

Quasiconformal Homogeneity of Genus Zero Surfaces ^{*}

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Abstract

A Riemann surface M is said to be K -quasiconformally homogeneous if for every two points $p, q \in M$, there exists a K -quasiconformal homeomorphism $f: M \rightarrow M$ such that $f(p) = q$. In this paper, we show there exists a universal constant $\mathcal{K} > 1$ such that if M is a K -quasiconformally homogeneous hyperbolic genus zero surface other than \mathbb{D}^2 , then $K \geq \mathcal{K}$. This answers a question by Gehring and Palka [10]. Further, we show that a non-maximal hyperbolic surface of genus $g \geq 2$ is not K -quasiconformally homogeneous for any finite $K \geq 1$.

1 Introduction and statement of results

Let M be a Riemann surface and let $\mathcal{F}_K(M)$ be the family of all K -quasiconformal homeomorphisms of M . Then M is said to be K -quasiconformally homogeneous if the family $\mathcal{F}_K(M)$ is transitive; that is, given any two points $p, q \in M$, there exists an element $f \in \mathcal{F}_K(M)$ such that $f(p) = q$. In 1976, the notion of quasiconformal homogeneity of Riemann surfaces was introduced by Gehring and Palka [10] in the setting of genus zero surfaces (and higher dimensional analogues). It was shown in [10] that the only genus zero surfaces admitting a transitive conformal family are exactly those which are not hyperbolic, i.e. the surfaces conformally equivalent to either \mathbb{P}^1 , \mathbb{C} , $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ or \mathbb{D}^2 . It was also shown, by means of examples, that there exist hyperbolic genus zero surfaces, homeomorphic to \mathbb{P}^1 minus a Cantor set, that are K -quasiconformally homogeneous, for some finite $K > 1$. Recently, this problem has received renewed interest. Using Sullivan's Rigidity Theorem, it was

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shown in [3] by Bonfert-Taylor, Canary, Martin and Taylor, that in dimension $n \geq 3$, there exists a universal constant $\mathcal{K}_n > 0$ such that for every K -quasiconformally homogeneous hyperbolic n -manifold other than \mathbb{D}^n , it must hold that $K \geq \mathcal{K}_n$. In [4], by Bonfert-Taylor, Bridgeman and Canary, a partial result was obtained in dimension two for a certain subclass of closed hyperbolic surfaces satisfying a fixed-point condition. In the setting of genus zero surfaces, notions similar to (but stronger than) quasiconformal homogeneity have been studied by MacManus, Näkki and Palka in [11] and further developed in [6] and [5] by Bonfert-Taylor, Canary, Martin, Taylor and Wolf.

In this paper, we answer the original question of Gehring and Palka for genus zero surfaces.

Theorem A (Genus zero surfaces). *There exists a constant $\mathcal{K} > 1$, such that if M is a K -quasiconformally homogeneous hyperbolic genus zero surface other than \mathbb{D}^2 , then $K \geq \mathcal{K}$.*

The outline of the proof of Theorem A is as follows. First, we restrict our attention to *short geodesics*, that is, simple closed geodesics which are close in length to the infimum of the lengths of all simple closed geodesics on our surface M . For $K > 1$ small enough, using K -quasiconformal homogeneity, we show there exist intersections of short simple closed geodesics in a small neighbourhood of any preassigned point. Using this information, we construct a configuration of three intersecting short simple closed geodesics, see the three-circle Lemma below. By a combinatorial argument, we show that if M is near conformally homogeneous, these configurations can not exist, leading to the desired contradiction.

As the only genus zero surfaces homogeneous with respect to a conformal family are conformally equivalent to either $\mathbb{P}^1, \mathbb{C}, \mathbb{C}^*$ or \mathbb{D}^2 , we thus have the following corollary of Theorem A.

Corollary 1.1. *There exists a constant $\mathcal{K} > 1$ such that if M is a K -quasiconformally homogeneous genus zero surface with $K < \mathcal{K}$, then M is conformally equivalent to either $\mathbb{P}^1, \mathbb{C}, \mathbb{C}^*$ or \mathbb{D}^2 .*

This paper is organised as follows. In section 2 we recall some standard material and background results that will be used in the remainder of this paper. In section 3 we present our proof of Theorem A. In section 4 we discuss the problem of obtaining a good estimate of the universal constant $\mathcal{K} > 1$ whose existence we establish in the proof of Theorem A.

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2 Preliminaries

Let us first recall some background material on hyperbolic surfaces and quasiconformal mappings and set notation used throughout the remainder of this paper.

2.1 Definitions

Let M be a hyperbolic Riemann surface. By uniformizing M , we may assume that $M = \mathbb{D}^2/\Gamma$ is covered by the Poincaré disk \mathbb{D}^2 . We endow M with the metric $d(\cdot, \cdot)$, induced from the canonical hyperbolic metric $\tilde{d}(\cdot, \cdot)$ on \mathbb{D}^2 . The *injectivity radius* $\iota(M)$ of M is the infimum over all $p \in M$ of the largest radius for which the exponential map at p is a injective. Define $\lambda(M)$ to be the infimum of the lengths of all simple closed geodesics on M . We have that $\lambda(M) \geq 2\iota(M)$. We denote by $D(p, \rho) \subset M$ the closed hyperbolic disk with center p and radius ρ . Let $\mathcal{F}_K(M)$ be the family of all K -quasiconformal homeomorphisms of M . Then M is said to be K -quasiconformally homogeneous if the family $\mathcal{F}_K(M)$ is transitive; that is, given any two points $p, q \in M$, there exists an element $f \in \mathcal{F}_K(M)$ such that $f(p) = q$.

Let us also recall the following. An open Riemann surface M is said to be *extensible* or *non-maximal* if it can be embedded in another Riemann surface M_0 as a proper subregion; that is, if there exists a conformal mapping of M onto a proper subregion of M_0 . If M is not extensible, then it is called *maximal*. Every open Riemann surface is contained in a maximal Riemann surface (including infinite genus Riemann surfaces), see [2]. A non-maximal Riemann surface of genus 0, i.e. embedded in the Riemann sphere, is also called a *planar domain*. Proposition 2.6 below shows that a non-maximal hyperbolic surface of genus $1 \leq g \leq \infty$ is not K -quasiconformally homogeneous for any $K \geq 1$. Therefore, as far as quasiconformal homogeneity of hyperbolic surfaces is concerned, one can further restrict to either:

- (i) hyperbolic genus zero surfaces, or
- (ii) maximal surfaces of genus $2 \leq g \leq \infty$.

In this paper, we restrict attention to hyperbolic genus zero surfaces.

Given a closed curve $\gamma \in M$, we denote by $[\gamma]$ the homotopy class of γ in M . The *geometric intersection number* of isotopy classes of two closed curves $\alpha, \beta \in \pi_1(M)$ is defined by

$$i(\alpha, \beta) = \min \#\{\gamma \cap \gamma'\}, \quad (1)$$

where the minimum is taken over all closed curves $\gamma, \gamma' \subset M$ with $[\gamma] = \alpha$ and $[\gamma'] = \beta$. In other words, the geometric intersection number is the least number of intersections between curves representing the two homotopy classes. Let us recall some standard facts about simple closed curves, and in particular simple closed geodesics, on genus zero surfaces, see e.g. [8].

Lemma 2.1 (Curves on genus zero surfaces). *Let M be a genus zero surface.*

- (i) *Every simple closed curve $\gamma \subset M$ separates M into exactly two connected components.*
- (ii) *If $\gamma, \gamma' \subset M$ are two simple closed curves, then $i(\gamma, \gamma')$ is even.*
- (iii) *If $\gamma, \gamma' \subset M$ are non-homotopic closed geodesics, then the closed curve $\alpha \subset M$ formed by any two subarcs $\eta \subset \gamma$ and $\eta' \subset \gamma'$ connecting two points of $\gamma \cap \gamma'$ is homotopically non-trivial.*

2.2 Geometrical estimates

The following lemma describes the asymptotic behaviour of the injectivity radius $\iota(M)$ of a K -quasiconformally homogeneous hyperbolic surface in terms of K , see [3].

