A Cayley-Hamiltonian Theorem for Periodic Finite Band Matrices

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Abstract. Let K be a doubly infinite, self–adjoint matrix which is finite band (i.e. $K_{jk} = 0$ if $|j-k| > m$) and periodic ($KS^n = S^nK$ for some n where $(Su)_j = u_{j+1}$) and non–degenerate (i.e. $K_{jj+m} \neq 0$ for all j). Then, there is a polynomial, $p(x,y)$, in two variables with $p(K, S^n) = 0$. This generalizes the tridiagonal case where $p(x,y) = y^2 - y\Delta(x) + 1$ where Δ is the discriminant. I hope Pavel Exner will enjoy this birthday bouquet.

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1. Introduction–The Magic Formula

Let J be a doubly infinite, self–adjoint, tridiagonal Jacobi matrix (i.e. $J_{jk} = 0$ if $|j-k| > 1$ and $J_{jj+1} > 0$) that is periodic, i.e. if

$$(Su)_j = u_{j+1} \tag{1}$$

then for some $n \in \mathbb{Z}_+$, $S^nJ = JS^n$. There is a huge literature on this subject – see Simon [7], Chapter 5.

$(J-E)u = 0$ is a second order difference equation, so there is a linear map $T(E) : \mathbb{C}^2 \to \mathbb{C}^2$ so that if u_0, u_1 are given, then $T(E)\left(\begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_n \\ u_{n+1} \end{smallmatrix}\right)$ for the solution of $(J-E)u = 0$. $\Delta(E) = \mathrm{Tr}(M(E))$ is called the discriminant of J. We note that $\det(T(E)) = 1$ so $T(E)$ has eigenvalues λ and λ^{-1} and $\Delta(E) = \lambda + \lambda^{-1}$. If $\Delta(E) \in (-2,2)$, then $\lambda = e^{i\theta}$ for some θ in $\pm(0,\pi)$ and then $Ju = Eu$ has Floquet solutions, $u^{\pm}$ obeying $u^{\pm}_{j+nk} = e^{\pm ik\theta}u^{\pm}_j$. These are bounded and there are only bounded solutions if $\Delta(E) \in [-2,2]$. Thus $\mathrm{spec}(J) = \Delta([-2,2])$. One often writes this relation as

$$\Delta(E) = 2\cos(\theta) \tag{2}$$

In [2], Damanik, Killip and Simon emphasized and exploited the operator form of (2), namely

$$\Delta(J) = S^n + S^{-n} \tag{3}$$

This follows from (2) and the view of J as a direct integral. More importantly, what they called the "magic formula", [2] shows that a two sided, not apriori periodic, Jacobi matrix, which obeys (3), is periodic and in the isospectral torus of J.

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A Laurent matrix is a finite band doubly infinite matrix that is constant along diagonals, so a polynomial in S and S^{-1}. $S^n + S^{-n}$ is an example of such a matrix. The current paper had its genesis in a question asked me by Jonathan Breuer and Maurice Duits. They asked if K is finite band and periodic but not tridiagonal if there is a polynomial Q so that $Q(K)$ is a Laurent matrix. They guessed that Q might be connected to the trace of a transfer matrix.

While I don't have a formal example where I can prove there is no such Q, I have found a related result which strongly suggests that, in general, the answer is no. I found an object which replaces Δ for more general K which is width $2m+1$ (i.e. $K_{jk} = 0$ if $|j-k| > m$), self-adjoint and non-degenerate in the sense that for all j, $K_{jj+m} \neq 0$. Namely we prove the existence of a polynomial, $p(x, y)$, in x and y of degree $2m$ in y, so that $p(K, S^n) = 0$. In the Jacobi case,

$$p(x, y) = y^2 - y\Delta(x) + 1 \tag{4}$$

so that $p(J, S^n) = 0$ is equivalent to (3).

We prove this theorem and begin the exploration of this object in Section 2. That a scalar polynomial vanishes when the variable is replaced by an operator is the essence of the Cayley-Hamiltonian theorem which says that a matrix obeys its secular equation. This was proven in 1853 by Hamilton [4] for the two special cases of three dimensional rotations and for multiplication by quaternions and stated in general by Cayley [1] in 1858 who proved it only for 2×2 matrices although he said he'd done the calculation for 3×3 matrices. In 1878, Frobenius [3] proved the general result and attributed it to Cayley. We regard our main result, Theorem 2.1, as a form of the Cayley-Hamiltonian Theorem.

The magic formula had important precursors in two interesting papers of Naǐman [5, 6]. These papers are also connected to our work here.

It is a pleasure to present this paper to Pavel Exner for his 70^{th} birthday. I have long enjoyed his contributions to areas of common interest. I recall with fondness the visit he arranged for me in Prague. He was a model organizer of an ICMP. And he is an all around sweet guy. Happy birthday, Pavel.

2. Main Result

By a width $2m+1$ matrix, $m \in \{1, 2, \dots\}$, we mean a doubly infinite matrix, K, with $K_{jk} = 0$ if $|j-k| > m$. If $\sup |K_{jk}| < \infty$, K defines a bounded operator on $\ell^2(\mathbb{Z})$ which we also denote by K. We say that K is *non-degenerate* if $K_{jj\pm m} \neq 0$ for all j. K is periodic (with period n) if $S^n K = K S^n$, where S is the unitary operator given by (1).

