

## Classification of continuously transitive circle groups

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Let  $G$  be a closed transitive subgroup of  $\text{Homeo}(\mathbb{S}^1)$  which contains a non-constant continuous path  $f: [0, 1] \rightarrow G$ . We show that up to conjugation  $G$  is one of the following groups:  $\text{SO}(2, \mathbb{R})$ ,  $\text{PSL}(2, \mathbb{R})$ ,  $\text{PSL}_k(2, \mathbb{R})$ ,  $\text{Homeo}_k(\mathbb{S}^1)$ ,  $\text{Homeo}(\mathbb{S}^1)$ . This verifies the classification suggested by Ghys in [5]. As a corollary we show that the group  $\text{PSL}(2, \mathbb{R})$  is a maximal closed subgroup of  $\text{Homeo}(\mathbb{S}^1)$  (we understand this is a conjecture of de la Harpe). We also show that if such a group  $G < \text{Homeo}(\mathbb{S}^1)$  acts continuously transitively on  $k$ -tuples of points,  $k > 3$ , then the closure of  $G$  is  $\text{Homeo}(\mathbb{S}^1)$  (cf [1]).

[37E10](#); [22A05](#), [54H11](#)

### 1 Introduction

Let  $\text{Homeo}(\mathbb{S}^1)$  denote the group of orientation preserving homeomorphisms of  $\mathbb{S}^1$  which we endow with the uniform topology. Let  $G$  be a subgroup of  $\text{Homeo}(\mathbb{S}^1)$  with the topology induced from  $\text{Homeo}(\mathbb{S}^1)$ . We say that  $G$  is transitive if for every two points  $x, y \in \mathbb{S}^1$ , there exists a map  $f \in G$ , such that  $f(x) = y$ . We say that a group  $G$  is closed if it is closed in the topology of  $\text{Homeo}(\mathbb{S}^1)$ . A continuous path in  $G$  is a continuous map  $f: [0, 1] \rightarrow G$ .

Let  $\text{SO}(2, \mathbb{R})$  denote the group of rotations of  $\mathbb{S}^1$  and  $\text{PSL}(2, \mathbb{R})$  the group of Möbius transformations. The first main result we prove describes transitive subgroups of  $\text{Homeo}(\mathbb{S}^1)$  that contain a non constant continuous path.

**Theorem 1.1** *Let  $G$  be a transitive subgroup of  $\text{Homeo}(\mathbb{S}^1)$  which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:*

- (1)  $G$  is conjugate to  $\text{SO}(2, \mathbb{R})$  in  $\text{Homeo}(\mathbb{S}^1)$ .
- (2)  $G$  is conjugate to  $\text{PSL}(2, \mathbb{R})$  in  $\text{Homeo}(\mathbb{S}^1)$ .
- (3) For every  $f \in \text{Homeo}(\mathbb{S}^1)$  and each finite set of points  $x_1, \dots, x_n \in \mathbb{S}^1$  there exists  $g \in G$  such that  $g(x_i) = f(x_i)$  for each  $i$ .

- (4)  $G$  is a cyclic cover of a conjugate of  $\mathrm{PSL}(2, \mathbb{R})$  in  $\mathrm{Homeo}(\mathbb{S}^1)$  and hence conjugate to  $\mathrm{PSL}_k(2, \mathbb{R})$  for some  $k > 1$ .
- (5)  $G$  is a cyclic cover of a group satisfying condition 3 above.

Here we write  $\mathrm{PSL}_k(2, \mathbb{R})$  and  $\mathrm{Homeo}_k(\mathbb{S}^1)$  to denote the cyclic covers of the groups  $\mathrm{PSL}(2, \mathbb{R})$  and  $\mathrm{Homeo}(\mathbb{S}^1)$  respectively, for some  $k \in \mathbb{N}$ .

The proof begins by showing that the assumptions of the theorem imply that  $G$  is continuously 1-transitive. This means that if we vary points  $x, y \in \mathbb{S}^1$  in a continuous fashion, then we can choose corresponding elements of  $G$  which map  $x$  to  $y$  that also vary in a continuous fashion. In Theorems 3.8 and 3.10 we show that this leads us to two possibilities, either  $G$  is conjugate to  $\mathrm{SO}(2, \mathbb{R})$ , or  $G$  is a cyclic cover of a group which is continuously 2-transitive.

We then analyse groups which are continuously 2-transitive and show that they are in fact all continuously 3-transitive. Furthermore, if such a group is not continuously 4-transitive, we show that it is a convergence group and hence conjugate to  $\mathrm{PSL}(2, \mathbb{R})$ . On the other hand if it is continuously 4-transitive, then we use an induction argument to show that it is continuously  $n$ -transitive for all  $n \geq 4$ . This implies that for every  $f \in \mathrm{Homeo}(\mathbb{S}^1)$  and each finite set of points  $x_1, \dots, x_n \in \mathbb{S}^1$  there exists a group element  $g$  such that  $g(x_i) = f(x_i)$  for each  $i$ .

The remaining possibilities, namely cases 2 and 3, arise when the aforementioned cyclic cover is trivial.

In the case where the group  $G$  is also closed we can use Theorem 1.1 to make the following classification.

**Theorem 1.2** *Let  $G$  be a closed transitive subgroup of  $\mathrm{Homeo}(\mathbb{S}^1)$  which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:*

- (1)  $G$  is conjugate to  $\mathrm{SO}(2, \mathbb{R})$  in  $\mathrm{Homeo}(\mathbb{S}^1)$ .
- (2)  $G$  is conjugate to  $\mathrm{PSL}_k(2, \mathbb{R})$  in  $\mathrm{Homeo}(\mathbb{S}^1)$  for some  $k \geq 1$ .
- (3)  $G$  is conjugate to  $\mathrm{Homeo}_k(\mathbb{S}^1)$  in  $\mathrm{Homeo}(\mathbb{S}^1)$  for some  $k \geq 1$ .

The above theorem provides the classification of closed, transitive subgroups of  $\mathrm{Homeo}(\mathbb{S}^1)$  that contain a non-trivial continuous path. This classification was suggested by Ghys for all transitive and closed subgroups of  $\mathrm{Homeo}(\mathbb{S}^1)$  (See [5]).

One well known problem in the theory of circle groups is to prove that the group of Möbius transformations is a maximal closed subgroup of  $\mathrm{Homeo}(\mathbb{S}^1)$ . We understand

that this is a conjecture of de la Harpe (see [1]). The following theorem follows directly from our work and answers this question.

**Theorem 1.3**  $\text{PSL}(2, \mathbb{R})$  is a maximal closed subgroup of  $\text{Homeo}(\mathbb{S}^1)$ .

In the following five sections we develop the techniques needed to prove our results. Here we prove the results about the transitivity on  $k$ -tuples of points. In Section 7 we give the proofs of all the main results stated above.

## 2 Continuous Transitivity

Let  $G < \text{Homeo}(\mathbb{S}^1)$  be a transitive group of orientation preserving homeomorphisms of  $\mathbb{S}^1$ . We begin with some definitions which generalize the notion of transitivity.

Set,

$$P_n = \{(x_1, \dots, x_n) : x_i \in \mathbb{S}^1, x_i = x_j \iff i = j\}$$

to be the set of distinct  $n$ -tuples of points in  $\mathbb{S}^1$ . Two  $n$ -tuples

$$(x_1, \dots, x_n), (y_1, \dots, y_n) \in P_n$$

have matching orientations if there exists  $f \in \text{Homeo}(\mathbb{S}^1)$  such that  $f(x_i) = y_i$  for each  $i$ .

**Definition 2.1**  $G$  is  $n$ -transitive if for every pair  $(x_1, \dots, x_n), (y_1, \dots, y_n) \in P_n$  with matching orientations there exists  $g \in G$  such that  $g(x_i) = y_i$  for each  $i$ .

**Definition 2.2**  $G$  is uniquely  $n$ -transitive if it is  $n$ -transitive and for each pair  $(x_1, \dots, x_n), (y_1, \dots, y_n) \in P_n$  with matching orientations there is exactly one element  $g \in G$  such that  $g(x_i) = y_i$ . Equivalently, the only element of  $G$  fixing  $n$  distinct points is the identity.

Endow  $\mathbb{S}^1$  with the standard topology and  $P_n$  with the topology it inherits as a subspace of the  $n$ -fold Cartesian product  $\mathbb{S}^1 \times \dots \times \mathbb{S}^1$ . These are metric topologies. With the topology on  $P_n$  being induced by the distance function

$$d_{P_n}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max\{d_{\mathbb{S}^1}(x_i, y_i) : i = 1, \dots, n\},$$

where  $d_{\mathbb{S}^1}$  is the standard Euclidean distance function on  $\mathbb{S}^1$ .

Endow  $G$  with the uniform topology. This is also a metric topology, induced by the distance function,

$$d_G(g_1, g_2) = \sup\{\max\{d_{\mathbb{S}^1}(g_1(x), g_2(x)), d_{\mathbb{S}^1}(g_1^{-1}(x), g_2^{-1}(x))\} : x \in \mathbb{S}^1\}$$

A path in a topological space  $X$  is a continuous map  $\gamma: [0, 1] \rightarrow X$ . If  $\mathcal{X}: [0, 1] \rightarrow P_n$  is a path in  $P_n$  we will write  $x_i(t) = \pi_i \circ \mathcal{X}(t)$ , where  $\pi_i$  is projection onto the  $i$ -th component of  $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ , so that we can write  $\mathcal{X}(t) = (x_1(t), \dots, x_n(t))$ . We will call a pair of paths  $\mathcal{X}, \mathcal{Y}: [0, 1] \rightarrow P_n$  *compatible* if there exists a path  $h: [0, 1] \rightarrow \text{Homeo}(\mathbb{S}^1)$  with  $h(t)(x_i(t)) = y_i(t)$  for each  $i$  and  $t$ .

**Definition 2.3**  $G$  is continuously  $n$ -transitive if for every compatible pair of paths  $\mathcal{X}, \mathcal{Y}: [0, 1] \rightarrow P_n$  there exists a path  $g: [0, 1] \rightarrow G$  with the property that  $g(t)(x_i(t)) = y_i(t)$  for each  $i$  and  $t$ .

**Definition 2.4** A continuous deformation of the identity in  $G$  is a non constant path of homeomorphisms  $f_t \in G$  for  $t \in [0, 1]$  with  $f_0 = \text{id}$ .

We have the following lemma.

**Lemma 2.5** For  $n \geq 2$  the following are equivalent:

- (1)  $G$  is continuously  $n$ -transitive.
- (2)  $G$  is continuously  $n-1$ -transitive and the following holds. For every  $n-1$ -tuple  $(a_1, \dots, a_{n-1}) \in P_{n-1}$  and  $x \in \mathbb{S}^1 \setminus \{a_1, \dots, a_{n-1}\}$  there exists a continuous map  $F_x: I_x \rightarrow G$  satisfying the following conditions,
  - (a)  $F_x(y)$  fixes  $a_1, \dots, a_{n-1}$  for all  $y \in I_x$
  - (b)  $(F_x(y))(x) = y$  for all  $y \in I_x$
  - (c)  $F_x(x) = \text{id}$
 where  $I_x$  is the component of  $\mathbb{S}^1 \setminus \{a_1, \dots, a_{n-1}\}$  containing  $x$ .
- (3)  $G$  is continuously  $n-1$ -transitive and there exists  $(a_1, \dots, a_{n-1}) \in P_{n-1}$  with the following property. There is a component  $I$  of  $\mathbb{S}^1 \setminus \{a_1, \dots, a_{n-1}\}$ , a point  $\tilde{x} \in I$  and a continuous map  $F_{\tilde{x}}: I \rightarrow G$  satisfying the following conditions,
  - (a)  $F_{\tilde{x}}(y)$  fixes  $a_1, \dots, a_{n-1}$  for all  $y \in I$
  - (b)  $(F_{\tilde{x}}(y))(\tilde{x}) = y$  for all  $y \in I$
  - (c)  $F_{\tilde{x}}(\tilde{x}) = \text{id}$ .