Lemma 2.2. *Let M be a K -quasiconformally homogeneous hyperbolic surface and $\iota(M)$ its injectivity radius. Then $\iota(M)$ is uniformly bounded from below (for K bounded from above) and $\iota(M) \rightarrow \infty$ for $K \rightarrow 1$.*

Consequently, $\lambda(M)$ is uniformly bounded from below and $\lambda(M) \rightarrow \infty$ for $K \rightarrow 1$. We fix $K_0 > 1$ such that $\lambda(M) \geq 10$ for every K -quasiconformally homogeneous hyperbolic surface M with $K \leq K_0$.

Remark 1. If M is a K -quasiconformally homogeneous hyperbolic surface, then the fact that $\lambda(M) > 0$ implies M has no cusps and therefore, any essential simple closed curve $\alpha \subset M$ has a unique simple closed geodesic representative $\gamma \subset M$, see e.g. [14]. In particular, if $\gamma \subset M$ is a simple closed geodesic and $f: M \rightarrow M$ a homeomorphism, then the closed geodesic homotopic to $f(\gamma)$ exists and is simple.

A *pair of pants* is a surface homeomorphic to the sphere \mathbb{P}^1 with the interior of three mutually disjoint closed topological disks removed. Geometrically, it is the surface obtained by gluing two hyperbolic hexagons along their seams.

In what follows, we denote by $|\gamma|$ the hyperbolic length of a piecewise geodesic curve $\gamma \subset M$. Here by piecewise geodesic curve we mean a finite concatenation of geodesics arcs. We have the following uniform estimate on lengths of simple closed geodesics, see [9, Theorem 4.3.3].

Lemma 2.3. *Let M be a hyperbolic surface and γ a simple closed geodesic. Let $f: M \rightarrow M$ a K -quasiconformal homeomorphism and γ' the simple closed geodesic homotopic to $f(\gamma)$. Then*

$$\frac{1}{K}|\gamma| \leq |\gamma'| \leq K|\gamma|. \quad (2)$$

Further, we use the following classical result in the geometry of hyperbolic surfaces.

Collar Lemma. *Set $m(\ell) = \arcsin(1/(\sinh(\ell/2)))$. For a simple closed geodesic $\gamma \subset M$ of length $\ell = |\gamma|$, the set*

$$A(\gamma) = \{p \in M : d(p, \gamma) < m(\ell)\} \quad (3)$$

is an embedded annular neighbourhood of γ .

In the next two lemma's we recollect uniform approximation estimates of K -quasiconformal homeomorphisms, in particular the behaviour when $K \rightarrow 1$, see e.g. [9].

Lemma 2.4. *For every $K \geq 1$ and $0 < \rho < 1$, there exists a constant $C_1(K, \rho)$, depending only on K and ρ , such that if $f: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ is a K -quasiconformal homeomorphism and $D(p, \rho) \subset \mathbb{D}^2$ the closed hyperbolic disk of radius ρ centered at $p \in \mathbb{D}^2$, there exists a Möbius transformation $\mu \in \text{Möb}(\mathbb{D}^2)$ such that*

$$\tilde{d}(f(q), \mu(q)) \leq C_1(K, \rho) \quad (4)$$

for all $q \in D(p, \rho)$. For fixed $\rho > 0$, we have that $C_1(K, \rho) \rightarrow 0$ for $K \rightarrow 1$.

Proof. By normalizing with suitable Möbius transformations, we may assume that $f(0) = 0$ and $f(1) = 1$. As the family of K -quasiconformal homeomorphisms of \mathbb{D}^2 onto itself fixing 0 and 1 in \mathbb{D}^2 is a normal family, see e.g. [1, p. 32], by a standard argument of uniform convergence on compact subsets, there exists a function $C_1(K, \rho)$ with $C_1(K, \rho) \rightarrow 0$ if $K \rightarrow 1$ such that $\tilde{d}(f(z), z) \leq C_1(K, \rho)$. Thus (4) follows. \square

Further, we will utilize the following, see e.g. [7, Lemma 2].

Lemma 2.5. *For every $K \geq 1$, there exists a constant $C_2(K)$ depending only on K with the following property. Let M be a hyperbolic surface and $\gamma \subset M$ a simple closed geodesic. If $f: M \rightarrow M$ is a K -quasiconformal homeomorphism, then the geodesic γ' homotopic to $f(\gamma)$ has the property that*

$$d(f(\gamma), \gamma') \leq C_2(K). \quad (5)$$

Furthermore, $C_2(K) \rightarrow 0$ as $K \rightarrow 1$.

2.3 Non-maximal surfaces of positive genus

Using the geometrical estimates derived in the previous section, let us first prove the claim made in the introduction that non-maximal Riemann surfaces of genus $1 \leq g \leq \infty$ are not K -quasiconformally homogeneous for any finite $K \geq 1$.

Proposition 2.6 (Non-maximal surfaces of positive genus). *Let M be a non-maximal surface of genus $1 \leq g \leq \infty$. Then M is not K -quasiconformally homogeneous for any $K \geq 1$.*

Proof. To derive a contradiction, assume that M is K -quasiconformally homogeneous for some finite $K \geq 1$. As M is non-maximal, M is embedded in a maximal hyperbolic surface M_0 of genus $g \geq 1$. Let $\bar{p} \in M_0$ be an ideal boundary point of M and let $D \subset M_0$ be a small closed disk embedded in M_0 and centered at \bar{p} . There exists a sequence of points $p_n \in M \cap D$ so that $d_0(p_n, \bar{p}) \rightarrow 0$ (where d_0 is the metric on M_0) and thus $d(p_n, \partial D) \rightarrow \infty$, as \bar{p} is in the ideal boundary of M . On the other hand, as $g \geq 1$, there exists a non-separating simple closed geodesic $\gamma \subset M$. Mark a point $p \in \gamma$. By transitivity of the family $\mathcal{F}_K(M)$, for every $n \geq 1$, there exists an element $f_n \in \mathcal{F}_K(M)$ such that $f_n(p) = p_n$. Therefore the simple closed curve $f_n(\gamma)$ is non-separating for every $n \geq 1$ and thus

$$f_n(\gamma) \cap \partial D \neq \emptyset. \quad (6)$$

Indeed, otherwise we have that $f_n(\gamma) \subset M \cap D$, implying that $f_n(\gamma)$ is separating, as D is an embedded disk in the maximal surface M_0 and thus the connected components of $M \cap D$ are planar subsurfaces, contradicting that γ (and thus $f_n(\gamma)$) is non-separating. By Lemma 2.5, the geodesic γ_n homotopic to $f_n(\gamma)$ has to stay within a bounded distance of $f_n(\gamma)$ and therefore, for n large enough, the geodesic γ_n has the property that

$$\gamma_n \cap \partial D \neq \emptyset \quad (7)$$

by (6). It follows from (7) that

$$|\gamma_n| \geq 2(d(p_n, \partial D) - C_2(K)) \quad (8)$$

with $C_2(K)$ the uniform constant of Lemma 2.5; put in words, the geodesic γ_n has to enter $D \cap M$, pass close to p_n , and leave $D \cap M$ again. As $d(p_n, \partial D) \rightarrow \infty$ for $n \rightarrow \infty$, it follows that $|\gamma_n| \rightarrow \infty$ for $n \rightarrow \infty$, contradicting Lemma 2.3. Thus M can not be K -quasiconformally homogeneous for any finite $K \geq 1$. \square

3 Quasiconformal homogeneity of genus zero surfaces

In the remainder of our proof, let $K_0 > 1$ as defined in section 2 and let M be a K -quasiconformally homogeneous hyperbolic genus zero surface, with $1 < K \leq K_0$, and $\mathcal{F}_K(M)$ the family of all K -quasiconformal homeomorphisms of M , which is transitive by homogeneity of M .

3.1 The two-circle Lemma

In what follows, we focus on *short geodesics*, in the following sense.

Definition 1 (δ -short geodesics). Given $\delta > 0$, a simple closed geodesic $\gamma \subset M$ is said to be δ -short if $|\gamma| \leq (1 + \delta)\lambda(M)$.

