A Cayley-Hamiltonian Theorem for Periodic Finite Band Matrices

Barry Simon

Abstract. Let $K$ be a doubly infinite, self–adjoint matrix which is finite band (i.e. $K_{jk} = 0$ if $|j-k| > m$) and periodic ($KS^n = S^n K$ for some $n$ where $(Su)_j = u_{j+1}$) and non–degenerate (i.e. $K_{jj+m} \neq 0$ for all $j$). Then, there is a polynomial, $p(x,y)$, in two variables with $p(K, S^n) = 0$. This generalizes the tridiagonal case where $p(x, y) = y^2 - y \Delta(x) + 1$ where $\Delta$ is the discriminant. I hope Pavel Exner will enjoy this birthday bouquet.

2010 Mathematics Subject Classification. Primary 47B36, 47B39, 30C10
Keywords. Periodic Jacobi Matrices, Discriminant, Magic Formula.

1. Introduction–The Magic Formula

Let $J$ be a doubly infinite, self–adjoint, tridiagonal Jacobi matrix (i.e. $J_{jk} = 0$ if $|j-k| > 1$ and $J_{jj+1} > 0$) that is periodic, i.e. if

$$(Su)_j = u_{j+1} \quad (1)$$

then for some $n \in \mathbb{Z}_+$, $S^n J = JS^n$. There is a huge literature on this subject – see Simon [7], Chapter 5.

$(J - E)u = 0$ is a second order difference equation, so there is a linear map $T(E) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ so that if $u_0, u_1$ are given, then $T(E)(u_0) = (u_{n+1})$ for the solution of $(J - E)u = 0$. $\Delta(E) = \text{Tr}(M(E))$ is called the discriminant of $J$. We note that $\det(T(E)) = 1$ so $T(E)$ has eigenvalues $\lambda$ and $\lambda^{-1}$ and $\Delta(E) = \lambda + \lambda^{-1}$.

If $\Delta(E) \in (-2, 2)$, then $\lambda = e^{i\theta}$ for some $\theta$ in $\pm(0, \pi)$ and then $Ju = Eu$ has Floquet solutions, $u^\pm$ obeying $u_{j+nk}^\pm = e^{\pm ik\theta} u_j^\pm$. These are bounded and there are only bounded solutions if $\Delta(E) \in [-2, 2]$. Thus $\text{spec}(J) = \Delta([-2, 2])$. One often writes this relation as

$$\Delta(E) = 2\cos(\theta) \quad (2)$$

In [2], Damanik, Killip and Simon emphasized and exploited the operator form of (2), namely

$$\Delta(J) = S^n + S^{-n} \quad (3)$$

This follows from (2) and the view of $J$ as a direct integral. More importantly, what they called the “magic formula”, [2] shows that a two sided, not apriori periodic, Jacobi matrix, which obeys (3), is periodic and in the isospectral torus of $J$.

\*Research supported in part by NSF grant DMS-1265592 and in part by Israeli BSF Grant No. 2014337
A Laurent matrix is a finite band doubly infinite matrix that is constant along diagonals, so a polynomial in $S$ and $S^{-1}$. $S^n + S^{-n}$ is an example of such a matrix. The current paper had its genesis in a question asked me by Jonathan Breuer and Maurice Duits. They asked if $K$ is finite band and periodic but not tridiagonal if there is a polynomial $Q$ so that $Q(K)$ is a Laurent matrix. They guessed that $Q$ might be connected to the trace of a transfer matrix.

While I don’t have a formal example where I can prove there is no such $Q$, I have found a related result which strongly suggests that, in general, the answer is no. I found an object which replaces $\Delta$ for more general $K$ which is width $2m + 1$ (i.e. $K_{jk} = 0$ if $|j - k| > m$), self-adjoint and non-degenerate in the sense that for all $j$, $K_{jj+m} \neq 0$. Namely we prove the existence of a polynomial, $p(x, y)$, in $x$ and $y$ of degree $2m$ in $y$, so that $p(K, S^n) = 0$. In the Jacobi case,

$$p(x, y) = y^2 - y\Delta(x) + 1$$

so that $p(J, S^n) = 0$ is equivalent to (3).