- (4)  $G$  is continuously  $n - 1$ -transitive and there exists  $(a_1, \dots, a_{n-1}) \in P_{n-1}$  with the following property. There is a component  $I$  of  $\mathbb{S}^1 \setminus \{a_1, \dots, a_{n-1}\}$ , such that for each  $x \in I$  there exists a continuous deformation of the identity  $f_t$ , satisfying  $f_t(a_i) = a_i$  for each  $t$  and  $i$  and  $f_t(x) \neq x$  for some  $t$ .

**Proof** We start by showing  $[1 \Rightarrow 4]$ . As  $G$  is continuously  $n$ -transitive, it will automatically be continuously  $n - 1$  transitive. Take  $(a_1, \dots, a_{n-1}) \in P_{n-1}$  and  $x \in \mathbb{S}^1 \setminus \{a_1, \dots, a_{n-1}\}$ . Let  $I_x$  be the component of  $\mathbb{S}^1 \setminus \{a_1, \dots, a_{n-1}\}$  which contains  $x$ . Take  $y \in I_x \setminus \{x\}$  and let  $x_t$  be an injective path in  $I_x$  with  $x_0 = x$  and  $x_1 = y$ .

Let  $\mathcal{X}: [0, 1] \rightarrow P_n$  be the constant path defined by  $\mathcal{X}(t) = (a_1, \dots, a_{n-1}, x_0)$  and let  $\mathcal{Y}: [0, 1] \rightarrow P_n$  be the path defined by  $\mathcal{Y}(t) = (a_1, \dots, a_{n-1}, x_t)$ . Then since  $x_t \in I_x$  for every time  $t$  these form an compatible pair of paths. Consequently, there exists a path  $g_t \in G$  which fixes each  $a_i$  and such that  $g_t(x) = (x_t)$ . Defining  $f_t = g_t \circ (g_0^{-1})$  gives us the required continuous deformation of the identity.

We now show that  $[4 \Rightarrow 3]$ . For  $\tilde{x} \in I$  set  $K_{\tilde{x}}$  to be the set of points  $x \in I$  for which there is a path of homeomorphisms  $f_t \in G$  satisfying,

- (1)  $f_0 = \text{id}$
- (2)  $f_t(a_i) = a_i$  for each  $i$  and  $t$
- (3)  $f_1(\tilde{x}) = x$ .

Obviously,  $K_{\tilde{x}}$  will be a connected subset of  $I$  and hence an interval for each  $\tilde{x} \in I$ .

Choose  $\tilde{x} \in I$  and take  $x \in K_{\tilde{x}}$ . Let  $f_t$  and  $g_t$  be continuous deformations of the identity which fix the  $a_i$  for all  $t$  and such that  $f_{t_0}(x) \neq x$  for some  $t_0 \in (0, 1]$  and  $g_1(\tilde{x}) = x$ .  $f_t$  exists by the assumptions of condition 4. and  $g_t$  exists because  $x \in K_{\tilde{x}}$ . The following paths show that the interval between  $f_{t_0}(x)$  and  $(f_{t_0})^{-1}(x)$  is contained in  $K_{\tilde{x}}$ :

$$h_1(t) = \begin{cases} g_{2t} & t \in [0, 1/2] \\ f_{t_0(2t-1)} \circ g_1 & t \in [1/2, 1] \end{cases}$$

$$h_2(t) = \begin{cases} g_{2t} & t \in [0, 1/2] \\ (f_{t_0(2t-1)})^{-1} \circ g_1 & t \in [1/2, 1] \end{cases}$$

As  $x$  is contained in this interval and cannot be equal to either of its endpoints we see that  $K_{\tilde{x}}$  is open for every  $\tilde{x} \in I$ . On the other hand,  $\tilde{x} \in K_{\tilde{x}}$  for each  $\tilde{x} \in I$  and if  $x_1 \in K_{x_2}$  then  $K_{x_1} = K_{x_2}$ . Consequently, the sets  $\{K_{\tilde{x}} : \tilde{x} \in I\}$  form a partition of  $I$  and hence  $K_{\tilde{x}} = I$  for every  $\tilde{x} \in I$ .

We now construct the map  $F_{\tilde{x}}$ . To do this, take a nested sequence of intervals  $[x_n, y_n]$  containing  $\tilde{x}$  for each  $n$  and such that  $x_n, y_n$  converge to the endpoints of  $I$  as  $n \rightarrow \infty$ . We define  $F_{\tilde{x}}$  inductively on these intervals. Since  $K_{\tilde{x}} = I$  we can find a path of homeomorphisms  $f_t \in G$  satisfying,

- (1)  $f_0 = \text{id}$
- (2)  $f_t(a_i) = a_i$  for each  $i$  and  $t$
- (3)  $f_1(\tilde{x}) = x_1$ .

We now show that there exists a path  $\tilde{f}_t \in G$ , which also satisfies the above, but with the additional condition that the path  $\tilde{f}_t(\tilde{x})$  is simple.

To see this, let  $[x^*, \tilde{x}]$  be the largest subinterval of  $[x_1, \tilde{x}]$  for which there exists a path  $\tilde{f}_t \in G$  which satisfies,

- (1)  $\tilde{f}_0 = \text{id}$
- (2)  $\tilde{f}_t(a_i) = a_i$  for each  $i$  and  $t$
- (3)  $\tilde{f}_1(\tilde{x}) = x^*$
- (4)  $\tilde{f}_t(\tilde{x})$  is simple.

We want to show that  $x^* = x_1$ . Assume for contradiction that  $x^* \neq x_1$ . Then since  $x^* \in [x_1, \tilde{x}]$  there exists  $s \in [0, 1]$  such that  $f_s(\tilde{x}) = x^*$  and for small  $\epsilon > 0$ , we have that  $f_{s+\epsilon}(\tilde{x}) \notin [x^*, \tilde{x}]$ . Then if we concatenate the path  $\tilde{f}_t$  with  $f_{s+\epsilon} \circ f_s^{-1} \circ \tilde{f}_1$  for small  $\epsilon$  we can construct a simple path satisfying the same conditions as  $\tilde{f}_t$  but on a interval strictly bigger than  $[x^*, \tilde{x}]$ , this contradicts the maximality of  $x^*$  and we deduce that  $x^* = x_1$ .

We can use the path  $\tilde{f}_t$  to define a map  $F_{\tilde{x}}^1: [x_1, y_1] \rightarrow G$  satisfying,

- (1)  $F_{\tilde{x}}^1(y)$  fixes each  $a_i$  for each  $y \in I$
- (2)  $(F_{\tilde{x}}^1(y))(\tilde{x}) = y$  for all  $y \in I$
- (3)  $F_{\tilde{x}}^1(\tilde{x}) = \text{id}$ .

by taking paths of homeomorphisms that move  $\tilde{x}$  to  $x_1$  and  $y_1$  along simple paths in  $\mathbb{S}^1$ .

Now assume we have defined a map  $F_{\tilde{x}}^k: [x_k, y_k] \rightarrow G$  satisfying,

- (1)  $F_{\tilde{x}}^k(y)$  fixes each  $a_i$  for each  $y \in I$
- (2)  $(F_{\tilde{x}}^k(y))(\tilde{x}) = y$  for all  $y \in I$
- (3)  $F_{\tilde{x}}^k(\tilde{x}) = \text{id}$ .

We can use the same argument used to produce  $F_{\tilde{x}}^1$  to show that there exists a map  $\mathfrak{F}_{x_k} : [x_{k+1}, x_k] \rightarrow G$  such that  $\mathfrak{F}_{x_k}(x)$  fixes the  $a_i$  for each  $x$ ,  $\mathfrak{F}_{x_k}(x_k) = \text{id}$  and  $(\mathfrak{F}_{x_k}(x))(x_k) = x$ . Similarly there exists a map  $\mathfrak{F}_{y_k} : [y_k, y_{k+1}] \rightarrow G$  such that  $\mathfrak{F}_{y_k}(x)$  fixes the  $a_i$  for each  $x$ ,  $\mathfrak{F}_{y_k}(y_k) = \text{id}$  and  $(\mathfrak{F}_{y_k}(x))(y_k) = x$ .

This allows us to define,  $F_{\tilde{x}}^{k+1} : [x_{k+1}, y_{k+1}] \rightarrow G$  by:

$$F_{\tilde{x}}^{k+1}(x) = \begin{cases} F_{\tilde{x}}^k(x) & x \in [x_k, y_k] \\ (\mathfrak{F}_{x_k}(x)) \circ F_{\tilde{x}}^k(x_k) & x \in [x_{k+1}, x_k] \\ (\mathfrak{F}_{y_k}(x)) \circ F_{\tilde{x}}^k(y_k) & x \in [y_k, y_{k+1}] \end{cases}$$

Inductively, we can now define the full map  $F_{\tilde{x}} : I \rightarrow G$ .

We now show that [3  $\Rightarrow$  2]. So take  $x' \in I$  with  $x' \neq \tilde{x}$  and define  $F_{x'} : I \rightarrow G$  by

$$(1) \quad F_{x'}(y) = F_{\tilde{x}}(y) \circ (F_{\tilde{x}}(x'))^{-1}$$

Then  $F_{x'}$  satisfies,

- (1)  $F_{x'}(y)$  fixes  $a_1, \dots, a_{n-1}$  for all  $y \in I$
- (2)  $(F_{x'}(y))(x') = y$  for all  $y \in I$
- (3)  $F_{x'}(x') = \text{id}$ .

Moreover, we can use (1) to define a map  $F : I \times I \rightarrow G$  which is continuous in each variable and satisfies,

- (1)  $F(x, y)$  fixes  $a_1, \dots, a_{n-1}$  for all  $x, y \in I$
- (2)  $(F(x, y))(x) = y$  for all  $x, y \in I$
- (3)  $F(x, x) = \text{id}$  for all  $x \in I$ .

Now take  $x'$  to be a point in  $\mathbb{S}^1 \setminus I \cup \{a_1, \dots, a_{n-1}\}$  and let  $I'$  be the component of  $\mathbb{S}^1 \setminus \{a_1, \dots, a_{n-1}\}$  which contains  $x'$ . Then since  $G$  is continuously  $n - 1$ -transitive there exists  $g \in G$  which permutes the  $a_i$  so that  $g(I) = I'$ . Define  $F_{x'} : I' \rightarrow G$  by

$$F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}$$

for  $y \in I'$ . Then  $F_{x'}$  satisfies,

- (1)  $F_{x'}(y)$  fixes  $a_1, \dots, a_{n-1}$  for all  $y \in I$
- (2)  $(F_{x'}(y))(x') = y$  for all  $y \in I'$
- (3)  $F_{x'}(x') = \text{id}$ .