By Lemma 2.2, and the remark following it, for every $K \leq K_0$, there is a uniform lower bound on the length of simple closed geodesics on M . Fix

$$\delta_0 = \frac{1}{378}. \tag{9}$$

Definition 2 (Two-circle configuration). A *two-circle configuration* is a union of two δ_0 -short geodesics $\gamma_1, \gamma_2 \in M$ such that γ_1 and γ_2 intersect in exactly two points $p_1, p_2 \in M$.

Topologically a two-circle configuration is a union of two simple closed curves γ_1, γ_2 in the surface M , intersecting transversely in exactly two points and

$$M \setminus (\gamma_1 \cup \gamma_2)$$

consists of four connected components and the boundary of each component consists of two arcs. For future reference, let us label the four arcs $\eta_1, \eta_2 \subset \gamma_1$ and $\eta_3, \eta_4 \subset \gamma_2$ connecting the two intersection points p_1 and p_2 , see Figure 3.1.

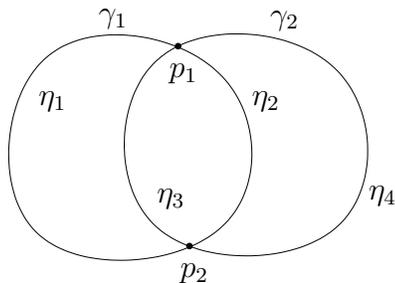


Figure 3.1: A two-circle configuration with labeling.

First, the following result, see also [12, Proposition 4.6].

Lemma 3.1. *There exists a uniform constant $r_0 > 0$ such that for a pair of pants $P \subset M$, there exists a $p \in P$ such that $D(p, r_0) \subset P$.*

Proof. Each pair of pants decomposes into two ideal triangles. As every ideal triangle contains a disk of radius $\frac{1}{2} \log 3$, every pair of pants therefore contains a disk of (at least) that radius. Thus we may take $r_0 = \frac{1}{2} \log 3$. \square

We first prove the existence of intersecting δ_0 -short geodesics on M for sufficiently small $K > 1$, as we build forth upon this result in the remainder of the proof.

Lemma 3.2 (Intersections of short geodesics). *There exists a constant $1 < K_1 \leq K_0$, such that if M is K -quasiconformally homogeneous with $1 < K \leq K_1$, then there exist δ_0 -short geodesics that intersect.*

Proof. To prove there exist intersecting δ_0 -short geodesics on M , we argue by contradiction. That is, suppose all δ_0 -short geodesics on M are mutually disjoint. Choose $1 < K_1 \leq K_0$ so that

$$K_1 \leq \frac{1 + \delta_0}{1 + \delta_0/2} \text{ and } C_2(K_1) \leq \frac{r_0}{2}, \quad (10)$$

with $C_2(K)$ the constant of Lemma 2.5 and r_0 the constant of Lemma 3.1.

Let us first observe that there exist infinitely many distinct δ_0 -short geodesics on M . Indeed, as $\lambda(M) > 0$, there exists a simple closed geodesic $\gamma_0 \subset M$ such that $|\gamma_0| \leq (1 + \delta_0/2)\lambda(M)$. Mark a point $p_0 \in \gamma_0$ and choose $f \in \mathcal{F}_K(M)$ such that $f(p_0) = q$, for a certain $q \in M$. By our choice of K_1 , cf. (10), combined with Lemma 2.3, the geodesic γ homotopic to $f(\gamma_0)$ is δ_0 -short, for every $f \in \mathcal{F}_K(M)$. As the surface is unbounded (in the hyperbolic metric), by transporting the geodesic γ_0 by different elements of $\mathcal{F}_K(M)$ sufficiently far apart, by Lemma 2.5, we see that there must indeed exist infinitely many different δ_0 -short curves. Denote Γ_0 the (countable) family of all δ_0 -short geodesics on M .

As all elements of Γ_0 lie in different homotopy classes, and all elements are mutually disjoint, we claim that the elements of Γ_0 are locally finite, in the sense that a compact subset of M only intersects finitely many distinct elements of Γ_0 . Indeed, suppose that a compact subset of M intersects infinitely many elements of Γ_0 . Label these geodesics γ_n , $n \in \mathbb{Z}$. By compactness, there exists an element $\gamma := \gamma_n$, for some $n \in \mathbb{Z}$, and a subsequence γ_{n_k} , with $n_k \neq n$, such that $d(\gamma, \gamma_{n_k}) \rightarrow 0$ for $k \rightarrow \infty$. As all these elements are mutually disjoint, we can find points $p_k \in \gamma_{n_k}$ such that $p_k \rightarrow p \in \gamma$, where, moreover, the vectors $v_k \in T_{p_k}M$ tangent to γ_{n_k} at p_k converge to the tangent vector $v \in T_pM$ of γ at p . As the lengths of the geodesics γ_{n_k} are uniformly bounded from above, by the Collar Lemma (see section 2), every curve γ_{n_k} is contained in a uniformly thick embedded annulus $A_k := A(\gamma_{n_k}) \subset M$. Conversely, as the lengths of the geodesics are uniformly bounded from above, and as the initial data (p_k, v_k) of γ_{n_k} converges to the initial data (p, v) of γ , the geodesics γ_{n_k} converge uniformly to γ . In particular, for sufficiently large k , γ is entirely contained in A_k . However, this implies that γ is homotopic to γ_{n_k} , a contradiction as these were all assumed to be mutually disjoint and thus non-homotopic.

Choose an element $\gamma_1 \in \Gamma_0$. As the elements of Γ_0 are locally finite, and as the distance between any two elements of Γ_0 is finite, the distance between γ_1 and the union of the elements $\Gamma_0 \setminus \{\gamma_1\}$ is therefore bounded from below and above. In particular, there exists a δ_0 -short geodesic with shortest distance to γ_1 (though this geodesic need not be unique). Denote one such geodesic by γ_2 . There exists a geodesic arc $\eta \subset M$ connecting γ_1 and γ_2 , with $|\eta| = d(\gamma_1, \gamma_2)$. Take a simple closed curve $\alpha \subset M$ homotopic to $\gamma_1 \cup \eta \cup \gamma_2$ and let γ' be the (not necessarily δ_0 -short) geodesic homotopic to α . As γ_1 and γ_2 are disjoint, and therefore in distinct homotopy classes, γ' is non-trivial. By Lemma 2.1 (iii), we have that

$$\gamma' \cap (\gamma_1 \cup \gamma_2) = \emptyset.$$

Let $P \subset M$ be the pair of pants bounded by the simple closed geodesics γ', γ_1 and γ_2 . As every pair of pants contains a unique simple geodesic arc connecting each pair of boundary geodesics of the pair of pants, $\eta \subset P$ is the unique geodesic arc in P joining γ_1 and γ_2 such that $|\eta| = d(\gamma_1, \gamma_2)$, see Figure 3.2.