We prove this theorem and begin the exploration of this object in Section 2.

That a scalar polynomial vanishes when the variable is replaced by an operator is the essence of the Cayley-Hamiltonian theorem which says that a matrix obeys its secular equation. This was proven in 1853 by Hamilton [4] for the two special cases of three dimensional rotations and for multiplication by quaternions and stated in general by Cayley [1] in 1858 who proved it only for $2 \times 2$ matrices although he said he’d done the calculation for $3 \times 3$ matrices. In 1878, Frobenius [3] proved the general result and attributed it to Cayley. We regard our main result, Theorem 2.1, as a form of the Cayley-Hamiltonian Theorem.

The magic formula had important precursors in two interesting papers of Naîman [5, 6]. These papers are also connected to our work here.

It is a pleasure to present this paper to Pavel Exner for his 70th birthday. I have long enjoyed his contributions to areas of common interest. I recall with fondness the visit he arranged for me in Prague. He was a model organizer of an ICMP. And he is an all around sweet guy. Happy birthday, Pavel.

2. Main Result

By a width $2m + 1$ matrix, $m \in \{1, 2, \ldots\}$, we mean a doubly infinite matrix, $K$, with $K_{jk} = 0$ if $|j - k| > m$. If $\sup |K_{jk}| < \infty$, $K$ defines a bounded operator on $l^2(\mathbb{Z})$ which we also denote by $K$. We say that $K$ is non-degenerate if $K_{jj+m} \neq 0$ for all $j$. $K$ is periodic (with period $n$) if $S^nK = KS^n$, where $S$ is the unitary operator given by (1).

We consider width $2m + 1$, non-degenerate, period-$n$ self-adjoint matrices. In that case, for any $E$, because $K$ is non-degenerate, $Ku = Eu$, as a finite difference equation, has an unique solution for each choice of $\{u_j\}_{j=0}^{2m-1}$. $T(E)$ will be defined as the map from $\{u_j\}_{j=0}^{2m-1}$ to $\{u_j\}_{j=n}^{n+2m-1}$ – it is a $2m \times 2m$ matrix. If $T(E)u = \lambda u$ for $\lambda \in \mathbb{C}$, $Ku = Eu$ has a Floquet solution with $u_{kn+j} = \lambda^k u_j$. If $T(E)$ is diagonalizable, the set of Floquet solutions is a basis for all solutions of $Ku = Eu$. 

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$$p(x, y) = y^2 - y\Delta(x) + 1$$
If $T(E)$ has Jordan anomalies (see [8] for background on linear algebra), there is a basis of modified Floquet solutions with some polynomial growth on top of the exponential $\lambda^k$.

The values of $\lambda$ are determined by

$$p(E, \lambda) = \det(\lambda 1 - T(E))$$

Since

$$\det(\lambda 1 - T(E)) = \lambda^{2m} \det(1 - \lambda^{-1} T(E))$$
$$= \lambda^{2m} \left( \sum_{j=0}^{2m} (-\lambda)^j \text{Tr}(\wedge^j(T(E))) \right)$$
$$= \sum_{j=0}^{2m} \lambda^j p_j(E)$$

where $\wedge^j$ is given by multilinear algebra ([8, Section 1.3]) with $\wedge^0(T(E)) = 1$ on $\mathbb{C}$ so its Trace is 1. Thus in (6),

$$p_{2m}(E) = 1, \quad p_j(E) = (-1)^j \text{Tr}(\wedge^{2m-j}(T(E)))$$

and $p_j$ is of degree at most $(2m - j)n$ in $E$.