We consider width $2m+1$, non-degenerate, period-n self–adjoint matrices. In that case, for any E, because K is non-degenerate, $Ku = Eu$, as a finite difference equation, has a unique solution for each choice of $\{u_j\}_{j=0}^{2m-1}$. $T(E)$ will be defined as the map from $\{u_j\}_{j=0}^{2m-1}$ to $\{u_j\}_{j=n}^{n+2m-1}$ – it is a $2m \times 2m$ matrix. If $T(E)u = \lambda u$ for $\lambda \in \mathbb{C}$, $Ku = Eu$ has a Floquet solution with $u_{kn+j} = \lambda^k u_j$. If $T(E)$ is diagonalizable, the set of Floquet solutions is a basis for all solutions of $Ku = Eu$.

If $T(E)$ has Jordan anomalies (see [8] for background on linear algebra), there is a basis of modified Floquet solutions with some polynomial growth on top of the exponential λ^k.

The values of λ are determined by

$$p(E,\lambda) = \det(\lambda\mathbf{1} - T(E)) \tag{5}$$

Since

$$\begin{aligned}\det(\lambda\mathbf{1} - T(E)) &= \lambda^{2m}\det(\mathbf{1} - \lambda^{-1}T(E)) \\ &= \lambda^{2m}\left(\sum_{j=0}^{2m}(-\lambda)^j \operatorname{Tr}(\wedge^j(T(E)))\right) \\ &= \sum_{j=0}^{2m}\lambda^j p_j(E) \end{aligned}\tag{6}$$

where $\wedge^j$ is given by multilinear algebra ([8, Section 1.3]) with $\wedge^0(T(E)) = \mathbf{1}$ on $\mathbb{C}$ so its Trace is 1. Thus in (6),

$$p_{2m}(E) = 1, \qquad p_j(E) = (-1)^j \operatorname{Tr}(\wedge^{2m-j}(T(E))) \tag{7}$$

and p_j is of degree at most $(2m-j)n$ in E.

Since S and K are commuting bounded normal operators, they have a joint spectral resolution which is supported precisely on the solutions of $p(E,\lambda) = 0$ with $|\lambda| = 1$. By the spectral theorem (equivalently, a direct integral analysis), we thus have the main result of this note:

Theorem 2.1. *Let K be a self-adjoint, non-degenerate, width $2m+1$, period n matrix. Then for p given by (6)/(7), we have that*

$$p(K, S^n) = 0 \tag{8}$$

We end with a number of comments:

(1) We used the self–adjointness of K to be able to exploit the spectral theorem. But just as the Cayley–Hamilton Theorem for finite matrices holds in the non-self-adjoint case, it seems likely that Theorem 2.1 is valid for general non-degenerate, periodic K, even if not self–adjoint

(2) Since $K_{jj-m} \neq 0$, the transfer matrix, $T(E)$ is invertible and thus $\det(T(E))$ has no zeros. Since it is a polynomial, it must be constant, that is $p_0(E)$ is a constant. It is thus of much smaller degree than the bound, $2mn$, obtained by counting powers of E.

(3) In many cases of interest, $T(E)$ will be symplectic, i.e., there exists an anti-symmetric Q on $\mathbb{C}^{2m}$ with $Q^2 = -\mathbf{1}$ so that $T(E)^t Q T(E) = Q$. Such a $T(E)$ has $T(E)^{-1}$ and $T(E)^t$ similar, so the eigenvalues $\{\lambda_j\}_{j=1}^{2m}$ can be ordered so that $\lambda_{2m+1-j} = \lambda_j^{-1}, j = 1,\ldots,m$. It follows that $\det(T(E)) = 1$ but even

more, we have that

$$\begin{aligned}\mathrm{Tr}(\wedge^k(T(E))) &= \sum_{j_1<\dots<j_k} \lambda_{j_1}\dots\lambda_{j_k} \\ &= \sum_{j_1<\dots<j_{2m-k}} \lambda_{j_1}^{-1}\dots\lambda_{j_{2m-k}}^{-1} \qquad (9) \\ &= \sum_{j_1<\dots<j_{2m-k}} \lambda_{j_1}\dots\lambda_{j_{2m-k}} \qquad (10) \\ &= \mathrm{Tr}(\wedge^{2m-k}(T(E)))\end{aligned}$$

and $p_{2m-k}(E) = p_k(E)$. In the above, (9) follows from the fact that the product of all the λ's is 1, and we can sum over the complements of all k–sets. (10) then uses the fact that $\lambda_{2m+1-j} = \lambda_j^{-1}, j = 1, \dots, m$.

(4) One can ask whether there is a magic formula in this case, i.e. does $p(\widetilde{K}, S^n) = 0$ imply that $\widetilde{K}$ is periodic and isospectral to K. There is already one subtlety one faces at the start. If $\widetilde{K}$ is not supposed apriori n–periodic, then $S^{nj}p_j(\widetilde{K})$ may not equal $p_j(\widetilde{K})S^{nj}$ so there is a question of what $p(\widetilde{K}, S^n) = 0$ means. Even if one supposes that $\widetilde{K}S^n = S^n\widetilde{K}$, $p(\widetilde{K}, S^n) = 0$ and the spectral theorem only implies that $\mathrm{spec}(\widetilde{K}) \subset \mathrm{spec}(K)$, so there is more to be proven. Indeed, the isospectral set in this case remains to be explored.

(5) It seems unlikely that there is another independent relation besides (8) between a polynomial in K and Laurent polynomial in S. In general one cannot hope that $p(K, S^n) = 0$ yields a polynomial in one variable so that $Q(K)$ is a Laurent polynomial in S^n but it remains to find an explicit example where one can prove that the Breuer-Duits question has a negative answer.

There are lots of interesting open questions connected to our main result, Theorem 2.1.

3. References

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