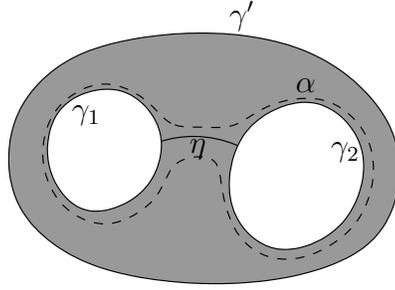


Figure 3.2: Proof of Lemma 3.2

Next, we claim that the interior of P is disjoint from any δ_0 -short geodesic. Indeed, let

$$\gamma_3 \in \Gamma_0 \setminus \{\gamma_1, \gamma_2\},$$

and suppose that $\gamma_3 \cap \text{Int}(P) \neq \emptyset$. As $\gamma_1, \gamma_2 \in \Gamma_0$, by assumption γ_3 can not intersect γ_1 or γ_2 . Thus, if $\gamma_3 \cap \text{Int}(P) \neq \emptyset$, then we necessarily have that $\gamma_3 \cap \gamma' \neq \emptyset$. Consider any two consecutive intersection points $q_1, q_2 \in \gamma_3$ of γ' with γ_3 and denote $\eta' \subset \gamma_3 \cap P$ the corresponding simple arc with endpoints q_1 and q_2 . We first show that we must have that $\eta' \cap \eta \neq \emptyset$. To show this, suppose that $\eta' \cap \eta = \emptyset$. Then $\eta' \cap (\gamma_1 \cup \eta \cup \gamma_2) = \emptyset$. Define $\gamma'_{1,2} \subset \gamma'$ to be the simple arcs which are the two connected components of $\gamma' \setminus \{q_1, q_2\}$. Consider the subsurfaces $M_1, M_2 \subset M$ bounded by $\gamma'_1 \cup \eta'$ and $\gamma'_2 \cup \eta'$ respectively and intersecting $\text{Int}(P)$. As $\eta' \cap (\gamma_1 \cup \eta \cup \gamma_2) = \emptyset$, $\gamma_1 \cup \eta \cup \gamma_2$ is contained in either M_1 or M_2 . However,

this implies that one of these subsurfaces is a topological disk, which contradicts Lemma 2.1 (iii). Therefore, we must have that $\eta' \cap \eta \neq \emptyset$. This in turn implies that

$$d(\gamma_3, \gamma_1) < d(\gamma_1, \gamma_2) \quad (11)$$

which contradicts the assumption that γ_2 is the closest δ_0 -short geodesic to γ_1 . Thus the interior of P is disjoint from any δ_0 -short geodesic.

By Lemma 3.1, there exists a point $p \in P$ such that $D(p, r_0) \subset P$. By the previous paragraph, the disk $D(p, r_0)$ is disjoint from any δ_0 -short geodesic. Take $f \in \mathcal{F}_K(M)$ such that $f(p_0) = p$. The geodesic γ'' homotopic to $f(\gamma_0)$ is δ_0 -short and, again by our choice of K_1 , combined with Lemma 2.5, the geodesic γ'' has the property that $\gamma'' \cap D(p, r_0) \neq \emptyset$. This contradicts our earlier conclusion that the interior of P is disjoint from δ_0 -short geodesics and thus there must exist δ_0 -short geodesics that intersect. \square

Lemma 3.3 (Two-circle Lemma). *Let M be K -quasiconformally homogeneous with $1 < K \leq K_1$ and let γ_1 and γ_2 be two intersecting δ -short geodesics, where $\delta < 1/3$. Then $\gamma_1 \cup \gamma_2$ is a two-circle configuration and the four arcs η_i , $1 \leq i \leq 4$, connecting the intersection points p_1 and p_2 have lengths*

$$\frac{\lambda(M)}{2} - \frac{\delta}{2}\lambda(M) \leq |\eta_i| \leq \frac{\lambda(M)}{2} + \frac{3\delta}{2}\lambda(M). \quad (12)$$

Proof. Let $\gamma_1, \gamma_2 \subset M$ be two δ -short geodesics that intersect. By Lemma 2.1 (ii), γ_1 and γ_2 intersect in an even number of points. To prove there can be no more than two intersection points, suppose that there are $2k$ intersection points, with $k \geq 2$. Label these points p_1, \dots, p_{2k} according to their cyclic ordering on γ_1 , relative to an orientation on γ_1 and an initial point. Define the arcs α_i , with $1 \leq i \leq 2k$, to be the connected components of

$$\gamma_1 \setminus \bigcup_{i=1}^{2k} p_i.$$

As $k \geq 2$ by assumption, at least one of these arcs has length at most $(1+\delta)\lambda(M)/4$. Without loss of generality, we may suppose that this is the case for α_1 . Then the endpoints of α_1 , p_1 and p_2 , cut the geodesic γ_2 into two connected components β_1 and β_2 . One of these components, say β_1 , has length at most $(1+\delta)\lambda(M)/2$. By Lemma 2.1 (iii), $\alpha_1 \cup \beta_1$ is a non-trivial closed curve. However, we have that

$$|\alpha_1 \cup \beta_1| \leq \frac{3(1+\delta)\lambda(M)}{4} < \lambda(M),$$

as $(1+\delta) < 4/3$, which is impossible. Thus γ_1 and γ_2 intersect in exactly two points.

To prove (12), we adopt the labeling in Figure 3.1. As $\gamma_1 = \eta_1 \cup \eta_2$ and $|\gamma_1| \leq (1 + \delta)\lambda(M)$, one of the arcs η_1 or η_2 has length at most $(1 + \delta)\lambda(M)/2$. We may assume this is the case for η_1 . As the closed curve $\eta_3 \cup \eta_1$ is homotopically non-trivial, we must have that

$$\frac{(1 + \delta)\lambda(M)}{2} + |\eta_3| \geq |\eta_1| + |\eta_3| \geq \lambda(M),$$

and thus

$$|\eta_3| \geq \frac{(1 - \delta)\lambda(M)}{2}. \quad (13)$$

Conversely, in order that $|\eta_3| + |\eta_4| \leq (1 + \delta)\lambda(M)$, by (13), we must have that

$$|\eta_4| \leq \left(\frac{1}{2} + \frac{3}{2}\delta\right)\lambda(M). \quad (14)$$

The other cases follow by symmetry. This finishes the proof. \square

In particular, the two-circle Lemma holds for all $\delta \leq 6\delta_0 < 4/3$. For future reference, we introduce the following.

Definition 3 (Tight pair of pants). A *tight pair of pants* is a pair of pants $P \subset M$ such that the three boundary curves are $3\delta_0$ -short geodesics.

We have the following corollary of the two-circle Lemma.

Corollary 3.4. *Let M be K -quasiconformally homogeneous with $1 < K \leq K_1$. Then there exists a tight pair of pants.*

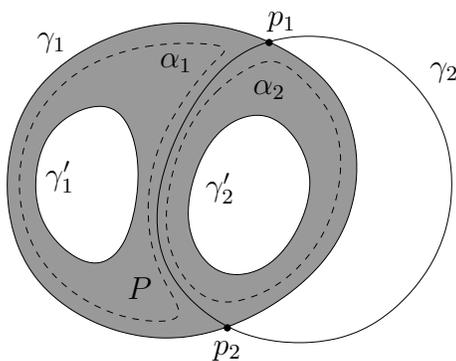


Figure 3.3: Proof of Corollary 3.4.

Proof. By the two-circle Lemma, for $K \leq K_1$, there exists δ_0 -short geodesics γ_1 and γ_2 that intersect in exactly two points. In the labeling of Figure 3.1, let α_1 be the simple closed curve $\eta_1 \cup \eta_3$ and γ'_1 be the simple closed geodesic homotopic to α_1 . Similarly, let α_2 be the simple closed curve $\eta_2 \cup \eta_3$ and γ'_2 be the simple closed geodesic homotopic to α_2 . By Lemma 2.1 (iii), the three geodesics γ'_1, γ'_2 and γ_1 are disjoint and thus the region bounded by these three simple closed geodesics is a pair of pants P , see Figure 3.3.

To prove P is a tight pair of pants, it suffices to show that γ'_i is $3\delta_0$ -short, for $i = 1, 2$, as γ_1 is δ_0 -short. It follows from (12) of the two-circle Lemma that

$$|\eta_1| + |\eta_3| \leq 2 \left(\frac{\lambda(M)}{2} + \frac{3\delta_0}{2} \lambda(M) \right) = (1 + 3\delta_0) \lambda(M). \quad (15)$$

Therefore, the length of γ'_1 is bounded by the length of $\eta_1 \cup \eta_3$, which is at most $(1 + 3\delta_0) \lambda(M)$. Similarly, considering the length of $\eta_2 \cup \eta_3$, we obtain that γ'_2 is $3\delta_0$ -short. Therefore, P is a tight pair of pants. \square

3.2 Definite angles of intersection

In what follows, we use the following notation. Let $\gamma_1, \gamma_2 \subset M$ be two simple closed geodesics that intersect at a point $p \in M$. The angle between the two geodesics at $p \in M$, denoted $\angle(\gamma_1, \gamma_2)_p$, is defined to be the minimum of $\angle(v_1, v_2)_p$ and $\angle(v_1, -v_2)_p$ where $v_1, v_2 \in T_p M$ is a tangent vector to γ_1, γ_2 respectively. In order to produce certain configurations of intersecting simple closed geodesics, we show that for all $K > 1$ sufficiently small, there exist $3\delta_0$ -short geodesics intersecting at a uniformly large angle. More precisely,

Lemma 3.5 (Definite angles of intersection). *There exists a constant $1 < K_2 \leq K_1$, such that if M is K -quasiconformally homogeneous with $1 < K \leq K_2$, then there exist two $3\delta_0$ -short geodesics $\gamma_1, \gamma_2 \subset M$, intersecting at a point $q \in M$, such that $\angle(\gamma_1, \gamma_2)_q \geq \pi/4$.*

The proof of Lemma 3.5 uses the following two auxiliary lemma's.