Now let  $(b_1, \dots, b_{n-1}) \in P_{n-1}$  have the same orientation as  $(a_1, \dots, a_{n-1})$  then since  $G$  is continuously  $n - 1$ -transitive there exists  $g \in G$  so that  $g(a_i) = b_i$  for each  $i$ . Let  $x' \in \mathbb{S}^1 \setminus \{b_1, \dots, b_{n-1}\}$  and let  $I'$  be the component of  $\mathbb{S}^1 \setminus \{b_1, \dots, b_{n-1}\}$  in which it lies. Define  $F_{x'}: I' \rightarrow G$  by

$$F_{x'}(y) = g \circ F_{g^{-1}(x')}(g^{-1}(y)) \circ g^{-1}$$

for  $y \in I'$ . Then  $F_{x'}$  satisfies,

- (1)  $F_{x'}(y)$  fixes  $b_1, \dots, b_{n-1}$  for all  $y \in I'$
- (2)  $(F_{x'}(y))(x') = y$  for all  $y \in I'$
- (3)  $F_{x'}(x') = \text{id}$

and we have that [3  $\Rightarrow$  2]

Finally we have to show that [2  $\Rightarrow$  1]. Let  $\mathcal{X}, \mathcal{Y}: [0, 1] \rightarrow P_n$  be an compatible pair of paths. We define  $\mathcal{X}': [0, 1] \rightarrow P_{n-1}$  by

$$\mathcal{X}'(t) = (x_1(t), \dots, x_{n-1}(t))$$

and  $\mathcal{Y}': [0, 1] \rightarrow P_{n-1}$  by

$$\mathcal{Y}'(t) = (y_1(t), \dots, y_{n-1}(t)).$$

Notice that  $\mathcal{X}'$  and  $\mathcal{Y}'$  will also be a compatible pair of paths. Furthermore, as  $G$  is continuously  $n - 1$ -transitive there will exist a path  $g': [0, 1] \rightarrow G$  such that  $g'(t)(x_i(t)) = y_i(t)$  for  $1 \leq i \leq n - 1$ .

The paths  $\mathcal{X}', \mathcal{Y}': [0, 1] \rightarrow P_{n-1}$  will also be compatible with the constant paths,

$$\mathcal{X}'_0: [0, 1] \rightarrow P_{n-1}$$

$$\mathcal{X}'_0(t) = \mathcal{X}'(0)$$

and

$$\mathcal{Y}'_0: [0, 1] \rightarrow P_{n-1}$$

$$\mathcal{Y}'_0(t) = \mathcal{Y}'(0)$$

respectively. So that there exist paths  $g'_x, g'_y: [0, 1] \rightarrow G$  with  $g'_x(x_i(0)) = x_i(t)$  and  $g'_y(y_i(0)) = y_i(t)$  for  $1 \leq i \leq n - 1$ . Furthermore, by pre composing with  $(g'_x(0))^{-1}$  and  $(g'_y(0))^{-1}$  if necessary, we can assume that  $g'_x(0) = g'_y(0) = \text{id}$ .

We now construct a path  $g_x: [0, 1] \rightarrow G$  which satisfies,

$$g_x(t)(x_i(0)) = x_i(t)$$

for  $1 \leq i \leq n$ . To do this let  $I$  be the component of  $\mathbb{S}^1 \setminus \{x_1(0), \dots, x_{n-1}(0)\}$  containing  $x_n(0)$ . By assumption we have a continuous map  $F_{x_n(0)}: I \rightarrow G$  satisfying



- (1)  $F_{x_n(0)}(y)$  fixes  $x_1(0), \dots, x_{n-1}(0)$  for all  $y \in I$
- (2)  $(F_{x_n(0)}(y))(x) = y$  for all  $y \in I$
- (3)  $F_{x_n(0)}(x) = \text{id}$ .

Define  $g_x: [0, 1] \rightarrow G$  by

$$g_x(t) = g'_x(t) \circ (F_{x_n(0)}((g'_x(t))^{-1}(x_n(t))))^{-1}.$$

Then  $g_x(t)(x_i(0)) = x_i(t)$  for  $1 \leq i \leq n$ . We can repeat this process with  $g'_y$  to construct a path  $g_y: [0, 1] \rightarrow G$  satisfying  $g_y(t)(y_i(0)) = y_i(t)$  for  $1 \leq i \leq n$ .

The map  $g'(0)$  which we defined earlier will map  $x_i(0)$  to  $y_i(0)$  for  $1 \leq i \leq n - 1$ . Moreover,  $g'(0)(x_n(0))$  will lie in the same component of  $\mathbb{S}^1 \setminus \{y_1(0), \dots, y_{n-1}(0)\}$  as  $y_n(0)$ . So we have a map  $F_{g'(0)(x_n(0))}(y_n(0))$  which maps  $g'(0)(x_n(0))$  to  $y_n(0)$  and fixes the other  $y_i(0)$ . Putting all of this together allows us to define  $g: [0, 1] \rightarrow G$  by

$$g(t) = g_y(t) \circ F_{g'(0)(x_n(0))}(y_n(0)) \circ g'(0) \circ (g_x(t))^{-1}.$$

This is a path in  $G$  which satisfies  $g_i(x_i(t)) = y_i(t)$  for each  $i$  and  $t$ . Since we can do this for any two compatible paths,  $G$  is continuously  $n$ -transitive and we have shown that  $[2 \Rightarrow 1]$ . □

**Proposition 2.6** *If  $G$  is 1-transitive and there exists a continuous deformation of the identity  $f_t: [0, 1] \rightarrow G$  in  $G$ , then  $G$  is continuously 1-transitive.*

**Proof** Let  $x_0 \in \mathbb{S}^1$  be such that  $f_{t_0}(x_0) \neq x_0$  for some  $t_0 \in [0, 1]$ . Take  $x \in \mathbb{S}^1$  then there exists  $g \in G$  such that  $g(x) = x_0$ . Consequently,  $g^{-1} \circ f_t \circ g$  is a continuous deformation of the identity which doesn't fix  $x$  for some  $t$ . Since these deformations exist for each  $x \in \mathbb{S}^1$  the proof follows in exactly the same way as  $[4 \Rightarrow 1]$  from the proof of [Lemma 2.5](#). □

From now on we will assume that  $G$  contains a continuous deformation of the identity, and hence is continuously 1-transitive.

### 3 The set $J_x$

**Definition 3.1** For  $x \in \mathbb{S}^1$  we define  $J_x$  to be the set of points  $y \in \mathbb{S}^1$  which satisfy the following condition. There exists a continuous deformation of the identity  $f_t \in G$  which fixes  $x$  for all  $t$  and such that  $f_{t_0}(y) \neq y$  for some  $t_0 \in [0, 1]$ .

It follows directly from this definition that  $x \notin J_x$ .

**Lemma 3.2**  $J_{f(x)} = f(J_x)$  for every  $f \in G$  and  $x \in \mathbb{S}^1$ .

**Proof** Let  $y \in J_{f(x)}$  and let  $f_t$  be the corresponding continuous deformation of the identity with  $f_{t_0}(y) \neq y$ . Then  $f^{-1} \circ f_t \circ f$  is also a continuous deformation of the identity which now fixes  $x$ , and for which  $f_{t_0}(f^{-1}(y)) \neq f^{-1}(y)$ . This means that  $f^{-1}(y) \in J_x$  and hence  $y \in f(J_x)$  so that  $J_{f(x)} \subset f(J_x)$ . The other inclusion is an identical argument.  $\square$

**Lemma 3.3**  $J_x$  is open for every  $x \in \mathbb{S}^1$ .

**Proof** Let  $y \in J_x$  and take  $f_t$  to be the corresponding continuous deformation of the identity with  $f_{t_0}(y) \neq y$  for some  $t_0 \in [0, 1]$ . Then since  $f_{t_0}$  is continuous there exists a neighborhood  $U$  of  $y$  such that  $f_{t_0}(z) \neq z$  for all  $z \in U$ . This implies that  $U \subset J_x$  and hence that  $J_x$  is open.  $\square$

**Lemma 3.4**  $J_x = \emptyset$  for every  $x \in \mathbb{S}^1$  or  $J_x$  has a finite complement for every  $x \in \mathbb{S}^1$ .

To prove this lemma we will use the Hausdorff maximality Theorem which we now recall.

**Definition 3.5** A set  $\mathcal{P}$  is partially ordered by a binary relation  $\leq$  if,

- (1)  $a \leq b$  and  $b \leq c$  implies  $a \leq c$
- (2)  $a \leq a$  for every  $a \in \mathcal{P}$
- (3)  $a \leq b$  and  $b \leq a$  implies that  $a = b$ .

**Definition 3.6** A subset  $\mathcal{Q}$  of a partially ordered set  $\mathcal{P}$  is totally ordered if for every pair  $a, b \in \mathcal{Q}$  either  $a \leq b$  or  $b \leq a$ . A totally ordered subset  $\mathcal{Q} \subset \mathcal{P}$  is maximal if for any member  $a \in \mathcal{P} \setminus \mathcal{Q}$ ,  $\mathcal{Q} \cup \{a\}$  is not totally ordered.

**Theorem 3.7** (Hausdorff Maximality Theorem) *Every nonempty partially ordered set contains a maximal totally ordered subset.*

We now prove [Lemma 3.4](#).

**Proof** Assume that there exists  $x \in \mathbb{S}^1$  for which  $J_x = \emptyset$ . Then for every  $y \in \mathbb{S}^1$  there exists a map  $g \in G$  such that  $g(x) = y$ . Consequently,

$$J_y = J_{g(x)} = g(J_x) = g(\emptyset) = \emptyset$$

for every  $y \in \mathbb{S}^1$ .

Assume that  $J_x \neq \emptyset$  for every  $x \in \mathbb{S}^1$  and let  $S_x = \mathbb{S}^1 \setminus J_x$  denote the complement of  $J_x$ . This means that  $S_x$  consists of the points  $y \in \mathbb{S}^1$  such every continuous deformation of the identity which fixes  $x$  also fixes  $y$ . The set  $\mathcal{P} = \{S_x : x \in \mathbb{S}^1\}$  is partially ordered by inclusion so that by [Theorem 3.7](#) there exists a maximal totally ordered subset,  $\mathcal{Q} = \{S_x : x \in A\}$ , where  $A$  is the appropriate subset of  $\mathbb{S}^1$ .

If we set  $\mathcal{S} = \bigcap_{x \in A} S_x$  then we have the following:

- (1)  $\mathcal{S} \neq \emptyset$
- (2) if  $x \in \mathcal{S}$  then  $S_x = \mathcal{S}$ .

[1](#) follows from the fact that  $\mathcal{S}$  is the intersection of a descending family of compact sets, and hence is nonempty.

To see that [2](#) is also true, fix  $x \in \mathcal{S}$ . Then from the definition of  $\mathcal{S}$ , we will have  $x \in S_a$  for each  $a \in A$ . In other words, if we take  $a \in A$ , then every continuous deformation of the identity which fixes  $a$  will also fix  $x$ . Furthermore, if  $y \in S_x$  then every continuous deformation of the identity which fixes  $a$  not only fixes  $x$  but  $y$  too, so that  $S_x \subset S_a$ . This is true for every  $a \in A$  so that  $S_x \subset \mathcal{S}$ . On the other hand, by the maximality of  $\mathcal{Q}$ , it must contain  $S_x$ . Consequently, if  $x \in \mathcal{S}$  then  $S_x = \mathcal{S}$ .

Fix  $x_0 \in \mathcal{S}$  and assume for contradiction that  $S_{x_0}$  is infinite. Take a sequence  $x_n \in S_{x_0}$  and let  $x_{n_k}$  be a convergent subsequence with limit  $x'$ . This limit will also be in  $S_{x_0}$  as it is closed. As  $J_{x_0}$  is a nonempty open subset of  $\mathbb{S}^1$  it will contain an interval  $(a, b)$  with  $a, b \in S_{x_0}$ . Take maps  $g_a, g_b \in G$  so that  $g_a(x') = a$  and  $g_b(x') = b$ . Since  $x', a \in S_{x_0}$  we have that,

$$g_a(S_{x_0}) = g_a(S_{x'}) = S_{g_a(x')} = S_a = S_{x_0}$$

and similarly for  $g_b$ . As a result  $g_a(x_n), g_b(x_n) \in S_{x_0}$  for each  $n$ , but  $g_a, g_b$  are orientation preserving homeomorphisms so that at least one of these points will lie in  $(a, b)$ , a contradiction.

