (cyan) eventually reach the center of the beamsplitter. Due to the special arrangement of the domains there are only two domain interfaces supporting right-moving states that propagate away from the center, whereas the remaining domain interface (lower-right interface) does not support right-moving states propagating outwards (Fig. 3b). In this way, helical transport can be clearly distinguished from non-helical transport, even though the initial excitation produces both polarizations.

We have verified the above scenario by calculating the linear response to a periodic driving force using finite element methods. An intrinsic damping factor was introduced to avoid elastic waves hitting the physical boundary. This helps to keep the simulated structure and the computational effort as small as possible. The result clearly shows that an excitation near port $A$ results in a propagation of the elastic waves to port $B$ and $D$, whereas no elastic waves will hit port $C$, thereby proving the helical nature of the elastic waves (Fig. 3d).

We use the beam-splitter setup also to test the resilience to disorder of the topological transport without having to rely on the tight-binding model. In an experiment, fabrication errors are expected to be the main source of disorder. In our finite element simulations, we model these fabrication errors as random variations of the snowflake radii. In a structure with lattice constant $5 \mu$m, we allow a variation of the radii of the order of $40\mu$m, see Fig. 4a. (This is one order of magnitude larger than the typical error using state of the art fabrication techniques.) The corresponding transmission plot, cf. Fig. 4b, shows that even in the presence of realistic disorder the beam-splitter like setup separates the right-moving elastic wave, guiding it to port $B$ and $D$ but not to port $C$.

VI. TIGHT-BINDING-MODEL FOR THE TRANSPORT SIMULATIONS

In our transport calculation, we have simulated the photonic crystal by a tight-binding model that displays the same symmetries as the phononic crystal and, thus, is described by the same effective Dirac equation (2). In the tight-binding model, a bosonic mode $\hat{\delta}_j$ is localized on each site $j$ of a honeycomb lattice and hopping occurs between nearest neighbors. The bosonic modes are identical but there are two different hopping rates: the hopping rate $J$ for the links connecting the sites within a hexagonal unit cell containing 6 sites, and the hopping
Figure 3. Full FEM simulation of helical transport, in a beamsplitter arrangement for elastic waves. By dividing the snowflake silicon slab into four regions, with opposite sections belonging to the same domain, one can establish the presence of helical transport. With an arrangement as depicted in (b), only the right moving state (cyan) propagates towards the center of the beamsplitter. As only two of the four domain interfaces support right-movers traveling outwards, no elastic waves propagate along the interface opposite to the excitation-interface. In contrast, non-helical edge-states would propagate symmetrically along all domain interfaces (c). Finite-element simulations of the full 3D snowflake crystal reveal the helicity of the edge states: The linear response to a localized periodic force (at frequency $f_{ex} = 1.463 \text{GHz}$ in this simulation) results in elastic waves propagating towards the ports A, B and D but not to port C. (d) depicts the corresponding rms-displacement field over one cycle of oscillation of the structure. Note that only the essential central part of the beamsplitter is shown here (the simulated structure is about twice as large). (e) illustrates a zoom into a domain interface, showing the exact geometry of the domain interfaces.

Figure 4. Full FEM simulation of the beamsplitter arrangement with an incorporated geometric disorder. (a) Distribution of the radii of the individual snowflakes. The radii vary randomly across the whole simulated setup that comprises more than 2000 snowflakes. The simulated geometrical disorder is roughly one order of magnitude larger than the disorder that can be achieved with present-day fabrication techniques. (b) Linear response of an elastic wave launched in the upper-left domain interface (white dot). For the design details see Fig. 3.

Figure 5. Scheme of the tight-binding model: A triangular lattice of hexagonal unit cells comprising six sites (grey triangles) with equal eigenfrequency $\Omega$. The single sites within one unit cell (indicated by the dashed line) are coupled with the hopping amplitudes $\tilde{J}$ (blue lines) whereas two sites of different unit cells are coupled by $J$ (red lines). Rate $J$ for the links connecting sites in neighboring unit cells, cf. the sketch in Fig. 5. We can map the tight-binding model to our finite element simulations taking into account that $q = \tilde{J} - J$, and $v = 3J/2$.

Besides sharing the same symmetries, there is an obvious connection between the tight-binding model and our snowflake phononic crystal: the phononic crystal can be viewed as an arrangement of triangular membranes on a honeycomb lattice connected by weak links. By changing the radius of the central snowflake, the links connecting the triangles also change, consequently leading to different couplings. In particular, smaller central snowflakes ($\Delta r < 0$) imply larger internal couplings ($J > J$).
results, displayed in Fig. 6, show that the uni-directional propagation of a polarized mechanical wave along a domain wall is not significantly perturbed for disorder as large as $\epsilon = 3\%$. This is already safely above the level of disorder ($\sim 1\%$) seen in the experimentally produced snowflake phononic crystals. In the present case, where we envisage structures of larger lattice constant, the fractional disorder level $\epsilon$ is expected to be even smaller.

**Figure 6.** Investigation of the effects of disorder in a finite size sample simulated by using the effective tight-binding model. A $p^{+}$-polarized mechanical wave is injected right at the mirror plane of a mirror-symmetrical sample. (The choice of a mirror-symmetrical arrangement ensures that left movers and right movers will experience the exact same boundary shape along their way if an excitation is injected exactly at the symmetry axis, which is helpful for analyzing the results). The resulting transmission is plotted for (a) no disorder, and (b,c,d) increasing levels of disorder, where $\epsilon$ is the fractional strength of the disorder. Equal colors in different subfigures correspond to equal values of the mechanical response.

**VII. EFFECT OF DISORDER**

We investigate the effects of disorder by simulating the propagation of mechanical waves in the effective tight-binding model. In our simulations, the tight-binding parameters $\Omega_i$ (onsite energies) and $J_i$ and $\tilde{J}_i$ (external and internal hoppings) vary randomly according to

$$\begin{align*}
\Omega_i &= \Omega + \xi_{i,1}\Omega, \\
J_i &= J + \xi_{i,2}J, \\
\tilde{J}_i &= \tilde{J} + \xi_{i,3}\tilde{J},
\end{align*}$$

where the index $i$ indicates the lattice site, and $\xi_{i,j}$ are independent random variables in the range $-\epsilon < \xi_{i,j} < \epsilon$. This is a reasonable choice as the main source of disorder are the fabrication-induced geometrical errors. We are, thus, basically assuming that a deviation from the ideal geometry affects all parameters in the effective tight-binding model to an approximately equal degree (though the sign and exact size of the change may be different for the different parameters; hence the assumed independence of the random variables). Our numerical