Lemma 3.6. *Let $H \subset \mathbb{D}^2$ be a right-angled hyperbolic hexagon. Let a, b, c be the sides of the alternate edges and a', b', c' the sides of the opposite edges. Suppose that $|a| = (1 + \epsilon_1)\ell$, $|b| = (1 + \epsilon_2)\ell$ and $|c| = (1 + \epsilon_3)\ell$ with $0 \leq \epsilon_i < 1/2$, $1 \leq i \leq 3$. For every $\epsilon > 0$, there exists $\ell_\epsilon > 0$ such that, if $\ell \geq \ell_\epsilon$, then the lengths of the sides a', b' and c' are at most ϵ .*

Proof. By the hyperbolic cosine law for right-angled hexagons (see [13]), we have that

$$\cosh(|c'|) = \frac{\cosh(|a|) \cosh(|b|) + \cosh(|c|)}{\sinh(|a|) \sinh(|b|)}, \quad (16)$$

which we can write as

$$\cosh(|c'|) = \frac{1}{\tanh(|a|)\tanh(|b|)} + \frac{\cosh(|c|)}{\sinh(|a|)\sinh(|b|)} \quad (17)$$

We have that

$$\frac{\cosh(|c|)}{\sinh(|a|)\sinh(|b|)} \asymp \frac{e^{(1+\epsilon_3)\ell}}{e^{(2+\epsilon_1+\epsilon_2)\ell}} \rightarrow 0, \quad \text{for } \ell \rightarrow \infty, \quad (18)$$

as $0 \leq \epsilon_i < 1/2$ for $1 \leq i \leq 3$. Further, as $\tanh(r) \rightarrow 1$ for $r \rightarrow \infty$, given any $\epsilon' > 0$, there exists an $\ell_{\epsilon'}$ such that, if $\ell \geq \ell_{\epsilon'}$, then

$$\cosh(|c'|) \leq 1 + \epsilon'. \quad (19)$$

Thus given an $\epsilon > 0$, choose $\ell_{\epsilon'}$ such that (19) is satisfied with $\epsilon := \cosh^{-1}(1 + \epsilon')$. For this $\ell_{\epsilon'}$ (with ϵ' depending on ϵ only), we have that

$$|c'| \leq \cosh^{-1}(1 + \epsilon') = \epsilon \quad (20)$$

Cyclically permuting a, b, c and a', b', c' gives a similar estimate for a' and b' and this finishes the proof. \square

Lemma 3.7. *Let $T \subset \mathbb{D}^2$ be an ideal triangle with boundary $\partial T = \gamma_1 \cup \gamma_2 \cup \gamma_3$ and barycenter $0 \in \mathbb{D}^2$. Let $\gamma \subset \mathbb{D}^2$ be a geodesic passing through $0 \in \mathbb{D}^2$. Then for an $i \in \{1, 2, 3\}$, γ intersects ∂T at a point $p \in \gamma_i$, such that $\pi/4 + \epsilon_0 \leq \angle(\gamma, \gamma_i)_p \leq \pi/2$, where $\epsilon_0 \approx 0.24$.*

Proof. Let us label the boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ of the ideal triangle T as in Figure 3.4. Let $l_0 \subset \mathbb{D}^2$ be the axis of symmetry of T , relative to the symmetry that exchanges γ_1 and γ_2 . The distance of $0 \in \mathbb{D}^2$ to the point of intersection of l_0 with γ_3 is $\frac{1}{2} \log 3$. Let l_1 be the axis perpendicular to the axis of symmetry relative to the symmetry that exchanges γ_2 and γ_3 , see Figure 3.4. Let θ_1 be the angle between the axis l_1 and the geodesic γ_3 . The angle between the axes l_0 and l_1 is $\pi/6$. Therefore, by the hyperbolic cosine law, the angle θ_1 between γ_3 and l_1 is given by

$$\cos(\theta_1) = \cosh\left(\frac{1}{2} \log 3\right) \sin\left(\frac{\pi}{6}\right).$$

It is readily verified that $\theta_1 = \pi/4 + \epsilon_0$, with $\epsilon_0 \approx 0.24$. Suppose that the geodesic γ that passes through $0 \in \mathbb{D}^2$ is such that $\gamma \cap \gamma_3 \neq \emptyset$. Denote p the point of intersection of γ with γ_3 . By symmetry of the configuration, we may assume that p is contained in the arc $\eta \subset \gamma_3$ cut out by the two intersection points of l_0 and l_1 with γ_3 , which, up to symmetry, represents the extremal case. Thus we have that $\pi/4 + \epsilon_0 \leq \angle(\gamma, \gamma_i)_p \leq \pi/2$. This finishes the proof of the Lemma. \square

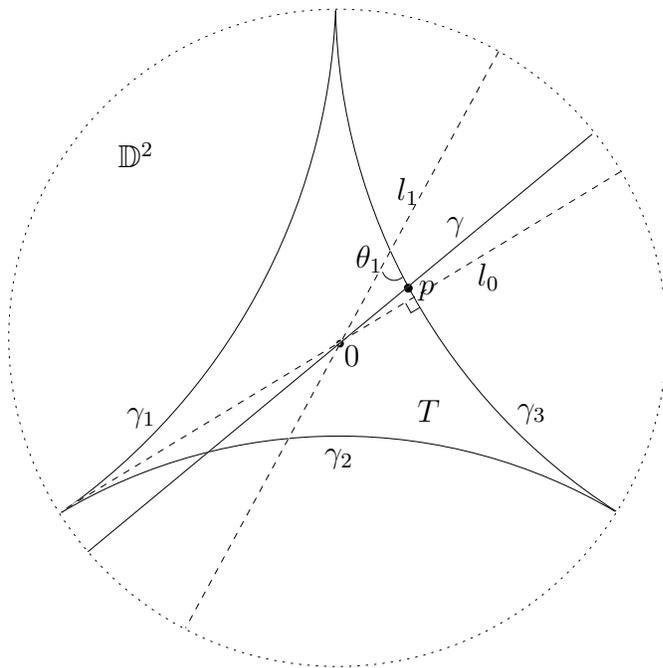


Figure 3.4: Proof of Lemma 3.7.

Proof of Lemma 3.5. By Corollary 3.4, there exists a tight pair of pants $P \subset M$, i.e. a pair of pants P with $\partial P = \gamma_1 \cup \gamma_2 \cup \gamma_3$, with the property that the three boundary geodesics $\gamma_1, \gamma_2, \gamma_3$ are $3\delta_0$ -short. Lifting the pair of pants P to the cover \mathbb{D}^2 , it unfolds to two right-angled hexagons $H, H' \subset \mathbb{D}^2$, each of which contains exactly a half of the component of the lift $\tilde{\gamma}_i$ of γ_i to \mathbb{D}^2 as its alternate boundary arcs, with $1 \leq i \leq 3$. Restrict to H and denote η_i with $1 \leq i \leq 3$ the alternating boundary arcs. As the length of η_i is exactly half of that of γ_i , with $1 \leq i \leq 3$, and these are $3\delta_0$ -short, the lengths η_i satisfy the length requirement of Lemma 3.6, so that, given any $\epsilon > 0$ there exists a $K > 1$ such that the lengths of the three sides of the hexagon H opposite to η_i are of length at most ϵ , as $\lambda(M) \rightarrow \infty$ for $K \rightarrow 1$ by Lemma 2.2.