We have shown that  $S_{x_0}$  is finite. If we now take any other point  $x \in \mathbb{S}^1$  then there exists a map  $g \in G$  such that  $g(x_0) = x$ . This means that the set  $S_x = S_{g(x_0)} = g(S_{x_0})$  will also be finite and we are done.  $\square$

**Theorem 3.8** *If  $J_x = \emptyset$  for all  $x \in \mathbb{S}^1$  then  $G$  is conjugate in  $\text{Homeo}(\mathbb{S}^1)$  to the group of rotations  $\text{SO}(2, \mathbb{R})$ .*

We require the following lemma for the proof of this Theorem.

**Lemma 3.9** *If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism which conjugates translations to translations, then it is an affine map.*

**Proof** Let  $f$  be a homeomorphism which conjugates translations to translations and set  $f_1 = T \circ f$  where  $T$  is the translation that sends  $f(0)$  to 0. Then  $f_1$  fixes 0 and also conjugates translations to translations. In particular there exists  $\alpha$  such that  $f_1$  conjugates  $x \mapsto x + 1$  to the map  $x \mapsto x + \alpha$ . Notice that  $\alpha \neq 0$  since the identity is only conjugate to itself.

Now define  $f_2 = f_1 \circ M_\alpha$  where  $M_\alpha(x) = \alpha x$ . A simple calculation shows that  $f_2$  conjugates  $x \mapsto x + 1$  to itself and conjugates translations to translations. Since  $f_2$  fixes 0 and conjugates  $x \mapsto x + 1$  to itself, we deduce that it must fix all the integer points.

Now, for  $n \in \mathbb{N}$  let  $\gamma \in \mathbb{R}$  be such that  $(f_2)^{-1} \circ T_{1/n} \circ f_2 = T_\gamma$  where  $T_\alpha(x) = x + \alpha$ . It follows that,

$$T_1 = (f_2)^{-1} \circ (T_{1/n})^n \circ f_2 = ((f_2)^{-1} \circ T_{1/n} \circ f_2)^n = (T_\gamma)^n$$

so that  $\gamma = 1/n$  and  $(f_2)^{-1} \circ T_{1/n} \circ f_2 = T_{1/n}$  for every  $n \in \mathbb{N}$ . Combining this with the fact that  $f_2$  fixes 0, we see that  $f_2$  must fix all the rational points and hence is the identity. This implies that  $f_1$  and hence  $f$  are affine.  $\square$

We can now prove [Theorem 3.8](#).

**Proof** Let  $\widehat{G} < G$  denote the path component of the identity in  $G$ . We are going to show that  $\widehat{G}$  is a compact group. Proposition 4.1 in [5] will then imply that it is conjugate in  $\text{Homeo}(\mathbb{S}^1)$  to a subgroup of  $\text{SO}(2, \mathbb{R})$ . Moreover, as  $\widehat{G}$  is 1-transitive it will be equal to the whole of  $\text{SO}(2, \mathbb{R})$ .

For  $x \in \mathbb{S}^1$  let  $\pi_x: \mathbb{R} \rightarrow \mathbb{S}^1$  be the usual projection map which sends each integer to  $x$  and for each integer translation  $T: \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $\pi_x \circ T = \pi_x$ .

If we fix  $x \in \mathbb{S}^1$  then since  $G$  is continuously 1-transitive we can choose a continuous path  $g: [0, 1] \rightarrow G$  such that  $g(t)(x) = \pi_x(t)$  and  $g(0) = \text{id}$ . Notice that this path is contained in  $\widehat{G}$  and  $g(1)$  is not necessarily the identity even though it fixes  $x$ .

For  $x \in \mathbb{S}^1$  we define a continuous map  $F_x: \mathbb{R} \rightarrow \widehat{G}$  by

$$F_x(t) = g(t - [t]) \circ g(1)^{[t]} \quad (*)$$

where  $[t]$  is the greatest integer less than or equal to  $t$ . Set  $f = F_x(1)$ . Note that  $F_x(n) = f^n$  for every  $n \in \mathbb{Z}$ .

We claim that  $F_x$  has the following properties,

- (1)  $F_x(t)(x) = \pi_x(t)$  for every  $t \in \mathbb{R}$
- (2)  $F_x(0) = \text{id}$
- (3) The map  $F_x$  is a surjection, that is  $F_x(\mathbb{R}) = \widehat{G}$
- (4) If the map  $f = F_x(1)$  is not equal to the identity map then  $F_x$  is a bijection

The first two properties follow directly from the definition. To see that the third property holds, let  $h_s$  be a path in  $\widehat{G}$ ,  $s \geq 0$ ,  $h_0 = \text{id}$ . Let  $\alpha(s) = h_s(x)$ . We have that  $\alpha$  is a continuous map from the non-negative reals  $\mathbb{R}^+$  into the circle. Since the set  $\mathbb{R}^+$  is contractible we can lift the map  $\alpha$  into the universal cover of the circle. That is, there is a map  $\beta: \mathbb{R}^+ \rightarrow \mathbb{R}$  such that  $\pi_x \circ \beta = \alpha$ . We have  $F_x(\beta(s))(x) = h_s(x)$ . Then  $(h_s^{-1} \circ F_x(\beta(s)))(x) = x$ . It follows from the assumption of the theorem that  $F_x(\beta(s)) = h_s$  and  $F_x$  is surjective. The map  $F_x$  is injective for  $0 \leq t < 1$ , because  $F_x(t)(x) = \pi_x(t)$ . If  $F_x(1)$  is not the identity, and since  $F_x(1)(x) = x$  we have that  $F_x(m) = F_x(n)$  if and only if  $m = n$ , for every two integers  $m, n$ . This implies the fourth property.

It follows from (\*), and the surjectivity of  $F_x$ , that  $\widehat{G}$  is a compact group if and only if the cyclic group generated by  $F_x(1) = f$  is a compact group. We will prove that  $f = \text{id}$ .

Assume that  $f$  is not the identity map. Since  $F_x$  is a bijection for each  $t \in \mathbb{R}$  there exists a unique  $s_n(t) \in \mathbb{R}$  such that,

$$f^n \circ F_x(t) \circ f^{-n} = F_x(s_n(t)). \tag{**}$$

This defines a function  $s_n: \mathbb{R} \rightarrow \mathbb{R}$  which we claim is continuous for each  $n$ . To see this, fix  $n$  and let  $t_m \in \mathbb{R}$  be a convergent sequence with limit  $t'$ . Since  $F_x$  is continuous,

$$f^n \circ F_x(t_m) \circ f^{-n} \longrightarrow f^n \circ F_x(t') \circ f^{-n}$$

and so  $F_x(s_n(t_m)) \rightarrow F_x(s_n(t'))$  as  $m \rightarrow \infty$ .

Now, if  $s_n(t_{m_k})$  is a convergent subsequence, with limit  $t_0$ , then using continuity  $F_x(s_n(t_{m_k}))$  will converge to  $F_x(t_0)$ . Since  $F_x$  is a bijection this gives us that  $t_0 = s_n(t')$ . Consequently, if the sequence  $s_n(t_m)$  were bounded, then it would converge to  $t'$ .

Assume now that the sequence  $s_n(t_m)$  is unbounded and take a divergent subsequence  $s_n(t_{m_k})$ . Consider the corresponding sequence,

$$F_x(s_n(t_{m_k})) = g(s_n(t_{m_k}) - [s_n(t_{m_k})]) \circ f^{[s_n(t_{m_k})]}.$$

Since  $s_n(t_{m_k}) - [s_n(t_{m_k})] \in [0, 1)$  for each  $m$ , there exists a subsequence  $t_{m_{k_l}}$  of  $t_{m_k}$  such that  $s_n(t_{m_{k_l}}) - [s_n(t_{m_{k_l}})]$  converges to some  $t_0 \in [0, 1]$ . Now since  $g$  is continuous and the sequence  $F_x(s_n(t_m))$  converges to a homeomorphism  $F_x(s_n(t'))$  we have that

$f^{[s_n(t_{m_k})]}$  converges to a homeomorphism as  $l \rightarrow \infty$ . However, as  $s_n(t_{m_k})$  is divergent  $[s_n(t_{m_k})]$  will be divergent too.

Let  $S_f$  denote the set of fixed points of  $f$ . Note that  $x \in S_f$ . Since we assume that  $f$  is not the identity we have that  $\mathbb{S}^1 \setminus S_f$  is non-empty. Let  $J$  be a component of  $\mathbb{S}^1 \setminus S_f$  and let  $a, b \in \mathbb{S}^1$  be its endpoints. Since  $f$  fixes  $J$ , and has no fixed points inside  $J$  we deduce that on compact subsets of  $J$  the sequence  $f^{[s_n(t_{m_k})]}$  converges to one of the endpoints and consequently, can not converge to a homeomorphism. This is a contradiction, so  $s_n(t_m)$  can not be unbounded and  $s_n$  is continuous.

Notice that  $s_n(0) = 0$  and if  $t \in \mathbb{Z}$  then  $F_x(t)$  will commute with the  $f^n$  so we have  $s_n(m) = m$  for all  $m \in \mathbb{Z}$ . This yields that  $s_n([0, 1]) = [0, 1]$  for every  $n \in \mathbb{Z}$ .

Let  $U_f \subset \mathbb{S}^1$  be the set defined as follows. We say that  $y \in U_f$  if there exists an open interval  $I$ ,  $y \in I$ , such that  $|f^n(I)| \rightarrow 0$ ,  $n \rightarrow \infty$ . Here  $|f^n(I)|$  denotes the length of the corresponding interval. The set  $U_f$  is open. We show that  $U_f$  is non-empty and not equal to  $\mathbb{S}^1$ . As before, let  $J$  be a component of  $\mathbb{S}^1 \setminus S_f$  and let  $a, b \in \mathbb{S}^1$  be its endpoints. Since  $f$  fixes  $J$ , and has no fixed points inside  $J$  we deduce that on compact subsets of  $J$  the sequence  $f^n$  converges to one of the endpoints, say  $a$ . This shows that  $J \subset U_f$ . Also, this shows that the point  $b$  does not belong to  $U_f$ .

Let  $y \in U_f$ , and let  $I$  be the corresponding open interval so that  $y \in I$  and  $|f^n(I)| \rightarrow 0$ ,  $n \rightarrow \infty$ . Set  $f^n(I) = I_n$ . Consider the interval  $F_x(s_n(t))(I_n)$ ,  $t \in [0, 1]$ . Since  $s_n([0, 1]) = [0, 1]$  we have that  $F_x(s_n([0, 1]))$  is a compact family of homeomorphisms. This allows us to conclude that  $|F_x(s_n(t))(I_n)| \rightarrow 0$ ,  $n \rightarrow \infty$ , uniformly in  $n$  and  $t \in [0, 1]$ . Set  $J_t = F_x(t)(I)$ . From (\*\*) we have that  $|f^n(J_t)| \rightarrow 0$ ,  $n \rightarrow \infty$ , for a fixed  $t \in [0, 1]$ . This implies that the point  $F_x(t)(y)$  belongs to the set  $U_f$  for every  $t \in [0, 1]$ .

Let  $J$  be a component of  $U_f$ , and let  $a, b$  be its endpoints. Note that the points  $a, b$  do not belong to  $U_f$ . Since  $F_x(t)$  is a continuous path and  $F_x(0) = \text{id}$ , for small enough  $t$  we have that  $F_x(t)(J) \cap J \neq \emptyset$ . Since  $F_x(t)(J) \subset U_f$ , and since  $a, b$  are not in  $U_f$  we have that  $F_x(t)(J) = J$ . By continuity this extends to hold for every  $t \in [0, 1]$ . But this means that  $F_x(t)(a) = a$  for every  $t \in [0, 1]$ . However, for appropriately chosen inverse  $t_0 = \pi_x^{-1}(a)$ , we have that  $F_x(t_0)(x) = a$ , which contradicts the fact that  $F_x(t_0)$  is a homeomorphism. This shows that  $f = \text{id}$ , and therefore we have proved that  $\widehat{G}$  is a compact group.