Since $S$ and $K$ are commuting bounded normal operators, they have a joint spectral resolution which is supported precisely on the solutions of $p(E, \lambda) = 0$ with $|\lambda| = 1$. By the spectral theorem (equivalently, a direct integral analysis), we thus have the main result of this note:

**Theorem 2.1.** Let $K$ be a self-adjoint, non-degenerate, width $2m + 1$, period $n$ matrix. Then for $p$ given by (6)/(7), we have that

$$p(K, S^n) = 0$$

We end with a number of comments:

(1) We used the self–adjointness of $K$ to be able to exploit the spectral theorem. But just as the Cayley–Hamilton Theorem for finite matrices holds in the non-self-adjoint case, it seems likely that Theorem 2.1 is valid for general non-degenerate, periodic $K$, even if not self–adjoint

(2) Since $K_{jj-m} \neq 0$, the transfer matrix, $T(E)$ is invertible and thus $\det(T(E))$ has no zeros. Since it is a polynomial, it must be constant, that is $p_0(E)$ is a constant. It is thus of much smaller degree than the bound, $2mn$, obtained by counting powers of $E$.

(3) In many cases of interest, $T(E)$ will be symplectic, i.e., there exists an anti-symmetric $Q$ on $\mathbb{C}^{2m}$ with $Q^2 = -1$ so that $T(E)^t Q T(E) = Q$. Such a $T(E)$ has $T(E)^{-1}$ and $T(E)^t$ similar, so the eigenvalues $\{\lambda_j\}_{j=1}^{2m}$ can be ordered so that $\lambda_{2m+1-j} = \lambda_j^{-1}$, $j = 1, \ldots, m$. It follows that $\det(T(E)) = 1$ but even
more, we have that
\[
\text{Tr}(\wedge^k(T(E))) = \sum_{j_1 < \cdots < j_k} \lambda_{j_1} \cdots \lambda_{j_k}
\]
\[
= \sum_{j_1 < \cdots < j_{2m-k}} \lambda_{j_1}^{-1} \cdots \lambda_{j_{2m-k}}^{-1} \tag{9}
\]
\[
= \sum_{j_1 < \cdots < j_{2m-k}} \lambda_{j_1} \cdots \lambda_{j_{2m-k}} \tag{10}
\]
\[
= \text{Tr}(\wedge^{2m-k}(T(E)))
\]
and \(p_{2m-k}(E) = p_k(E)\). In the above, (9) follows from the fact that the product of all the \(\lambda\)'s is 1, and we can sum over the complements of all \(k\)-sets. (10) then uses the fact that \(\lambda_{m+1-j} = \lambda_j^{-1}, j = 1, \ldots, m\).

(4) One can ask whether there is a magic formula in this case, i.e. does \(p(\tilde{K}, S^n) = 0\) imply that \(\tilde{K}\) is periodic and isospectral to \(K\). There is already one sublety one faces at the start. If \(\tilde{K}\) is not supposed apriori \(n\)-periodic, then \(S^n p_j(\tilde{K})\) may not equal \(p_j(\tilde{K})S^n\), so there is a question of what \(p(\tilde{K}, S^n) = 0\) means. Even if one supposes that \(\tilde{K}S^n = S^n\tilde{K}\), \(p(\tilde{K}, S^n) = 0\) and the spectral theorem only implies that \(\text{spec}(\tilde{K}) \subset \text{spec}(K)\), so there is more to be proven. Indeed, the isospectral set in this case remains to be explored.

(5) It seems unlikely that there is another independent relation besides (8) between a polynomial in \(K\) and Laurent polynomial in \(S\). In general one cannot hope that \(p(K, S^n) = 0\) yields a polynomial in one variable so that \(Q(K)\) is a Laurent polynomial in \(S^n\) but it remains to find an explicit example where one can prove that the Breuer-Duits question has a negative answer.

There are lots of interesting open questions connected to our main result, Theorem 2.1.

3. References


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