Therefore, by normalizing by a suitable element of $\text{Möb}(\mathbb{D}^2)$ if necessary, the hexagon $H \subset \mathbb{D}^2$ converges on a compact disk $D(0, 10)$ to an ideal triangle with barycenter $0 \in \mathbb{D}^2$, for $K \rightarrow 1$. In particular, by Lemma 3.7, there exists $1 < K_2 \leq K_1$, such that any geodesic $\gamma' \subset \mathbb{D}^2$ passing through $0 \in \mathbb{D}^2$ intersects one of the arcs η_i of H under an angle at least $\pi/4 + \epsilon_0/2$, with ϵ_0 as in Lemma 3.7. As geodesics in \mathbb{D}^2 passing near $0 \in \mathbb{D}^2$ are almost straight lines, there exists an $\epsilon_1 > 0$ such that

any geodesic $\gamma' \subset \mathbb{D}^2$ passing through the disk $D(0, \epsilon_1) \subset \mathbb{D}^2$ intersect one of the arcs η_i at an angle at least $\pi/4$. By choosing $K_2 > 1$ smaller if necessary, we can be sure that $C_2(K_2) \leq \epsilon_1$, where $C_2(K)$ is the constant of Lemma 2.5.

Let $\gamma_0 \subset M$ be a δ_0 -short geodesic and let $p_0 \in \gamma_0$. Choose the point $p \in P \subset M$ in the tight pair of pants, which without loss of generality we may assume to correspond to $0 \in \mathbb{D}^2$ in the lift. Choose an element $f \in \mathcal{F}_K(M)$ such that $f(p_0) = p$ and denote $\gamma \subset M$ the geodesic homotopic to $f(\gamma_0)$. By our choice of K_3 , the lift $\tilde{\gamma}$ of γ will intersect at least one of the three arcs η_1, η_2 or η_3 at an angle $\angle(\tilde{\gamma}, \eta_i)_q \geq \pi/4$. In other words, γ intersects one of the three boundary geodesics γ_1, γ_2 or γ_3 at an angle at least $\pi/4$. Further, as $K_2 \leq K_1$, we have that

$$K_2(1 + \delta_0) \leq K_1(1 + \delta_0) \leq 1 + 3\delta_0,$$

as $K_1(1 + \delta_0/2) \leq 1 + \delta_0$, and thus γ is $3\delta_0$ -short. This proves the Lemma. \square

3.3 The three-circle Lemma

A *triangle* $T \subset M$ is a subsurface of M such that its boundary consists of a simple closed curve comprised of three geodesic arcs. The triangle T is said to be *trivial* if the simple closed curve ∂T is homotopically trivial and *non-trivial* otherwise.

Definition 4 (Three-circle configuration). A *three-circle configuration* is a union of three $6\delta_0$ -short geodesics $\gamma_1, \gamma_2, \gamma_3$, such that each pair of geodesics intersect in exactly two points and the connected components $M \setminus \bigcup_{j=1}^3 \gamma_j$ consists of exactly 8 triangles, see Figure 3.5.

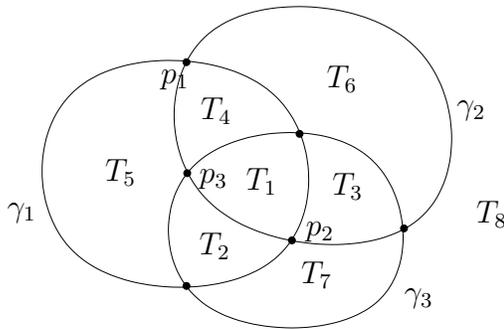


Figure 3.5: Three-circle configuration.

Lemma 3.8. *Let $\gamma_1, \gamma_2, \gamma_3 \subset M$ comprise a three-circle configuration and let $\{T_j\}_{j=1}^8$ be the collection of triangles associated to the configuration. If a triangle T_j for some $1 \leq j \leq 8$ is trivial, then the length of any of the three geodesic arcs that comprise ∂T_j is at most $\lambda(M)/7$.*

Proof. In what follows, we write the juxtaposition of arcs to denote the closed curve comprised by concatenating the arcs in counterclockwise direction. Suppose that a triangle $T := T_j$ for some $1 \leq j \leq 8$ is trivial and let the labeling be as given in Figure 3.6, where $\partial T = x_2 z_2 y_2$. The geodesics $\gamma_1, \gamma_2, \gamma_3$ are labeled $\gamma_s, \gamma_t, \gamma_k$ with 1, 2, 3 some permutation of the letters s, t, k , where arcs $x_i \subset \gamma_s, y_i \subset \gamma_t$ and $z_i \subset \gamma_k$ with $1 \leq i \leq 3$.

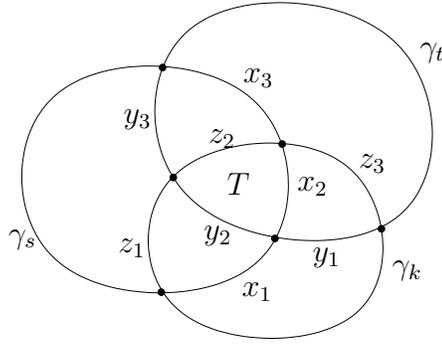


Figure 3.6: Proof of Lemma 3.8.

Define the simple closed curves

$$\alpha_1 = x_2 x_3 y_3 y_2, \quad \alpha_2 = x_1 x_2 z_2 z_1 \quad \text{and} \quad \alpha_3 = y_2 y_1 z_3 z_2. \quad (21)$$

By Lemma 2.1 (iii), the simple closed curves α_i , with $1 \leq i \leq 3$, are non-trivial. Further, as each of the three geodesics γ_s, γ_t and γ_k is $6\delta_0$ -short (by definition 4), by the two-circle Lemma we thus have that

$$\lambda(M) \leq |\alpha_i| \leq (1 + 18\delta_0)\lambda(M), \quad (22)$$

for $1 \leq i \leq 3$. Next, consider the simple closed curves

$$\beta_1 = x_3 y_3 z_2, \quad \beta_2 = x_1 y_2 z_1 \quad \text{and} \quad \beta_3 = y_1 z_3 x_2. \quad (23)$$

As the curves α_i are non-trivial, and the triangle T is trivial, the curve β_i is homotopic to α_i , for $1 \leq i \leq 3$, and therefore non-trivial. Moreover, as T is trivial, by the triangle-inequality applied to ∂T , we have that

$$|x_2| \leq |y_2| + |z_2|, \quad |y_2| \leq |x_2| + |z_2| \quad \text{and} \quad |z_2| \leq |x_2| + |y_2|. \quad (24)$$

Combining (22) with (24), it is readily verified that

$$\lambda(M) \leq |\beta_j| \leq (1 + 18\delta_0)\lambda(M), \quad (25)$$

for $1 \leq j \leq 3$. Summing up the lengths of the arcs that constitute the closed curves β_1, β_2 and β_3 , cf. (23), and reordering the terms, it follows that

$$3\lambda(M) \leq |x_1| + |x_2| + |x_3| + |y_1| + |y_2| + |y_3| + |z_1| + |z_2| + |z_3| \leq 3(1 + 18\delta_0)\lambda(M). \quad (26)$$

The length of α_1 can by (21) be expressed as the sum of the lengths of its constituent arcs and estimated by

$$\lambda(M) \leq |x_2| + |x_3| + |y_2| + |y_3| \leq (1 + 18\delta_0)\lambda(M). \quad (27)$$

Subtracting (27) from (26), we obtain the estimate

$$2\lambda(M) - 18\delta_0\lambda(M) \leq |x_1| + |y_1| + |z_1| + |z_2| + |z_3| \leq 2\lambda(M) + 54\delta_0\lambda(M) \quad (28)$$

However, adding up the lengths of β_2 and β_3 , we must have that

$$2\lambda(M) \leq |x_1| + |x_2| + |y_1| + |y_2| + |z_1| + |z_3| \leq 2(1 + 18\delta_0)\lambda(M) \quad (29)$$

Therefore, subtracting (29) from (28), we obtain

$$-54\delta_0\lambda(M) \leq |z_2| - (|x_2| + |y_2|) \leq 54\delta_0\lambda(M). \quad (30)$$

Repeating the same argument for α_2 and α_3 , one obtains

$$-54\delta_0\lambda(M) \leq |y_2| - (|x_2| + |z_2|) \leq 54\delta_0\lambda(M). \quad (31)$$

$$-54\delta_0\lambda(M) \leq |x_2| - (|y_2| + |z_2|) \leq 54\delta_0\lambda(M). \quad (32)$$

It then follows from (30), (31) and (32) that

$$|x_2| \leq 54\delta_0\lambda(M), \quad |y_2| \leq 54\delta_0\lambda(M) \quad \text{and} \quad |z_2| \leq 54\delta_0\lambda(M). \quad (33)$$

As $\delta_0 = 1/378$, it thus follows that the lengths of the arcs x_2, y_2 and z_2 have to be at most $\lambda(M)/7$. \square

In Lemma 3.10 below, we prove the existence of certain three-circle configurations satisfying additional geometrical properties. The proof uses the following geometric estimate.