To finish the argument, it remains to show that  $G = \widehat{G}$ . Let  $\Phi \in \text{Homeo}(\mathbb{S}^1)$  be a map which conjugates  $\widehat{G}$  to  $\text{SO}(2, \mathbb{R})$  and take  $g \in G \setminus \widehat{G}$ . Since  $\widehat{G}$  is a normal subgroup of  $G$ ,  $\Phi \circ g \circ \Phi^{-1}$  conjugates rotations to rotations. Lifting to the universal cover we get that every lift of  $\Phi \circ g \circ \Phi^{-1}$  conjugates translations to translations. If we choose one

then by Lemma 3.9 it will be affine. On the other hand, it must be periodic, and hence is a translation. So that  $\Phi \circ g \circ \Phi^{-1}$  is itself a rotation and we are done.  $\square$

**Theorem 3.10** *If  $J_x \neq \emptyset$  then one of the following is true:*

- (1)  $J_x = \mathbb{S}^1 \setminus \{x\}$  in which case  $G$  is continuously 2-transitive.
- (2) There exists  $R \in \text{Homeo}(\mathbb{S}^1)$  which is conjugate to a finite order rotation and satisfies  $R \circ g = g \circ R$  for every  $g \in G$ . Moreover,  $G$  is a cyclic cover of a group  $G_\Gamma$  which is continuously 2-transitive, where the covering transformations are the cyclic group generated by  $R$ .

**Proof** If  $J_x = \mathbb{S}^1 \setminus \{x\}$  then we are in case 4 of Lemma 2.5 with  $n = 2$ . In this situation we know that  $G$  will be continuously 2-transitive.

We already know that  $S_x = \mathbb{S}^1 \setminus J_x$  must contain  $x$  and by Lemma 3.4 must be finite. Moreover, as  $f(J_x) = J_{f(x)}$  the sets  $S_x$  contain the same number of points for each  $x \in \mathbb{S}^1$ . Define  $R: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  by taking  $R(x)$  to be the first point of  $S_x$  you come to as you travel anticlockwise around  $\mathbb{S}^1$ . Now take  $g \in G$  and  $x \in \mathbb{S}^1$ , then since  $J_{g(x)} = g(J_x)$  and  $g$  is orientation preserving  $R \circ g(x) = g \circ R(x)$  for all  $x \in \mathbb{S}^1$ .

We now show that  $R$  is a homeomorphism. To see this take any continuous path  $x_t \in \mathbb{S}^1$ , we will show that  $R(x_t) \rightarrow R(x_0)$  as  $t \rightarrow 0$ . Since  $G$  is continuously 1-transitive, there exists a continuous path  $g_t \in G$  satisfying  $g_t(x_t) = x_0$ , so that,

$$\lim_{t \rightarrow 0} R(x_t) = \lim_{t \rightarrow 0} (g_t)^{-1}(R(g_t(x_t))) = \lim_{t \rightarrow 0} (g_t)^{-1}(R(x_0)) = R(x_0).$$

where the first equality follows from the fact that  $R \circ g(x) = g \circ R(x)$  for all  $x \in \mathbb{S}^1$ . This shows that  $R$  is continuous. If we take  $y \notin J_x$  then  $J_x \subset J_y$ , and hence  $S_x \supset S_y$  but in this case since  $S_x$  and  $S_y$  contain the same number of points they will be equal. Consequently,  $R$  has an inverse defined by taking  $R^{-1}(x)$  to be the first point of  $S_x$  you come to by traveling clockwise around  $\mathbb{S}^1$  and this inverse is continuous by the same argument as for  $R$ . Consequently,  $R \in \text{Homeo}(\mathbb{S}^1)$ . Furthermore,  $R$  is of finite order equal to the number of points in  $S_x$  and hence conjugate to a rotation.

Let  $\Gamma$  denote the cyclic subgroup of  $\text{Homeo}(\mathbb{S}^1)$  generated by  $R$ . Define  $\pi: \mathbb{S}^1 \rightarrow \mathbb{S}^1/\Gamma \cong \mathbb{S}^1$ , in the usual way with  $\pi(x)$  being the orbit of  $x$  under  $\Gamma$ . Since  $R \circ g(x) = g \circ R(x)$  for all  $x \in \mathbb{S}^1$ , each  $g \in G$  defines a well defined homeomorphism of the quotient space  $\mathbb{S}^1/\Gamma$  which we call  $g_\Gamma$ . This gives us a homomorphism  $\pi_\Gamma: G \rightarrow \text{Homeo}(\mathbb{S}^1)$ , defined by  $\pi_\Gamma(g) = g_\Gamma$ . Let  $G_\Gamma$  denote the image of  $G$  under  $\pi_\Gamma$ , then  $G$  is a cyclic cover of  $G_\Gamma$ .

It remains to see that  $G_\Gamma$  is continuously 2–transitive. This follows from the fact that if we take  $x_0 \in \mathbb{S}^1$  then  $J_{\pi(x_0)} = \pi(J_{x_0})$ , where  $J_{\pi(x_0)}$  is the set of points that can be moved by continuous deformations of the identity in  $G_\Gamma$  which fix  $\pi(x_0)$ . Consequently,  $J_{\pi(x_0)} = \mathbb{S}^1 \setminus \{x_0\}$  so that  $G_\Gamma$  is continuously 2–transitive by the first part of this proposition.  $\square$

## 4 Implications of continuous 2–transitivity

We now know that if  $G$  is transitive and contains a continuous deformation of the identity then it is either conjugate to the group of rotations  $\text{SO}(2, \mathbb{R})$ , is continuously 2–transitive, or is a cyclic cover of a group which is continuously 2–transitive. For the rest of the paper we assume that  $G$  is continuously 2–transitive and examine which possibilities arise.

For  $n \geq 2$  and  $(x_1 \dots x_n) \in P_n$  we define  $J_{x_1 \dots x_n}$  to be the subset of  $\mathbb{S}^1$  containing the points  $x \in \mathbb{S}^1$  which satisfy the following condition. There exists a continuous deformation of the identity  $f_t \in G$ , with  $f_t(x_i) = x_i$  for each  $i$  and  $t$  and such that there exists  $t_0 \in [0, 1]$  with  $f_{t_0}(x) \neq x$ . This generalizes the earlier definition of  $J_x$  and we get the following analogous results.

**Lemma 4.1**  $J_{f(x_1) \dots f(x_n)} = f(J_{x_1 \dots x_n})$  for every  $f \in G$ .

**Lemma 4.2**  $J_{x_1 \dots x_n}$  is open.

We also have the following.

**Lemma 4.3** If  $J_{x_1 \dots x_n}$  is nonempty and  $G$  is continuously  $n$ –transitive, then it is equal to  $\mathbb{S}^1 \setminus \{x_1 \dots x_n\}$ .

**Proof** Assume that  $J_{x_1 \dots x_n} \subset \mathbb{S}^1 \setminus \{x_1, \dots, x_n\}$  is nonempty. By Lemma 4.2 it is also open and hence is a countable union of open intervals. Pick one of these, and call its endpoints  $b_1$  and  $b_2$ . Assume for contradiction that at least one of  $b_1$  and  $b_2$  is not one of the  $x_i$ . Interchanging  $b_1$  and  $b_2$  if necessary we can assume that this point is  $b_1$ . Since  $G$  is continuously  $n$ –transitive there exist elements of  $G$  which cyclically permute the  $x_i$ . Using these elements and the fact that  $J_{f(x_1) \dots f(x_n)} = f(J_{x_1 \dots x_n})$  for every  $f \in G$ , we can assume without loss of generality that  $b_1$  and hence the whole interval lies in the component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_n\}$  whose endpoints are  $x_1$  and  $x_2$ .



We now claim that  $J_{b_1, b_2, x_3, \dots, x_n} \supset J_{x_1 \dots x_n}$ . To see this, take  $x \in J_{x_1 \dots x_n}$ , then there exists a continuous deformation of the identity  $f_t$  which fixes  $x_1, \dots, x_n$  and for which there exists  $t_0$  such that  $f_{t_0}(x) \neq x$ . Now since  $b_1, b_2 \notin J_{x_1 \dots x_n}$ ,  $f_t$  must also fix  $b_1$  and  $b_2$  for all  $t$ , consequently we can use  $f_t$  to show that  $x \in J_{b_1, b_2, x_3, \dots, x_n}$ . In particular, this means that  $J_{b_1, b_2, x_3, \dots, x_n}$  contains the whole interval between  $b_1$  and  $b_2$ .

Take  $g \in G$  which maps  $\{b_1, b_2\}$  to  $\{x_1, x_2\}$  and fixes the other  $x_i$ , such an element exists as  $G$  is continuously  $n$ -transitive. Then,

$$J_{x_1, x_2, x_3, \dots, x_n} = J_{g(b_1), g(b_2), g(x_3), \dots, g(x_n)} = g(J_{b_1, b_2, x_3, \dots, x_n})$$

so that  $J_{x_1 \dots x_n}$  must contain the whole interval between  $x_1$  and  $x_2$ . This is a contradiction, since  $b_1$  lies between  $x_1$  and  $x_2$  but is not in  $J_{x_1 \dots x_n}$ .  $\square$

**Proposition 4.4** *Let  $G$  be continuously  $n$ -transitive for some  $n \geq 2$  and suppose there exist  $n$  distinct points  $a_1, \dots, a_n \in \mathbb{S}^1$  and a continuous deformation of the identity  $g_t \in G$ , which fixes each  $a_i$  for all  $t$ . Then  $G$  is continuously  $n + 1$  transitive.*

**Proof**  $J_{a_1 \dots a_n} \neq \emptyset$  so by [Lemma 4.3](#)  $J_{a_1 \dots a_n} = \mathbb{S}^1 \setminus \{a_1, \dots, a_n\}$ . We can now apply [Lemma 2.5](#) to see that  $G$  is continuously  $n + 1$ -transitive.  $\square$

**Corollary 4.5** *If  $G$  is continuously 2-transitive and there exists  $g \in G \setminus \{\text{id}\}$  with an open interval  $I \subset \mathbb{S}^1$  such that the restriction of  $g$  to  $I$  is the identity, then  $G$  is continuously  $n$ -transitive for every  $n \geq 2$ .*

**Proof** Let  $I \subset \mathbb{S}^1$  be a maximal interval on which  $g$  acts as the identity, so that if  $I' \supset I$  is another interval containing  $I$  then  $g$  doesn't act as the identity on  $I'$ . Let  $a$  and  $b$  be the endpoints of  $I$  and let  $a_t$  and  $b_t$  be continuous injective paths with  $a_0 = a, b_0 = b$  and  $a_t, b_t \notin I$  for each  $t \neq 0$ . This is possible because  $g \neq \text{id}$  so that  $\mathbb{S}^1 \setminus I$  will be a closed interval containing more than one point. Let  $g_t$  be a continuous path in  $G$  so that  $g_0 = \text{id}$ ,  $g_t(a) = a_t$  and  $g_t(b) = b_t$ , such a path exists as  $G$  is continuously 2-transitive. Consider the path  $h_t = g^{-1} \circ g_t \circ g \circ g_t^{-1}$  since  $g_0 = \text{id}$  we get  $h_0 = \text{id}$ . Now  $g_t \circ g \circ g_t^{-1}$  acts as the identity on the interval between  $a_t$  and  $b_t$  and by maximality of  $I$ ,  $g^{-1}$  will not act as the identity for  $t \neq 0$ . Consequently,  $h_t$  is a continuous deformation of the identity which acts as the identity on  $I$ . So if  $G$  is continuously  $k$ -transitive for  $k \geq 2$ , by taking  $k$ -points in  $I$  and using [Proposition 4.4](#) we get that  $G$  is  $k + 1$ -transitive. As a result, since  $G$  is continuously 2-transitive it will be  $n$ -transitive for every  $n \geq 2$ .  $\square$

$\text{SO}(2, \mathbb{R})$  is an example of a subgroup of  $\text{Homeo}(\mathbb{S}^1)$  which is continuously 1-transitive but not continuously 2-transitive. However, as the next result shows, there are no subgroups of  $\text{Homeo}(\mathbb{S}^1)$  which are continuously 2-transitive but not continuously 3-transitive.