Lemma 3.9. *There exists a constant $1 < K_3 \leq K_2$, such that if M is K -quasi-conformally homogeneous with $1 < K \leq K_3$, then the following holds. Let $\gamma_1, \gamma_2 \subset M$ be two $3\delta_0$ -short geodesics intersecting at a point $p \in \gamma_1 \cap \gamma_2$, such that $\angle(\gamma_1, \gamma_2)_p \geq \pi/4$. Let $\gamma_3 \subset M$ be a geodesic and let $q_0 \in \gamma_3$. Let $f \in \mathcal{F}_K(M)$ with $f(p) = q_0$ and let γ'_1, γ'_2 be the geodesic homotopic to $f(\gamma_1)$ and $f(\gamma_2)$ respectively. Then both γ'_1 and γ'_2 are $6\delta_0$ -short and at least one of γ'_1 or γ'_2 intersects γ_3 transversely at a point $q \in \gamma_3$ with $d(q, q_0) \leq 1/20$.*

Proof. First, by choosing $1 < K_3 \leq K_2$, we have that

$$K_3(1 + 3\delta_0) \leq 1 + 6\delta_0,$$

as $K_2 \leq K_1$ and $K_1(1 + 3\delta_0) \leq 1 + 6\delta_0$. Let $\gamma_1, \gamma_2 \subset M$ be two $3\delta_0$ -short geodesics intersecting at a point $p \in \gamma_1 \cap \gamma_2$, such that $\angle(\gamma_1, \gamma_2)_p \geq \pi/4$. Let $\gamma_3 \subset M$ be a geodesic and let $q_0 \in \gamma_3$, where $f(p) = q_0$. Then the geodesics γ'_1, γ'_2 homotopic to $f(\gamma_1)$ and $f(\gamma_2)$ respectively are $6\delta_0$ -short, as γ_1 and γ_2 are $3\delta_0$ -short. By Lemma 2.4, f is approximated by a Möbius transformation on a compact disk. Further, by Lemma 2.5, the geodesic γ'_i stays close to $f(\gamma_i)$, for $i = 1, 2$. Therefore, by choosing K_3 small enough, we have that

- (i) γ'_1 and γ'_2 intersect at a point $q_1 \in M$ with $d(q_1, q_0) \leq 1/100$, where $q_0 \in \gamma_3$, and
- (ii) $\theta := \angle(\gamma'_1, \gamma'_2)_{q_1} \geq \frac{\pi}{5}$.

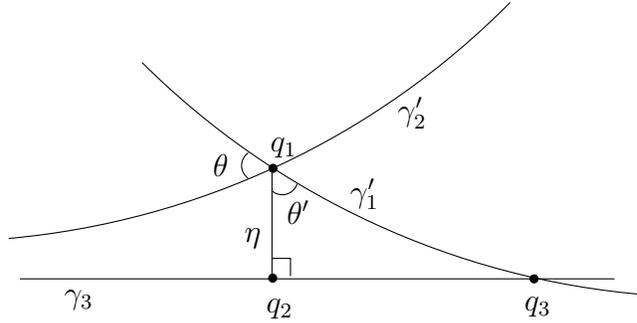


Figure 3.7: Proof of Lemma 3.9.

Let $\eta \subset M$ be the arc emanating from q_1 projecting perpendicularly onto γ_3 at the point $q_2 \in \gamma_3$. As $d(q_1, q_0) \leq 1/100$ and $q_2 \in \gamma_3$, we have that $|\eta| \leq 1/100$. Furthermore, as $\theta \geq \pi/5$, at least one of the geodesics γ'_i with $i = 1, 2$ intersects the

arc η at an angle at most $2\pi/5$. Without loss of generality, we may suppose this is the case for γ'_1 , i.e. that

$$\theta' := \angle(\eta, \gamma'_1)_{q_1} \leq \frac{2\pi}{5},$$

see Figure 3.7. If we consider the (embedded) geodesic triangle with vertices q_1, q_2 and q_3 , combined with $\theta' \leq 2\pi/5$ and $|\eta| \leq 1/100$, it follows from the hyperbolic sine law that

$$d(q_1, q_3) \leq 4/100.$$

As $d(q_1, q_0) \leq 1/100$, we have that $d(q_0, q_2) \leq 1/100$. Further, we have that $d(q_2, q_3) \leq d(q_1, q_3)$ and thus

$$d(q_0, q_3) \leq d(q_0, q_2) + d(q_2, q_3) \leq 1/100 + 4/100 = 1/20.$$

Thus setting $q := q_3$ finishes the proof. \square

Lemma 3.10 (Three-circle Lemma). *If M is K -quasiconformally homogeneous for $1 < K \leq K_3$, then there exists a three-circle configuration consisting of simple closed geodesics $\gamma_1, \gamma_2, \gamma_3 \subset M$, such that*

- (i) $\gamma_1 \cup \gamma_2$ is a two-circle configuration, and
- (ii) γ_3 is a $6\delta_0$ -short geodesic intersecting the arc $\eta_3 \subset \gamma_2$ at a point p_3 for which

$$\left(\frac{1}{4} - \frac{1}{20}\right) \lambda(M) \leq d(p_j, p_3) \leq \left(\frac{1}{4} + \frac{1}{20}\right) \lambda(M),$$

with $j = 1, 2$, in the labeling of Figure 3.1, and

- (iii) γ_3 intersects the interior of the arc η_i in exactly one point for every $1 \leq i \leq 4$.

Proof. As $K_3 \leq K_1$, by Lemma 3.3, there exists a two-circle configuration, comprised of two δ_0 -short geodesics $\gamma_1, \gamma_2 \subset M$. Label the configuration according to Figure 3.1. Mark a point $q \in \eta_3 \subset \gamma_2$ such that $d(q, p_1) = d(q, p_2)$. As $K_3 \leq K_2$, by Lemma 3.5, there exist two $3\delta_0$ -short geodesics γ_3, γ_4 intersecting at a point $p \in M$, such that

$$\angle(\gamma_3, \gamma_4)_p \geq \pi/4.$$

Applying Lemma 3.9 to γ_3 and γ_4 and the target point $q \in \eta_3 \subset \gamma_2$, there exists a $6\delta_0$ -short geodesic γ' intersecting γ_2 transversely at a point $q' \in \gamma_2$ such that $d(q, q') \leq 1/20$. Therefore, setting $p_3 := q'$ and $\gamma_3 := \gamma'$, the conditions (i) and (ii) of Lemma 3.10 are satisfied.