**Proposition 4.6** *If  $G$  is continuously 2–transitive, then it is continuously 3–transitive.*

**Proof** Let  $a, b \in \mathbb{S}^1$  be distinct points. Construct two injective paths  $a(t), b(t)$  in  $\mathbb{S}^1$  with disjoint images, such that  $a(0) = a$ ,  $b(0) = b$  and such that  $a(t)$  and  $b(t)$  lie in the same component of  $\mathbb{S}^1 \setminus \{a, b\}$  for  $t \in (0, 1]$ . We label this component  $I$  and the other  $I'$ .

Since  $G$  is continuously 2–transitive, there exists a path  $g(t) \in G$  such that  $g(0) = \text{id}$ ,  $g(t)(a) = a(t)$  and  $g(t)(b) = b(t)$  for every  $t$ . Now for every  $t$  the restriction of  $g(t)$  to the closure of  $I$ , is a continuous map of a closed interval into itself, and hence must have a fixed point,  $c(t)$ . This point will normally not be unique, but since  $g(t)$  is continuous, for a small enough time interval we can choose it to depend continuously on  $t$ . Likewise for the restriction of  $g(t)^{-1}$  to the closure of  $I'$ , for a small enough time interval we can choose a path of fixed points  $d(t)$ , which must therefore also be fixed points for  $g(t)$ .

Now pick points  $c \in I$  and  $d \in I'$ . Using continuous 2–transitivity of  $G$  construct a path  $h(t) \in G$  such that  $h(t)(c) = c(t)$  and  $h(t)(d) = d(t)$ . Then  $h_t^{-1} \circ g(t) \circ h_t$  is only the identity when  $t = 0$  because the same is true of  $g(t)$  and we have constructed a continuous deformation of the identity which fixes  $c$  and  $d$  for all  $t$ . Consequently we can use [Proposition 4.4](#) to show that  $G$  is continuously 3–transitive.  $\square$

## 5 Convergence Groups

**Definition 5.1** A subgroup  $G$  of  $\text{Homeo}(\mathbb{S}^1)$  is a convergence group if for every sequence of distinct elements  $g_n \in G$ , there exists a subsequence  $g_{n_k}$  satisfying one of the following two properties:

- (1) There exists  $g \in G$  such that,

$$\lim_{k \rightarrow \infty} g_{n_k} = g \quad \text{and} \quad \lim_{k \rightarrow \infty} g_{n_k}^{-1} = g^{-1}$$

uniformly in  $\mathbb{S}^1$ .

- (2) There exist points  $x_0, y_0 \in \mathbb{S}^1$  such that,

$$\lim_{k \rightarrow \infty} g_{n_k} = x_0 \quad \text{and} \quad \lim_{k \rightarrow \infty} g_{n_k}^{-1} = y_0$$

uniformly on compact subsets of  $\mathbb{S}^1 \setminus \{y_0\}$  and  $\mathbb{S}^1 \setminus \{x_0\}$  respectively.

The notion of convergence groups was introduced by Gehring and Martin [4] and they have proceeded to play a central role in geometric group theory. The following theorem has been one of the most important and we shall make frequent use of it.

**Theorem 5.2**  $G$  is a convergence group if and only if it is conjugate in  $\text{Homeo}(\mathbb{S}^1)$  to a subgroup of  $\text{PSL}(2, \mathbb{R})$ .

This Theorem was proved by Gabai in [3]. Prior to that, Tukia [7] proved this result in many cases and Hinkkanen [6] proved it for non discrete groups. Casson and Jungreis proved it independently using different methods [2]. See [2], [3], [7] for references to other papers in this subject.

For the rest of this section we shall assume that  $G$  is continuously  $n$ -transitive, but not continuously  $n + 1$ -transitive for some  $n \geq 3$ .

Take  $(x_1, \dots, x_{n-1}) \in P_{n-1}$  and define

$$G_0 = \{g \in G : g(x_i) = x_i \quad i = 1, \dots, n - 1\}.$$

Choose a component  $I$  of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$  and denote its closure by  $\bar{I}$ . We construct a homomorphism  $\Phi: G_0 \rightarrow \text{Homeo}(\mathbb{S}^1)$  as follows. Take  $g \in G_0$ , then since  $g$  fixes the endpoints of  $I$  and is orientation preserving, we can restrict it to a homeomorphism  $g'$  of  $\bar{I}$ . By identifying the endpoints of  $\bar{I}$  we get a copy of  $\mathbb{S}^1$  and we define  $\Phi(g)$  to be the homeomorphism of  $\mathbb{S}^1$  that  $g'$  descends to under this identification. We label the identification point  $\bar{x}$  and set  $\mathcal{G}_0 = \Phi(G_0)$  to be the image of  $G_0$  under  $\Phi$ .

In this situation [Lemma 2.5](#) implies the following. For every  $x \in I$ , there exists a continuous map  $F_x: \mathbb{S}^1 \setminus \bar{x} \rightarrow \mathcal{G}_0$  satisfying the properties,

- (1)  $(F_x(y))(x) = y \quad \forall y \in \mathbb{S}^1 \setminus \bar{x}$
- (2)  $F_x(x) = \text{id}$ .

**Proposition 5.3**  $\Phi: G_0 \rightarrow \mathcal{G}_0$  is an isomorphism.

**Proof** Surjectivity is trivial. If we assume that  $\Phi$  is not injective then there will exist  $g \in G_0$  which is non-trivial and acts as the identity on  $I$ . Then by [Corollary 4.5](#)  $G$  will be  $n + 1$  transitive, a contradiction.  $\square$

Let  $\widehat{G}_0$  denote the path component of the identity in  $G_0$ , we now analyze the group  $\widehat{\mathcal{G}}_0 = \Phi(\widehat{G}_0)$ .

**Proposition 5.4**  $\widehat{\mathcal{G}}_0$  is a convergence group.

**Proof** Choose  $x \in I$  then we know there exists a continuous map  $F_x: \mathbb{S}^1 \setminus \bar{x} \rightarrow \mathcal{G}_0$  satisfying the properties,

- (1)  $(F_x(y))(x) = y \quad \forall y \in \mathbb{S}^1 \setminus \bar{x}$
- (2)  $F_x(x) = \text{id}$ .

Now since  $F_x(x) = \text{id}$  and  $F_x$  is continuous, the image of  $F_x$  will lie entirely in  $\widehat{\mathcal{G}}_0$ .

In fact,  $F_x$  gives a bijection between  $\mathbb{S}^1 \setminus \bar{x}$  and  $\widehat{\mathcal{G}}_0$ . To see this we first observe that injectivity follows directly from condition 1. To see that it is also surjective, take  $g \in \widehat{\mathcal{G}}_0$ . Then there exists a path  $g_t \in \widehat{\mathcal{G}}_0$  for  $t \in [0, 1]$  with  $g_0 = \text{id}$  and  $g_1 = g$ . So that  $g_t(x)$  is a path in  $\mathbb{S}^1 \setminus \bar{x}$  from  $x$  to  $g(x)$ . Consider the path  $(F_x(g_t(x)))^{-1} \circ g_t$  in  $\widehat{\mathcal{G}}_0$ , it fixes  $x$  for every  $t$ , and so must be the identity for each  $t$ . Otherwise, by [Proposition 4.4](#),  $G$  would be continuously  $n + 1$ -transitive, which would contradict our assumptions. As a result  $g = F_x(g(x))$  so  $F_x$  is a bijection, with inverse given by evaluation at  $x$ .

Fix  $x_0 \in \mathbb{S}^1 \setminus \bar{x}$ , let  $g_n$  be a sequence of elements of  $\widehat{\mathcal{G}}_0$  and consider the sequence of points  $g_n(x_0)$ , since  $\mathbb{S}^1$  is compact  $g_n(x_0)$  has a convergent subsequence  $g_{n_k}(x_0)$  converging to some point  $x'$ . If  $x' \neq \bar{x}$  then by continuity of  $F_{x_0}$ ,  $g_{n_k}$  will converge to  $F_{x_0}(x')$ . Now if there does not exist a subsequence of  $g_n(x_0)$  converging to some  $x' \neq \bar{x}$ , then take a subsequence  $g_{n_k}$  such that  $g_{n_k}(x_0)$  converges to  $\bar{x}$ . If we can show that  $g_{n_k}(x)$  converges to  $\bar{x}$  for every  $x \in \mathbb{S}^1 \setminus \bar{x}$  then we shall be done.

Suppose for contradiction that there exists  $x \in \mathbb{S}^1 \setminus \bar{x}$  such that  $g_{n_k}(x)$  does not converge to  $\bar{x}$ . Then there exists a subsequence of  $g_{n_k}(x)$  which converges to  $x' \neq \bar{x}$ , but then by the previous argument the corresponding subsequence of  $g_{n_k}$  will converge to the homeomorphism  $F_x(x')$ . This is a contradiction since  $F_x(x')(x_0)$  would have to equal  $\bar{x}$ .  $\square$

**Corollary 5.5** *Let  $g$  be an element of  $\widehat{\mathcal{G}}_0$ . If  $g$  fixes a point in  $\mathbb{S}^1 \setminus \bar{x}$  then it is the identity.*

**Proof** Let  $x \in \mathbb{S}^1 \setminus \bar{x}$  be a fixed point of  $g$ . From the previous proof we know that  $F_x: I \rightarrow \widehat{\mathcal{G}}_0$  is a bijection. So that  $F_x(g(x)) = g$ , but  $g$  fixes  $x$  so that  $g = F_x(x) = \text{id}$ .  $\square$

**Corollary 5.6** *The restriction of the action of  $\widehat{\mathcal{G}}_0$  to  $\mathbb{S}^1 \setminus \bar{x}$  is conjugate to the action of  $\mathbb{R}$  on itself by translation.*

**Proof** By [Theorem 5.2](#) and [Proposition 5.4](#)  $\widehat{\mathcal{G}}_0$  is conjugate in  $\text{Homeo}(\mathbb{S}^1)$  to a subgroup of  $\text{PSL}(2, \mathbb{R})$  which fixes the point  $\bar{x}$ . Moreover, from [Corollary 5.5](#) this is the only point fixed by a non trivial element. By identifying  $\mathbb{S}^1$  with  $\mathbb{R} \cup \{\infty\}$  so that  $\bar{x}$  is identified with  $\{\infty\}$  in the usual way, we see that  $\widehat{\mathcal{G}}_0$  is conjugate to a subgroup of the

Möbius group acting on  $\mathbb{R} \cup \{\infty\}$ . Since every element will fix  $\{\infty\}$ , their restriction to  $\mathbb{R}$  will be an element of  $\text{Aff}(\mathbb{R})$  acting without fixed points, so can only be a translation. On the other hand the group must act transitively on  $\mathbb{R}$  and so must be the full group of translations. This gives the result.  $\square$

**Proposition 5.7** *The restriction of the action of  $G_0$  to  $I$  is conjugate to the action of a subgroup of the affine group  $\text{Aff}(\mathbb{R})$  on  $\mathbb{R}$ . In particular, each non trivial element of  $G_0$  can act on  $I$  with at most one fixed point.*

**Proof** The restriction of  $\widehat{G}_0$  to  $\mathbb{S}^1 \setminus \bar{x}$  is isomorphic to the restriction of  $\widehat{G}_0$  to  $I$ . So that by [Corollary 5.6](#) there exists a homeomorphism  $\phi: I \rightarrow \mathbb{R}$  which conjugates the restriction of  $\widehat{G}_0$  to  $I$ , to the action of  $\mathbb{R}$  on itself by translation. Take  $h \in G_0 \setminus \widehat{G}_0$  then  $h' = \phi \circ h \circ \phi^{-1}$  is a self-homeomorphism of  $\mathbb{R}$ . Since  $\widehat{G}_0$  is a normal subgroup of  $G_0$ ,  $h'$  conjugates every translation to another one and so by [Lemma 3.9](#) is itself an affine map and the proof is complete.  $\square$

Let  $g$  be a nontrivial element of  $G_0$ , then  $g \in \widehat{G}_0$  if and only if it acts on each component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$  as a conjugate of a non trivial translation. Furthermore, if  $g \notin \widehat{G}_0$  then it acts on each component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$  as a conjugate of a affine map which is not a translation, each of which must have a fixed point. This situation cannot actually arise as the next proposition will show.

**Proposition 5.8**  $G_0 = \widehat{G}_0$

**Proof** Let  $g \in G_0 \setminus \widehat{G}_0$ , then  $g$  acts on each component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$  as a conjugate of a affine map which is not a translation. Consequently,  $g$  will have a fixed point in each component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$ . Label the fixed points of  $g$  in the components of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$  whose boundaries both contain  $x_1$  as  $y_1$  and  $y_2$ . Since  $G$  is  $n$ -transitive, there exists a map  $g'$  which sends  $y_1$  to  $x_1$  and fixes all the other  $x_i$ . Then  $g' \circ g \circ (g')^{-1}$  fixes all the  $x_i$  and hence is an element of  $G_0$ . On the other hand,  $g' \circ g \circ (g')^{-1}$  also fixes  $g'(y_1)$  and  $g'(y_2)$  which lie in the same component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$ , this is impossible since every non-trivial element of  $G_0$  can only have one fixed point in each component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$ .  $\square$

**Corollary 5.9** *The restriction of the action of  $G_0$  to  $I$  is conjugate to the action of  $\mathbb{R}$  on itself by translation. In particular the action is free.*

We finish this section by comparing the directions that a non-trivial element of  $G_0$  moves points in different components of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$ . So endow  $\mathbb{S}^1$  with the anti-clockwise orientation, this gives us an ordering on any interval  $I \subset \mathbb{S}^1$ , where for distinct points  $x, y \in I$ ,  $x \prec y$  if one travels in an anti-clockwise direction to get from  $x$  to  $y$  in  $I$ . Let  $g \in G_0 \setminus \{\text{id}\}$  if  $I$  is a component of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$  then we shall say that  $g$  acts *positively* on  $I$  if  $x \prec g(x)$  and *negatively* if  $x \succ g(x)$  for one and hence every  $x \in I$ .

Let  $I$  and  $I'$  be the two components of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$  whose boundaries contain  $x_i$ . Labeled so that in the order on the closure of  $I$ ,  $x \prec x_i$  for each  $x \in I$ , whereas in the order on the closure of  $I'$ ,  $x_i \prec x$  for each  $x \in I'$ . Then we have the following,

**Proposition 5.10** *Let  $g$  be a non trivial element of  $G_0$ , if  $g$  acts positively on  $I$  then it acts negatively on  $I'$  and if  $g$  acts negatively on  $I$  then it acts positively on  $I'$ .*

**Proof** Let  $x, x' \in I$  and  $y, y' \in I'$  be points such that  $x \prec x'$  and  $y \succ y'$ . There exists  $g \in G$  fixing  $x_1, \dots, x_{i-1}$  and  $x_{i+1}, \dots, x_{n-1}$  and sending  $x$  to  $x'$  and  $y$  to  $y'$ . This map will have a fixed point  $\tilde{x}$  between  $x'$  and  $y'$ , since it maps the interval between them into itself.

Let  $g' \in G$  fix  $x_1, \dots, x_{i-1}$  and  $x_{i+1}, \dots, x_{n-1}$  and send  $\tilde{x}$  to  $x_i$ . Then  $g_0 = g' \circ g \circ (g')^{-1}$  will fix  $x_1, \dots, x_{n-1}$  and hence lie in  $G_0$ . Moreover,  $g_0$  acts positively on  $I$  and negatively on  $I'$ .

Now let  $g_1 \in G_0$  be any non-trivial element which acts positively on  $I$ . Then there exists a path  $g_t$  in  $G_0$  from  $g_0 = g' \circ g \circ (g')^{-1}$  to  $g_1$ , so that  $g_t \neq \text{id}$  for any  $t$ . Since  $g_t$  is never the identity and  $g_0$  acts negatively on  $I'$ ,  $g_1$  must also act negatively on  $I'$ .

If  $h \in G_0$  is a non-trivial element which acts negatively on  $I$ , then  $h^{-1}$  will act positively on  $I$ . So that, by the above argument,  $h^{-1}$  will act negatively on  $I'$ . This means that  $h$  will act positively on  $I'$  as required.  $\square$

**Corollary 5.11** *If  $G$  is  $n$ -transitive but not  $n + 1$ -transitive for  $n \geq 3$  then  $n$  is odd.*

**Proof** Let  $g$  be a non-trivial element of  $G_0$  which acts positively on some component  $I$  of  $\mathbb{S}^1 \setminus \{x_1, \dots, x_{n-1}\}$ . Then by [Proposition 5.10](#) as we travel around  $\mathbb{S}^1$  in an anti-clockwise direction the manner in which it acts on each component will alternate between negative and positive. Consequently, if  $n$  was even, when we return to  $I$  we would require that  $g$  acted negatively on  $I$ , a contradiction, so  $n$  is odd.  $\square$

## 6 Continuous 3–transitivity and beyond

We begin this section by analyzing the case where  $G$  is continuously 3–transitive but not continuously 4–transitive. We shall show that such a group is a convergence group and consequently conjugate to a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ .

Fix distinct points  $x_0, y_0 \in \mathbb{S}^1$  and define

$$G_0 = \{g \in G : g(x_0) = x_0, g(y_0) = y_0\}$$

$$\bar{G} = \{g \in G : g(x_0) = x_0\}$$

then we have the following propositions.

**Proposition 6.1**  $G_0$  is a convergence group.

**Proof** From [Corollary 5.9](#), we know that the restriction of  $G_0$  to each of the components of  $\mathbb{S}^1 \setminus \{x_0, y_0\}$  is conjugate to the action of  $\mathbb{R}$  on itself by translation. Let  $g_n$  be a sequence of distinct elements of  $G_0$  and take a point  $x \in \mathbb{S}^1 \setminus \{x_0, y_0\}$ . Then the sequence of points  $g_n(x)$  will have a convergent subsequence  $g_{n_k}(x)$ . If this sequence converges to  $x_0$  or  $y_0$ , then from [Proposition 5.10](#) so will the sequences  $g_{n_k}(y)$  for all  $y \in \mathbb{S}^1 \setminus \{y_0\}$  or  $\mathbb{S}^1 \setminus \{x_0\}$  respectively.

Let  $I_x$  be the component of  $\mathbb{S}^1 \setminus \{x_0, y_0\}$  containing  $x$ . Assume that the sequence of points  $g_{n_k}(x)$  converges to a point  $x' \in I_x$ . Now let  $y$  be a point in the other component,  $I_y$  of  $\mathbb{S}^1 \setminus \{x_0, y_0\}$ , and consider the sequence of points  $g_{n_k}(y)$  in  $I_y$ . If it had a subsequence which converged to  $x_0$  or  $y_0$  then the sequence  $g_{n_k}(x)$  would have to as well. This is impossible so  $g_{n_k}(y)$  must stay within a compact subset of  $I_y$  and hence  $g_{n_k}$  has a subsequence,  $g_{n_{k_l}}$  for which  $g_{n_{k_l}}(y)$  converges to some point  $y' \in I_y$ .

By [Corollary 5.9](#) there exist self homeomorphisms of  $I_x$  and  $I_y$  to which the sequence  $g_{n_{k_l}}$  converges uniformly on  $I_x$  and  $I_y$  respectively. Gluing these together at  $x_0$  and  $y_0$  gives us an element of  $\mathrm{Homeo}(\mathbb{S}^1)$  which  $g_{n_k}$  converges to uniformly. Consequently,  $G_0$  is a convergence group.  $\square$

**Proposition 6.2**  $\bar{G}$  is a convergence group.

**Proof** Let  $f_n$  be a sequence of elements of  $\bar{G}$ . If for every  $y \in \mathbb{S}^1 \setminus \{x_0\}$  every convergent subsequence of  $f_n(y)$  converges to  $x_0$  then we would be done. So assume that this is not the case, take  $y \in \mathbb{S}^1 \setminus \{x_0\}$  such that the sequence of points  $f_n(y)$  has a convergent subsequence  $f_{n_k}(y)$  converging to some point  $\tilde{y} \neq x_0$ . Let  $I$  be a small open interval around  $\tilde{y}$ , not containing  $x_0$  then since  $G$  is continuously 3–transitive, there exists a map  $F_{\tilde{y}}: I \rightarrow \bar{G}$  satisfying the following,

- (1)  $F_{\tilde{y}}(x)(\tilde{y}) = x$  for all  $x \in I$
- (2)  $F_{\tilde{y}}(\tilde{y})$  is the identity.

Let  $g_1, g_2 \in \bar{G}$  satisfy  $g_1(\tilde{y}) = y_0$  and  $g_2(y_0) = y$  consider the sequence,

$$h_k = g_1 \circ F_{\tilde{y}}(f_{n_k}(y))^{-1} \circ f_{n_k} \circ g_2$$

of elements of  $\bar{G}$ . They all fix  $y_0$ , and since  $g_1 \circ F_{\tilde{y}}(f_{n_k}(y))^{-1}$  converges to  $g_1$  as  $k \rightarrow \infty$  we have the following.

- (1) If  $h_k$  contains a subsequence  $h_{k_l}$  such that there exists a homeomorphism  $h$  with,

$$\lim_{l \rightarrow \infty} h_{k_l} = h \quad \text{and} \quad \lim_{l \rightarrow \infty} (h_{k_l})^{-1} = h^{-1}$$

then so does  $f_{n_k}$ .

- (2) Furthermore, if there exist points  $x', y' \in \mathbb{S}^1$  and a subsequence  $h_{k_l}$  of  $h_k$  such that,

$$\lim_{l \rightarrow \infty} h_{k_l} = x' \quad \text{and} \quad \lim_{l \rightarrow \infty} (h_{k_l})^{-1} = y'$$

uniformly on compact subsets of  $\mathbb{S}^1 \setminus \{y'\}$  and  $\mathbb{S}^1 \setminus \{x'\}$  respectively, then so does  $f_{n_k}$  ( $x'$  and  $y'$  will be replaced by  $g_1^{-1}(x')$  and  $g_1^{-1}(y')$ ).

Now, since  $G_0$  is a convergence group, one of the above situations must occur. Consequently,  $\bar{G} = \{g \in G : g(x_0) = x_0\}$  is a convergence group.  $\square$

**Proposition 6.3** *If  $G$  is a subgroup of  $\text{Homeo}(\mathbb{S}^1)$  which is continuously 3-transitive but not continuously 4-transitive then  $G$  is a convergence group.*

**Proof** This proof is almost identical to the previous one but we write it out in full for clarity.

Choose  $x_0 \in \mathbb{S}^1$  and let  $f_n$  be a sequence of elements of  $G$ . Then since  $\mathbb{S}^1$  is compact, the sequence of points  $f_n(x_0)$  will have a convergent subsequence,  $f_{n_k}(x_0)$ , converging to some point  $\tilde{x}$ . Let  $I$  be a small open interval around  $\tilde{x}$ , then since  $G$  is continuously 3-transitive, there exists a map  $F_{\tilde{x}}: I \rightarrow G$  satisfying the following,

- (1)  $F_{\tilde{x}}(x)(\tilde{x}) = x$  for all  $x \in I$
- (2)  $F_{\tilde{x}}(\tilde{x})$  is the identity.