We are left with showing that condition (iii) of Lemma 3.10 is satisfied. That is, we need to show that γ_3 intersects the interior of the arc η_i in exactly one point for every $1 \leq i \leq 4$. To this end, we first show that γ_3 can not intersect the arc η_3

(including the boundary points p_1 and p_2) more than once. To show this is indeed impossible, suppose that γ_3 intersects the arc η_3 in a point $p' \in \eta_3$ other than p_3 . Let $\alpha_1, \alpha_2 \subset \eta_3$ be the connected components of $\eta_3 \setminus \{p_3\}$. We may assume that $p' \in \alpha_1$, the case when $p' \in \alpha_2$ is similar. Therefore, if we let $\alpha' \subset \alpha_1$ the subarc with endpoints p_3 and p' , where we include the case that $p' = p_1$, then it follows that

$$|\alpha'| \leq \left(\frac{1}{4} + \frac{1}{20}\right) \lambda(M).$$

The two points p_3 and p' cut γ_3 into two component arcs β_1 and β_2 , one component of which is of length at most $(1 + 6\delta_0)\lambda(M)/2$; without loss of generality, we may suppose this is the case for β_1 . Then the closed curve $\alpha' \cup \beta_1$ is homotopically nontrivial and

$$|\alpha' \cup \beta_1| \leq \left(\frac{1}{4} + \frac{1}{20}\right) \lambda(M) + \frac{(1 + 6\delta_0)\lambda(M)}{2} = \left(\frac{3}{4} + \frac{1}{20} + 3\delta_0\right) \lambda(M) < \lambda(M),$$

as $\delta_0 = 1/378$, which is a contradiction. Therefore, γ_3 intersects γ_2 at the point p_3 , but does not intersect the arc η_3 in any point other than p_3 , and γ_3 does not pass through p_1 or p_2 .

As γ_3 intersects γ_1 , by Lemma 2.1 (ii), there has to exist at least one more intersection point of γ_3 with γ_2 . By the above argument, all other intersection points are contained in the interior of the arc η_4 . By the two-circle Lemma, applied to $\delta = 6\delta_0$, γ_3 intersects γ_2 only twice, and therefore γ_3 intersects the interior of η_4 exactly once. Similarly, as γ_3 intersects the arc η_3 exactly once, γ_3 has to intersect the interior of the arc η_1 and η_2 at least once. Again by the two-circle Lemma, applied to $\delta = 6\delta_0$, as the total number of intersection points of γ_3 with γ_1 is exactly two, γ_3 has to intersect the interior of the arc η_1 and η_2 exactly once. Thus condition (iii) is indeed satisfied and this proves the Lemma. \square

3.4 Proof of Theorem A

The endgame of the proof of Theorem A. is a combinatorial argument layered on the three-circle configuration of the Three-Circle Lemma. Thus, let $1 < K \leq K_3$ with K the quasiconformal homogeneity constant of M , with K_3 the constant which we obtained in the Three-Circle Lemma in order to ensure the existence of a three-circle configuration.

Lemma 3.11. *Let $\gamma_1, \gamma_2, \gamma_3$ be the three-circle configuration of Lemma 3.10. Then the triangle T_j is non-trivial, for every $1 \leq j \leq 8$.*

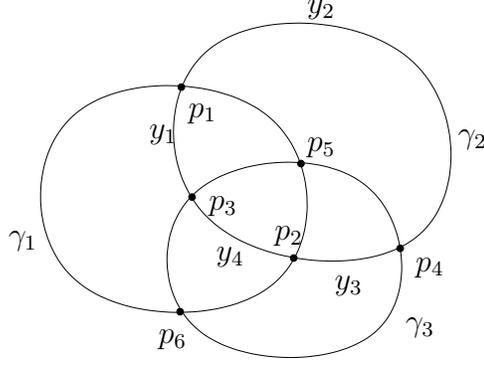


Figure 3.8: Proof of Lemma 3.11.

Proof. By Lemma 3.10, the $6\delta_0$ -short geodesic γ_3 intersects γ_2 at a point $p_3 \in \eta_3 \subset \gamma_1$ and

$$\left(\frac{1}{4} - \frac{1}{20}\right) \lambda(M) \leq d(p_k, p_3) \leq \left(\frac{1}{4} + \frac{1}{20}\right) \lambda(M), \quad (34)$$

with $k = 1, 2$, as given in Figure 3.8. Let y_i with $1 \leq i \leq 4$ be the connected components of $\gamma_2 \setminus \{p_1, p_2, p_3, p_4\}$. It suffices to show that

$$|y_i| > \frac{\lambda(M)}{6}, \text{ for } 1 \leq i \leq 4. \quad (35)$$

To prove sufficiency, note that every triangle ∂T_j , with $1 \leq j \leq 8$, contains exactly one edge y_i for some $1 \leq i \leq 4$. Now, if (35) holds, then by Lemma 3.8, ∂T_j has to be non-trivial, as otherwise all three edges of ∂T_j have to be of length less than $\lambda/7 < \lambda/6$.

To prove (35), first note that, by (34), the arcs y_1 and y_4 satisfy requirement (35). Therefore, we are left with proving the estimate for y_2 and y_3 . As both γ_1 and γ_2 are δ_0 -short, and γ_3 is $6\delta_0$ -short, the two-circle Lemma applied to the pairs of geodesics γ_1, γ_2 and γ_2, γ_3 gives respectively

$$\frac{\lambda(M)}{2} - \frac{\delta_0 \lambda(M)}{2} \leq |y_2| + |y_3| \leq \frac{\lambda(M)}{2} + \frac{3\delta_0 \lambda(M)}{2}, \quad (36)$$

and

$$\frac{\lambda(M)}{2} - \frac{6\delta_0 \lambda(M)}{2} \leq |y_3| + |y_4| \leq \frac{\lambda(M)}{2} + \frac{18\delta_0 \lambda(M)}{2}. \quad (37)$$

Combining (36) and (37), it follows that

$$-\frac{19\delta_0 \lambda(M)}{2} \leq |y_2| - |y_4| \leq \frac{19\delta_0 \lambda(M)}{2}. \quad (38)$$

As $|y_4| = d(p_3, p_2)$, combining (38) with (34), one obtains

$$|y_2| \geq \lambda(M) \left(\frac{1}{4} - \frac{1}{20} - \frac{19\delta_0}{2} \right) > \frac{\lambda(M)}{6}, \quad (39)$$

as $\delta_0 = 1/378$. By symmetry, the same estimate holds for the arc $|y_3|$. This concludes the proof. \square

Let us now conclude the proof.

Proof of Theorem A. Let M be K -quasiconformally homogeneous with $1 < K \leq K_3$ and let $\gamma_1, \gamma_2, \gamma_3$ be the three-circle configuration of Lemma 3.10. By Lemma 3.11, all triangles T_j with $1 \leq j \leq 8$ have to be non-trivial. Therefore, the length of ∂T_j has to be at least $\lambda(M)$ for every $1 \leq j \leq 8$. Adding up the lengths of all ∂T_j , $1 \leq j \leq 8$, means we count every boundary arc of a triangle T_j with multiplicity two and thus

$$2 \sum_{i=1}^3 |\gamma_i| = \sum_{j=1}^8 |\partial T_j| \geq 8\lambda(M). \quad (40)$$

However, as the geodesics γ_i , with $1 \leq i \leq 3$, are (at most) $6\delta_0$ -short by construction, the total length of these three geodesics counted with multiplicity two is bounded by

$$2 \sum_{i=1}^3 |\gamma_i| \leq 2 \cdot 3(1 + 6\delta_0)\lambda(M) = 6(1 + 6\delta_0)\lambda(M) < 7\lambda(M), \quad (41)$$

as $36\delta_0 < 1$. The contradictory claims (40) and (41) finish the proof. \square

4 Concluding remarks

It would be interesting to obtain an explicit value for the universal constant $\mathcal{K} > 1$, whose existence was shown in the proof of Theorem A. In [10], examples are given of K -quasiconformally homogeneous genus zero surfaces, for which explicit estimates are computed of the quasiconformal homogeneity constant K of M , which give the following upper bound on the universal constant \mathcal{K} ,

$$1 < \mathcal{K} \leq \left(\frac{e}{s} \right)^4, \quad (42)$$

where $s \approx 0.483$. The geometrical estimates in our proof can in many cases be improved to give explicit estimates of the quantities involved, except for the preliminary Lemma 2.2, whose known proof is based on a normal family argument (see [3]). Using these estimates, can one find a sharper estimate for \mathcal{K} ?

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