Let  $g \in G$  send  $\tilde{x}$  to  $x_0$  and consider the sequence,

$$h_k = g \circ F_{\tilde{x}}(f_{n_k}(x_0))^{-1} \circ f_{n_k}$$

of elements of  $G$ . They all fix  $x_0$ , and since  $g \circ F_{\tilde{x}}(f_{n_k}(x_0))^{-1}$  converges to  $g$  as  $k \rightarrow \infty$  we have the following.



- (1) If  $h_k$  contains a subsequence  $h_{k_l}$  such that there exists a homeomorphism  $h$  with,

$$\lim_{l \rightarrow \infty} h_{k_l} = h \quad \text{and} \quad \lim_{l \rightarrow \infty} (h_{k_l})^{-1} = h^{-1}$$

then so does  $f_{n_k}$ .

- (2) Furthermore, if there exist points  $x', y' \in \mathbb{S}^1$  and a subsequence  $h_{k_l}$  of  $h_k$  such that,

$$\lim_{l \rightarrow \infty} h_{k_l} = x' \quad \text{and} \quad \lim_{l \rightarrow \infty} (h_{k_l})^{-1} = y'$$

uniformly on compact subsets of  $\mathbb{S}^1 \setminus \{y'\}$  and  $\mathbb{S}^1 \setminus \{x'\}$  respectively, then so does  $f_{n_k}$  ( $x'$  and  $y'$  will be replaced by  $g^{-1}(x')$  and  $g^{-1}(y')$ ).

Now, since  $\bar{G} = \{g \in G : g(x_0) = x_0\}$  is a convergence group  $G$  is too. □

We now look at the case where  $G$  is continuously 4-transitive. In this case, we show that  $G$  must be  $n$ -transitive for every  $n \in \mathbb{N}$ .

**Theorem 6.4** *If  $G$  is continuously  $n$ -transitive for  $n \geq 4$ , then it is continuously  $n + 1$ -transitive.*

**Proof** Fix  $n \geq 4$  and assume for contradiction that  $G$  is continuously  $n$ -transitive but not continuously  $n + 1$ -transitive. Take  $(a_1, \dots, a_{n-2}) \in P_{n-2}$  and define,

$$\bar{G} = \{g \in G : g(a_i) = a_i \quad \forall i\}$$

Let  $I$  be a component of  $\mathbb{S}^1 \setminus \{a_1, \dots, a_{n-2}\}$ . Construct a homomorphism  $\Psi : \bar{G} \rightarrow \text{Homeo}(\mathbb{S}^1)$  in the same way as  $\Phi : G_0 \rightarrow \text{Homeo}(\mathbb{S}^1)$  was constructed in Section 5. Explicitly, take  $g \in \bar{G}$ , restrict it to a self homeomorphism of  $I$  and identify the endpoints to get an element of  $\text{Homeo}(\mathbb{S}^1)$ .

Let  $\bar{\mathcal{G}}$  denote the image of  $\bar{G}$  under  $\Psi$ . Then as in Proposition 5.3  $\bar{\mathcal{G}}$  is isomorphic to  $\bar{G}$ . Using the arguments from the earlier Propositions in this section we can show that  $\bar{\mathcal{G}}$  is a convergence group and hence conjugate to a subgroup of  $\text{PSL}(2, \mathbb{R})$ . On the other hand,  $\bar{\mathcal{G}}$  is 2-transitive on  $I$  and every element fixes the identification point. This means that the action of  $\bar{G}$  on  $I$  must be conjugate to the action of  $\text{Aff}(\mathbb{R})$  on  $\mathbb{R}$ .

Let  $I$  and  $I'$  be two components of  $\mathbb{S}^1 \setminus \{a_1, \dots, a_{n-2}\}$  and let  $\phi : I \rightarrow \mathbb{R}$  be a homeomorphism which conjugates the action of  $\bar{G}$  on  $I$  to the action of  $\text{Aff}(\mathbb{R})$  on  $\mathbb{R}$ . Let  $a_{n-1}, a'_{n-1}$  be two distinct points in  $I'$ . Consider the groups

$$G_0 = \{g \in \bar{G} : g(a_{n-1}) = a_{n-1}\}$$

and

$$G'_0 = \{g \in \bar{G} : g(a'_{n-1}) = a'_{n-1}\}$$

They each act transitively on  $I$  and by [Corollary 5.5](#) and [Proposition 5.8](#) without fixed points. Consequently,  $\phi$  conjugates both of these actions to the action of  $\mathbb{R}$  on itself by translation. Let  $g \in G_0$  and  $g' \in G'_0$  be elements which are conjugated to  $x \mapsto x + 1$  by  $\phi$ . Then  $g^{-1} \circ g'$  acts on  $I$  as the identity. However, if it is equal to the identity, then  $g' = g$  fixes  $a_{n-1}$  and  $a'_{n-1}$ , this is impossible as non-trivial elements of  $\bar{G}$  can have at most one fixed point in  $I'$ . So  $g^{-1} \circ g$  is a non-trivial element of  $G$  which acts as the identity on  $I$  and so by [Corollary 4.5](#) we have that  $G$  is continuously  $n + 1$ -transitive.  $\square$

## 7 Summary of Results

**Theorem 7.1** *Let  $G$  be a transitive subgroup of  $\text{Homeo}(\mathbb{S}^1)$  which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:*

- (1)  $G$  is conjugate to  $\text{SO}(2, \mathbb{R})$  in  $\text{Homeo}(\mathbb{S}^1)$ .
- (2)  $G$  is conjugate to  $\text{PSL}(2, \mathbb{R})$  in  $\text{Homeo}(\mathbb{S}^1)$ .
- (3) For every  $f \in \text{Homeo}(\mathbb{S}^1)$  and each finite set of points  $x_1, \dots, x_n \in \mathbb{S}^1$  there exists  $g \in G$  such that  $g(x_i) = f(x_i)$  for each  $i$ .
- (4)  $G$  is a cyclic cover of a conjugate of  $\text{PSL}(2, \mathbb{R})$  in  $\text{Homeo}(\mathbb{S}^1)$  and hence conjugate to  $\text{PSL}_k(2, \mathbb{R})$  for some  $k > 1$ .
- (5)  $G$  is a cyclic cover of a group satisfying condition 3 above.

**Proof** Let  $f: [0, 1] \rightarrow G$  be a non constant continuous path. Then

$$f(0)^{-1} \circ f: [0, 1] \rightarrow G$$

is a continuous deformation of the identity in  $G$ . Consequently, [Proposition 2.6](#) tells us that  $G$  is continuously 1-transitive.

If  $J_x = \emptyset$  for every  $x \in \mathbb{S}^1$  then by [Theorem 3.8](#)  $G$  is conjugate to  $\text{SO}(2, \mathbb{R})$  in  $\text{Homeo}(\mathbb{S}^1)$ . If  $J_x \neq \emptyset$  for some and hence all  $x \in \mathbb{S}^1$  then by [Theorem 3.10](#)  $G$  is either continuously 2-transitive or is a cyclic cover of a group  $G'$  which is continuously 2-transitive.

So assume that  $G$  is continuously 2-transitive, then by [Proposition 4.6](#) it is continuously 3-transitive. If moreover  $G$  is not continuously 4-transitive, then by [Proposition 6.3](#)

it is a convergence group and hence conjugate to a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ . On the other hand, since  $G$  is continuously 3–transitive, it is 3–transitive, and hence must be conjugate to the whole of  $\mathrm{PSL}(2, \mathbb{R})$ .

If we now assume that  $G$  is continuously 4–transitive then by [Theorem 6.4](#) it is continuously  $n$ –transitive and hence  $n$ –transitive for every  $n \in \mathbb{N}$ . So if we take  $f \in \mathrm{Homeo}(\mathbb{S}^1)$  and a finite set of points  $x_1, \dots, x_n \in \mathbb{S}^1$  there exists  $g \in G$  such that  $g(x_i) = f(x_i)$  and we are done.  $\square$

**Theorem 7.2** *Let  $G$  be a closed transitive subgroup of  $\mathrm{Homeo}(\mathbb{S}^1)$  which contains a non constant continuous path. Then one of the following mutually exclusive possibilities holds:*

- (1)  $G$  is conjugate to  $\mathrm{SO}(2, \mathbb{R})$  in  $\mathrm{Homeo}(\mathbb{S}^1)$ .
- (2)  $G$  is conjugate to  $\mathrm{PSL}_k(2, \mathbb{R})$  in  $\mathrm{Homeo}(\mathbb{S}^1)$  for some  $k \geq 1$ .
- (3)  $G$  is conjugate to  $\mathrm{Homeo}_k(\mathbb{S}^1)$  in  $\mathrm{Homeo}(\mathbb{S}^1)$  for some  $k \geq 1$ .

**Proof** Since  $G$  is a transitive subgroup of  $\mathrm{Homeo}(\mathbb{S}^1)$  which contains a non constant continuous path, [Theorem 7.1](#) applies. It remains to show that if  $G$  satisfies condition 3 in [Theorem 7.1](#) then its closure is  $\mathrm{Homeo}(\mathbb{S}^1)$ .

To see this, let  $f$  be an arbitrary element of  $\mathrm{Homeo}(\mathbb{S}^1)$ . If we can find a sequence of elements of  $G$  which converges uniformly to  $f$  then we shall be done. So let  $\{a_n : n \in \mathbb{N}\}$  be a countable and dense set of points in  $\mathbb{S}^1$ . Choose a sequence of maps  $g_n \in G$  so that  $g_n(a_k) = f(a_k)$  for  $1 \leq k \leq n$ . Then  $g_n$  will converge uniformly to  $f$  so that the closure of  $G$  will equal  $\mathrm{Homeo}(\mathbb{S}^1)$ .  $\square$

**Theorem 7.3**  $\mathrm{PSL}(2, \mathbb{R})$  is a maximal closed subgroup of  $\mathrm{Homeo}(\mathbb{S}^1)$ .

**Proof** Let  $G$  be a closed subgroup of  $\mathrm{Homeo}(\mathbb{S}^1)$  containing  $\mathrm{PSL}(2, \mathbb{R})$ . Then  $G$  is 3–transitive and by applying [Theorem 7.2](#) we can see that  $\mathrm{Homeo}(\mathbb{S}^1)$  and  $\mathrm{PSL}(2, \mathbb{R})$  are the only possibilities for  $G$ .  $\square$

## References

- [1] **M Bestvina**, *Questions in geometric group theory* Available at <http://www.math.utah.edu/~bestvina/>
- [2] **A Casson, D Jungreis**, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. 118 (1994) 441–456 [MR1296353](#)
- [3] **D Gabai**, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) 136 (1992) 447–510 [MR1189862](#)
- [4] **F W Gehring, G J Martin**, *Discrete quasiconformal groups I*, Proc. London Math. Soc. (3) 55 (1987) 331–358 [MR896224](#)
- [5] **É Ghys**, *Groups acting on the circle*, Enseign. Math. (2) 47 (2001) 329–407 [MR1876932](#)
- [6] **A Hinkkanen**, *Abelian and nondiscrete convergence groups on the circle*, Trans. Amer. Math. Soc. 318 (1990) 87–121 [MR1000145](#)
- [7] **P Tukia**, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. 391 (1988) 1–54 [MR961162](#)

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